INDIAN INSTITUTE OF TECHNOLOGY DELHI

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NPTEL ONLINE CERTIFICATION COURSE

Stochastic Processes - I

Module 7: Brownian Motion and its Applications

Lecture-04

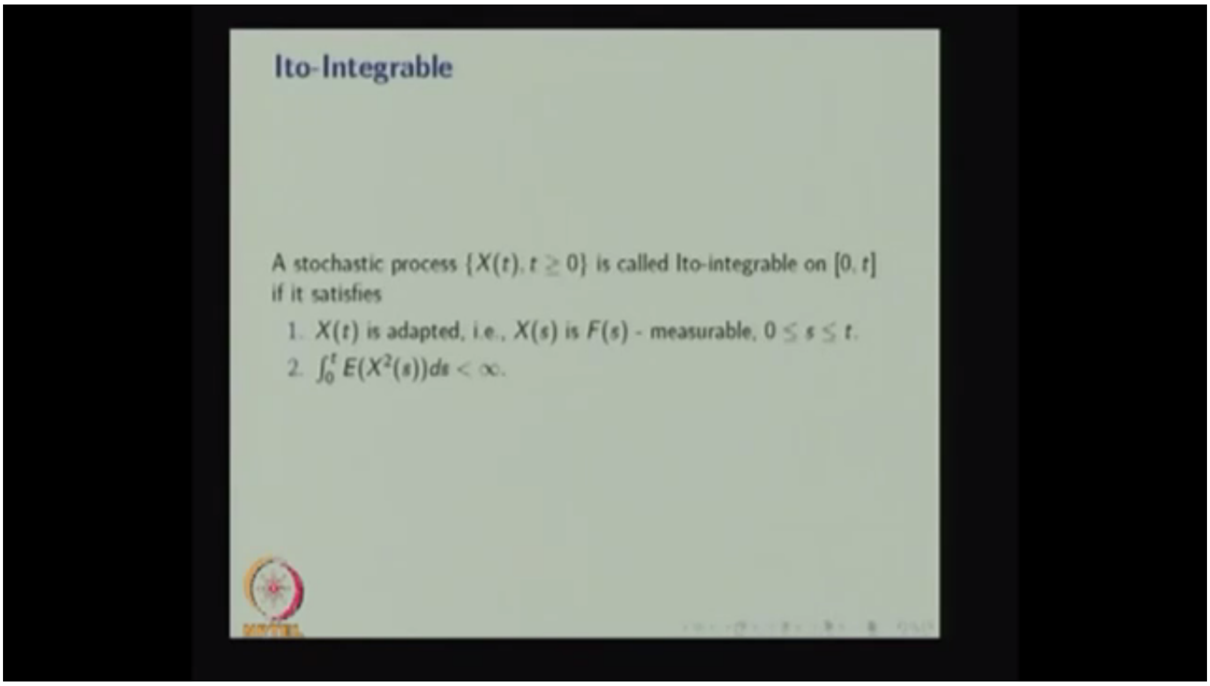
Ito Integrals (contd.)

With

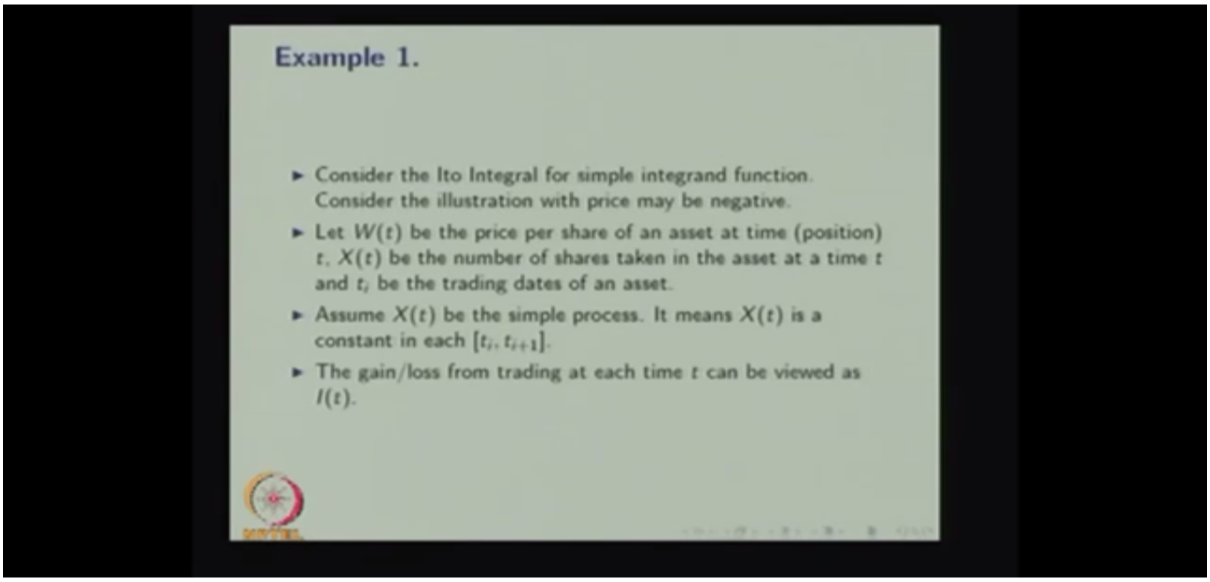
Professor S. Dharmaraja

Department of Mathematics

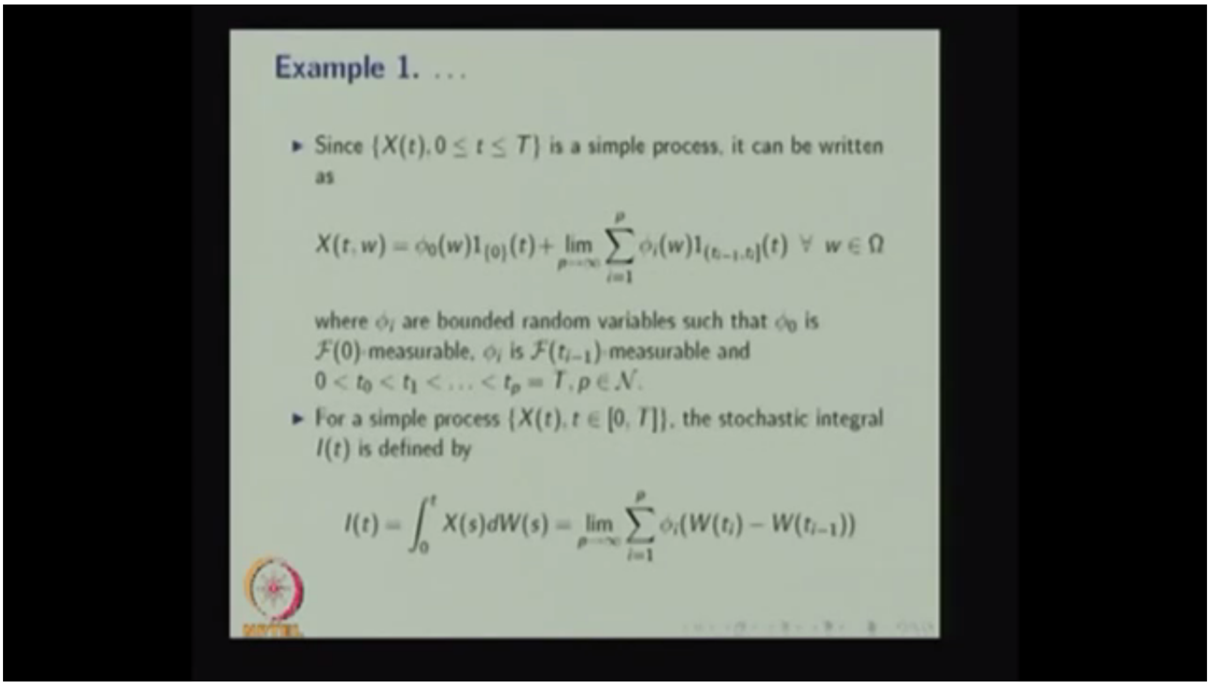
Indian Institute of Technology Delhi



Here, since the expression on the right-hand side is the sum of n random variables, therefore, the limit here refers to one of the mode of convergence of a sequence of random variables that is limit in the sense of probability in distribution almost surely or in earth mean. Then we can say the Ito integral, whenever we have a stochastic process is Ito integral, it has to be adopted that is a measurable as well as a it should be a mean square integral. When these two conditions are satisfied then we can say the stochastic processes is Ito integral, and Ito integral which we have discussed in the first line, $I(t)=\int_0^t X(u)\,dW(t)$. So a stochastic process is called the Ito integral on the interval 0 to t if it satisfies the first condition that means X(s) is f(s) measurable and it is mean square integral. That is $\int_0^t$ expectation of X^2ds has to be finite.



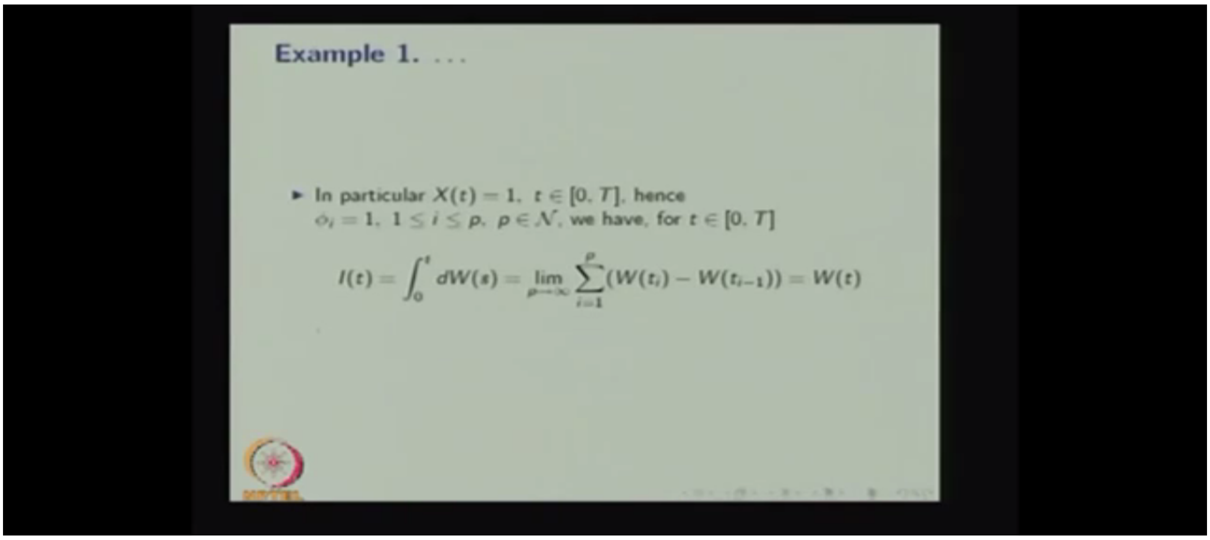
Now we are going to discuss a few examples. Through that we are going to study the Ito integral. First, we consider the simple situation. Consider the Ito integral for a simple integrant function that means, for example, you can see illustration with the price may be negative. Let W(t) be the price per share of a certain type. t and X(t) be the number of shares taken in the certain time t and t_i be the trading dates of an asset. Assume that X(t) be a simple process. That means it takes a constant value in the interval t_i to t_{i+1}. The gain or loss, there's a possibility the price may go down and so on, maybe negative also. So with that assumption, the gain or loss from trading at each time can be viewed as I(t) because I(t) is nothing but $\int_0^t X(t)dW(t)$. Therefore, the gain or loss from trading at each time can be viewed as I(t).



So since it is a simple process, you can write it X(t,w) where w is belonging to Ω, you can write down the simple process in this fashion where φ_i will be the bounded random variables such that φ_0 is F(0) measurable and φ_i is $F(t_{i-1})$ measurable. Because it is taking a constant running between the interval, t, it takes a constant between the interval t_i to t_{i+1} therefore, you can write down the simple process X(t) in this form.

So for a simple process X(t) the stochastic integral I(t) that is nothing but the gain or loss that I(t) is nothing but $\int_0^t X(s)dW(s)$ that is nothing but $\lim_{p\to\infty} \sum_{i=1}^{p} \varphi i(W(ti)-W(ti-1))$

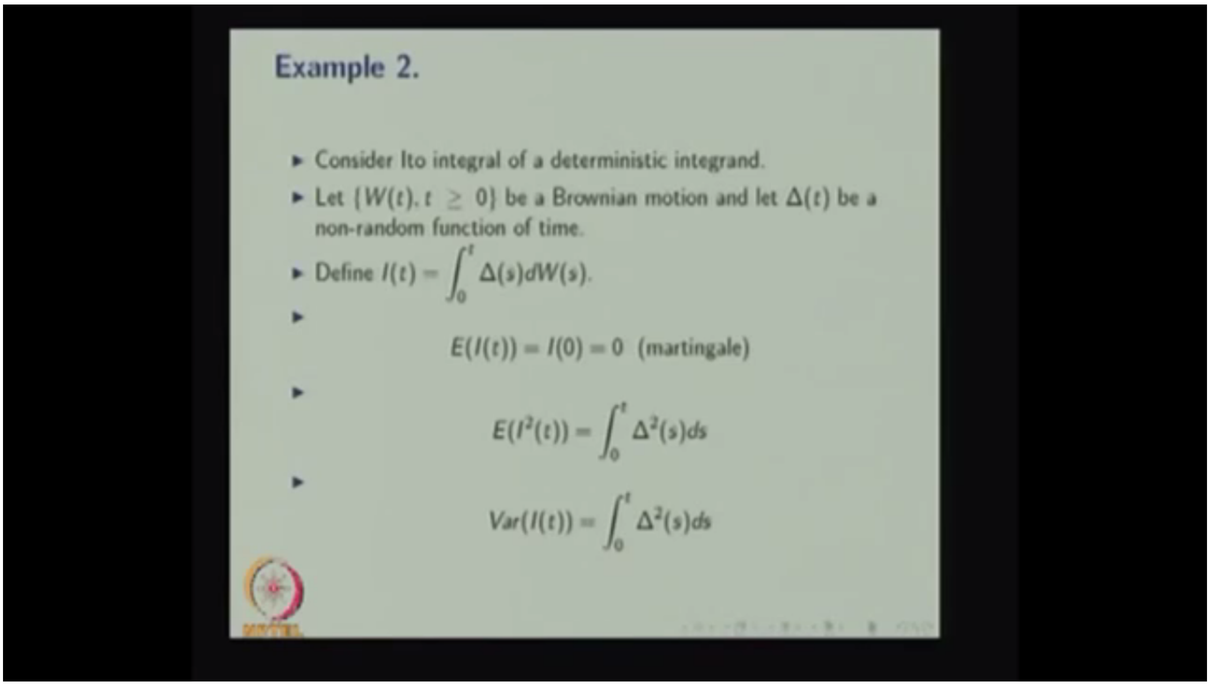
In this limit, the first term of the above equation that is a φ_i indicator function of t does not contribute anything. Hence W(0) = 0.



In particular, if you take X(t) = 1 for all the interval then this I is equal to 1 then φ_i is equal to 1 for all the interval. Therefore, the I(t) is nothing but since X(t) is equal to 1 you are just integrating 1 with respect to W(s). Therefore, you will get W(t).

We calculated Ito integral for a simple integrant. As a general integrant can be expressed as a limit of simple integrant. Therefore, the Ito integral of general integrals can be obtained by taking the limit of sequence of Ito integral of the simple integrals.

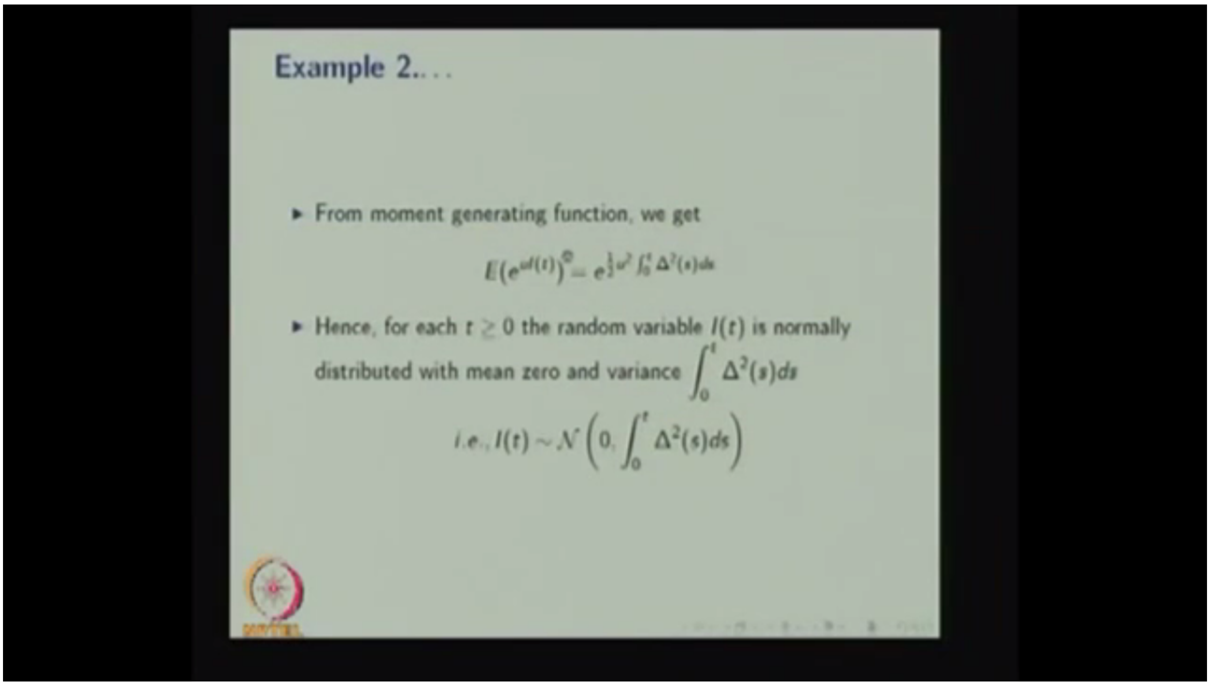
This is a simple situation in which integrant is a simple function.



Suppose the integrant is a deterministic function, not a random. The Δt is not a random. Now we are defining the Ito integral, I(t) that is $\int_0^t \Delta(s)dW(s)$. Since expectation of I(t) is nothing but a I(0) that is equal to 0, because the Ito integral satisfies the martingale property. This we are going to discuss at the end of this lecture, the Ito integral is going to be martingale. So this property is used to here; otherwise you can directly find out expectation of I(t) that you will end up 0 or by using the martingale property it will be I(0) but I(0) is equal to 0.

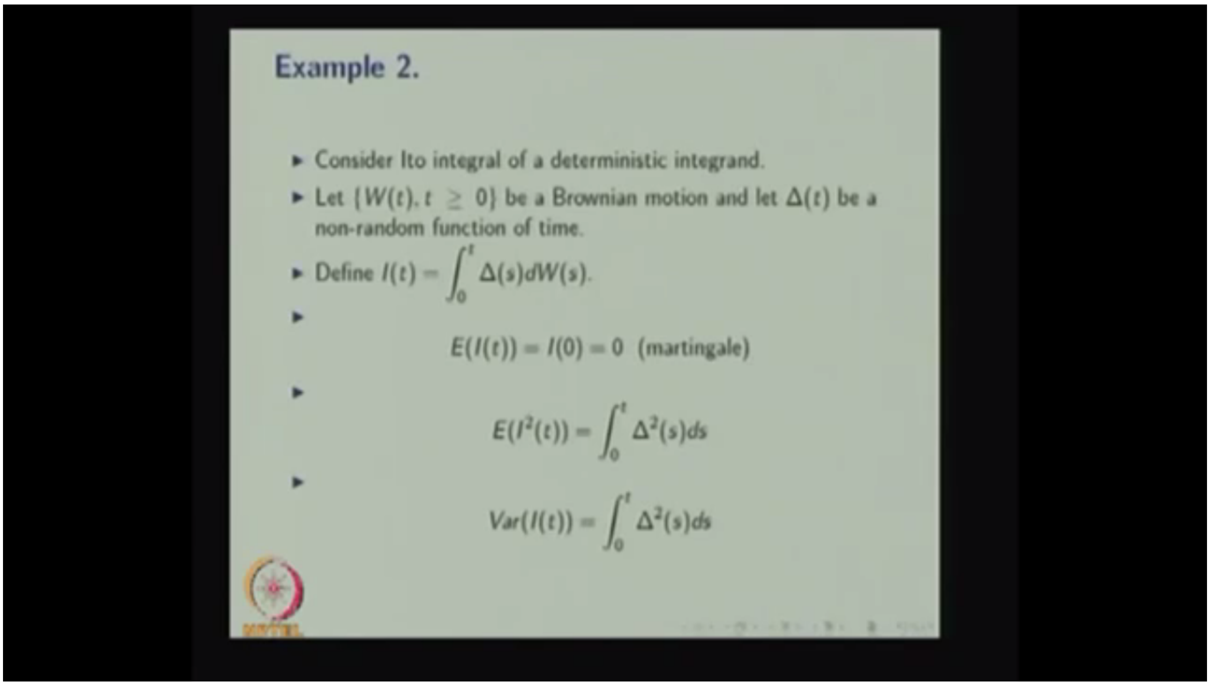
You can find out the variance of I(t) also. For that, you have to find out expectation of $I^2(t)$ square of T that is nothing but expectation of $\int_0^t \Delta(u)dW(s)$ then you can find out the expectation of $I^2(t)$. So that land up $\Delta^2(s)t(s)$. So using this, we'll find out the variance. Variance is equal to expectation of $I^2(t)$ minus expectation of $I(t)^2$ Since expectation of I(t)=0, therefore variance of I(t) is same as expectation of $I^2(t)$ that is $\int_0^t \Delta^2 sd(s)$ or you can use the isometric property for finding the expectation of I^2 that is $\int_0^t \Delta^2 sd(s)$. Since Δ(t) is a non-random function, the expectation of Δ^2 is same as a $\Delta^2(s)$.

So one can use the definition and find out this expectation or you can use the isometric property and use the expectation of a constant, the same as a constant. Therefore, it is $\int_0^t \Delta^2 sd(s)$. So in this example we are finding the expectation and the variance of (t). I of

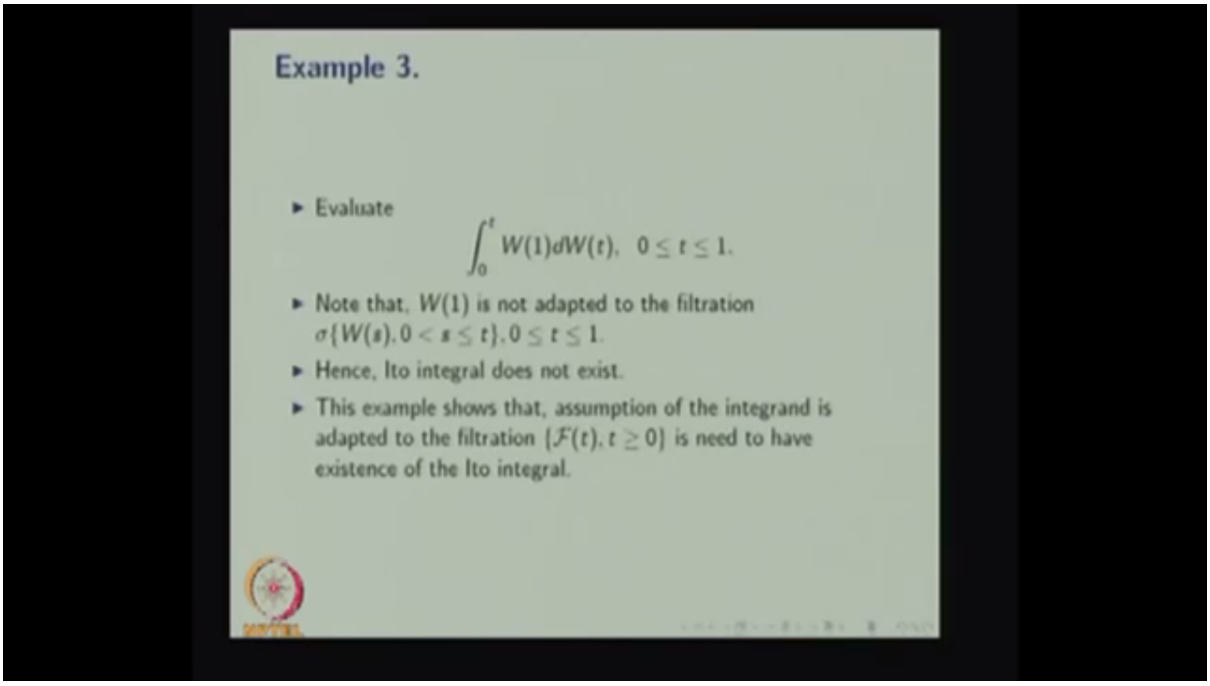


But you can find out the distribution of I(t). You can find out the distribution using -- so find out the moment-generating function of I(t). You know that moment generating function for any random variable is nothing but expectation of e^{ux}. So here you are finding the distribution of the random variable I(t) for fixed t. Therefore, it is expectation of $e^{u}I(t)$. You can find out this value as $e^{1/2u^2}$.

So once you are able to find out the moment-generating function for a fixed t, you can compare the moment generating function of some standard distribution, then we can conclude what is a distribution of I(t). Here, the moment generating function is $e^{1/2u^2}$ and this is of the form the moment generating function of normal distribution. Therefore, we can conclude for each t, I(t) is normally distributed with the mean. Now you have to compare this moment generating function with the moment generating function for function of normal distributed random variable with the mean u and variance σ^2. By comparing the MGFs, you can conclude what is the mean and variance of I(t). So here, you will come to the conclusion the I(t) is normal distributed random variable with the mean 0 and the variance. Variance is $\int_0^t \Delta^2 sd(s)$. For fixed t, it is normally distributed.



So in this example, we are finding the mean and variance of I(t) as well as what is the distribution of I(t) also. We are finding the mean and variance as well as the distribution of I(t) when the integrant is non-random function, when the integrant is non-random function of time.



Third example, here we are evaluating $\int_0^t W(1)\,dW(t)$. Note that W(1) is not adapted to the filtration that is f(t), σ{W(s)} for the interval 0 to 1. As the t increases from 0 to 1, W(1) is a not adapted to the filtration F(t) where F(t) is the natural filtration of Wiener process. So since it is not satisfying the condition of integrant is adopted, therefore this is the Ito integral does not exist.

So this example shows that the assumption of integrant is adapted to the filtration is needed to have the existence of Ito integral. So here, the Ito integral does not exist.

For Further Details Contact

Coordinator Educational Technology Cell
Indian Institute of Technology Roorkee
Roorkee – 247 667

E Mail:-etcell@iitr.ernet.in, iitrke@gmail.com

Website: www.nptel.iitm.ac.in

Acknowledgement
Prof. Ajit Kumar Chaturvedi
Director, IIT Roorkee
NPTEL Coordinator
IIT Roorkee
Prof. B. K Gandhi
Subject Expert
Dr. Gaurav Dixit
Department of Management Studies

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