INDIAN INSTITUTE OF TECHNOLOGY DELHI

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NPTEL ONLINE CERTIFICATION COURSE

Stochastic Processes - I

Module 7: Brownian Motion and its Applications

Lecture-04

Ito Integrals

With

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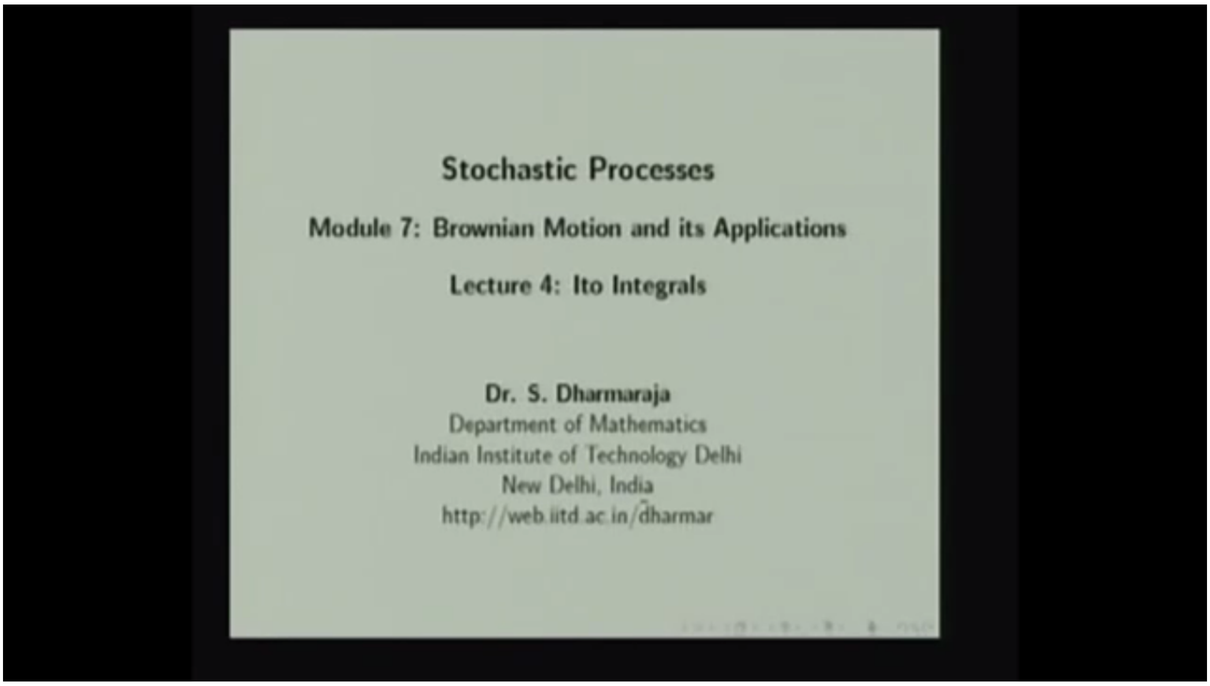
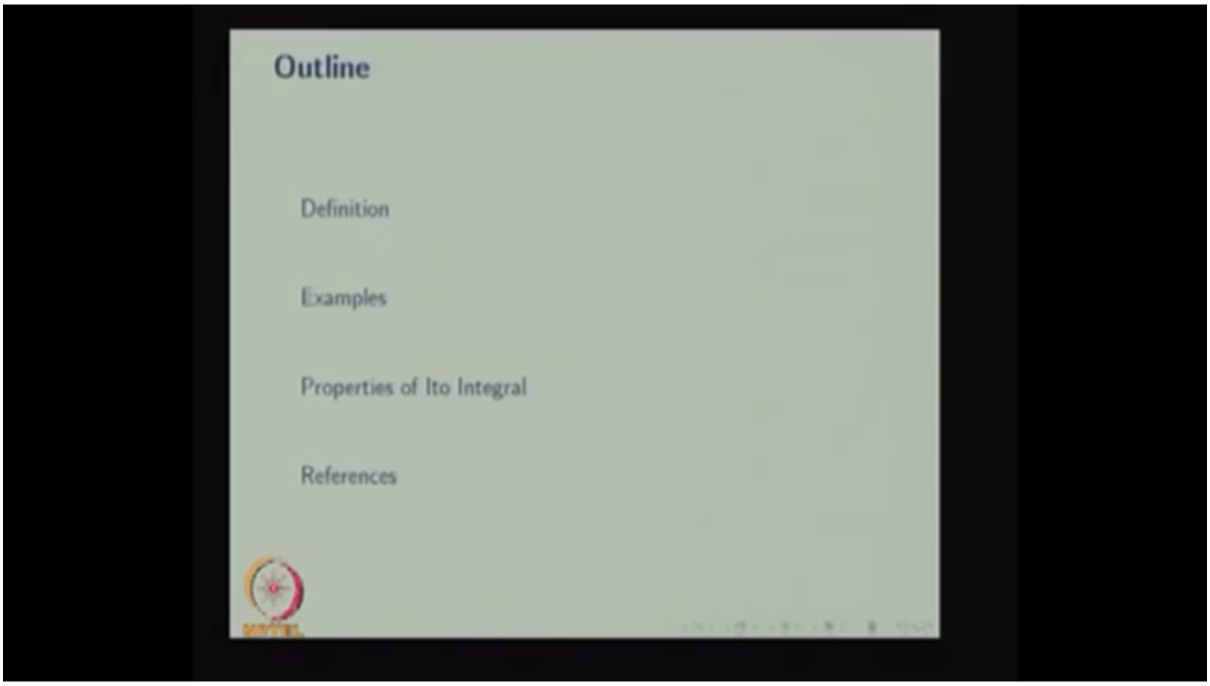
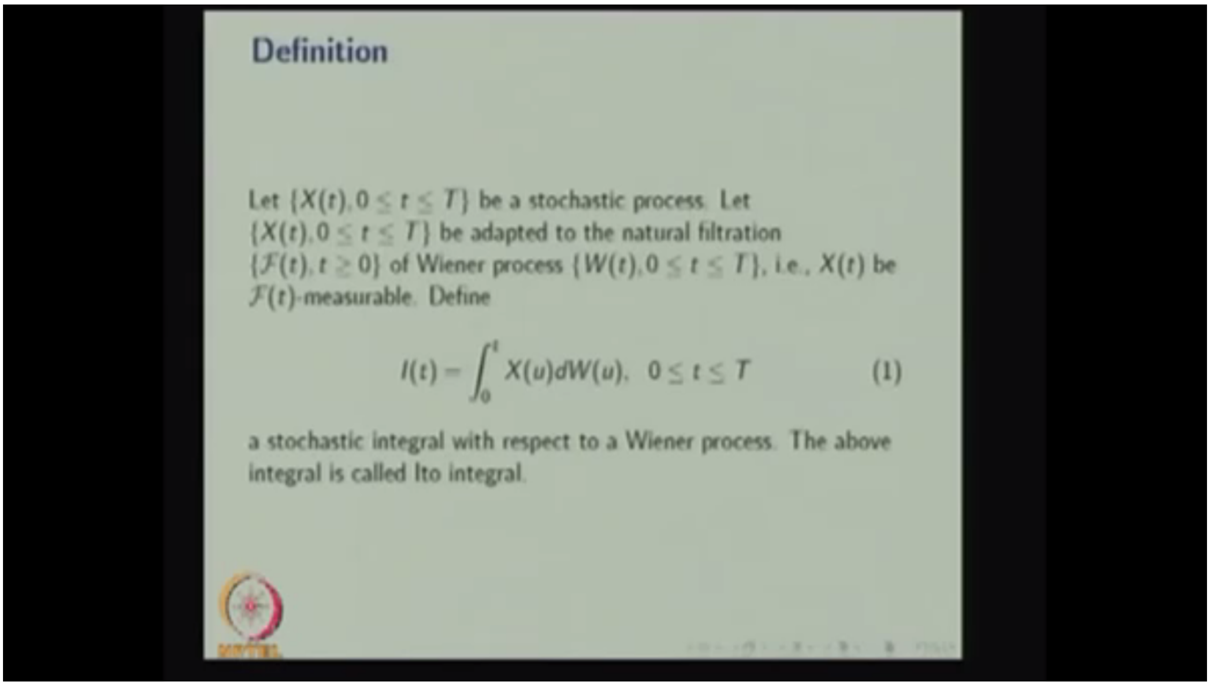


This is Stochastic Processes, Module 7, Brownian Motion and its Applications. Lecture 4, Ito Calculus Ito integrals. In the last three lectures we have discussed the definition and properties of Brownian motion in the first lecture. In the second lecture, we have discussed Geometric Brownian motion and the other process derived from the Brownian motion, and their properties also. The third lecture we have discussed stochastic differential equations. In this lecture we are going to discuss Ito integrals.



First, we are going to start with the definition of Ito integral. Then we say the stochastic process is this Ito integral. Then we are going to discuss what is Ito process. Then followed by these definitions, we are going to discuss few examples. Followed by the examples we are going to discuss the properties of Ito integrals. So with that this lecture will be completed.

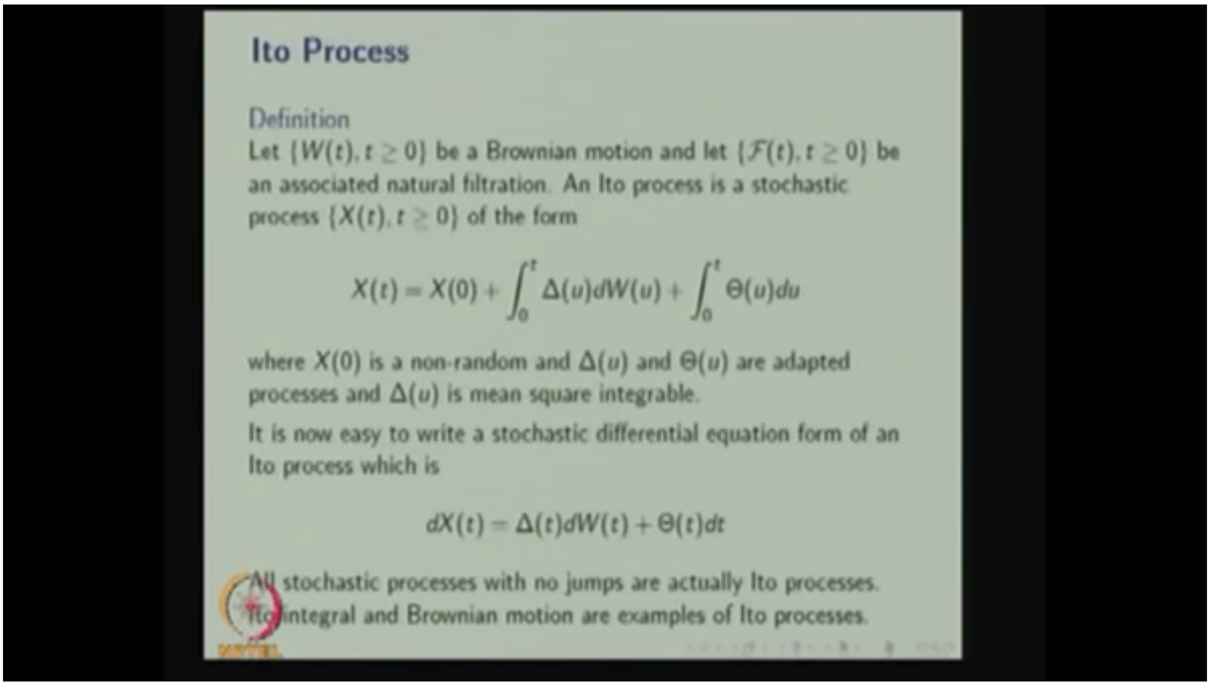


This is the definition of Ito integral. Let X(t) be a stochastic process which is adapted to the natural filtration F(t) of a Wiener process W(t). That is X(t) be a F(t) measurable. Define I(t) that is nothing but integration between the limit 0 to t X(u) integration with respect to W(u). That t lies between 0 to T where t is a positive constant.

I(t) be a stochastic integral with respect to a Wiener process. The above integral is called the Ito integral. A detailed the interpretation and the motivation of this can be found in the reference books. So the Ito integral I(t) is defined in the form of integration 0 to t and the integrant with respect to the integral with respect to the Wiener process, W(t). The Wiener process already discussed, the Wiener process or Brownian motion is discussed in Lecture 1, you can find in Lecture 1, what are all the properties of Wiener process and so on. The Wiener process W(t) is starting with the standard one, W_0 0 to 0 and it has increments of stationary as well as independent and the increments are normally distributed with the mean 0 or the standard Wiener process and the variants is an increment, the difference, for s is less than t, the variance is t-s.

So here we have a stochastic process, X(t) which is defined between the interval 0 to T. The same stochastic process is adopted to the natural filtration, F(t). Already we know that the Wiener process is adopted to the W(t) and here X(t) is also adopted to the W(t) that means for fixed t, X(t) is a F(t) measure. That means in the Sigma field generated by the X(t) for fixed t, will be contained in F(t). That means the information accumulated at time T that is sufficient to find out the value of X(t) for fixed t. So the X(t) is F(t) measurable for all t between the interval, 0 to T.

Then we can define this integral will be call it as a Ito integral. So the X(t) has to be F(t) measurable and F(t) is a natural filtration for the Wiener process, W(t), then for all values of T between the interval 0 to T, I(t) is called stochastic integral because this is not the issued integral; this is a stochastic integral with respect to the Weiner process, W(t). This is the definition of Ito integral.



Now we are going to discuss the Ito process. Let W(t) be the Brownian motion and F(t) be the associated natural filtration. A Ito process is a stochastic process, X(t), is of the form

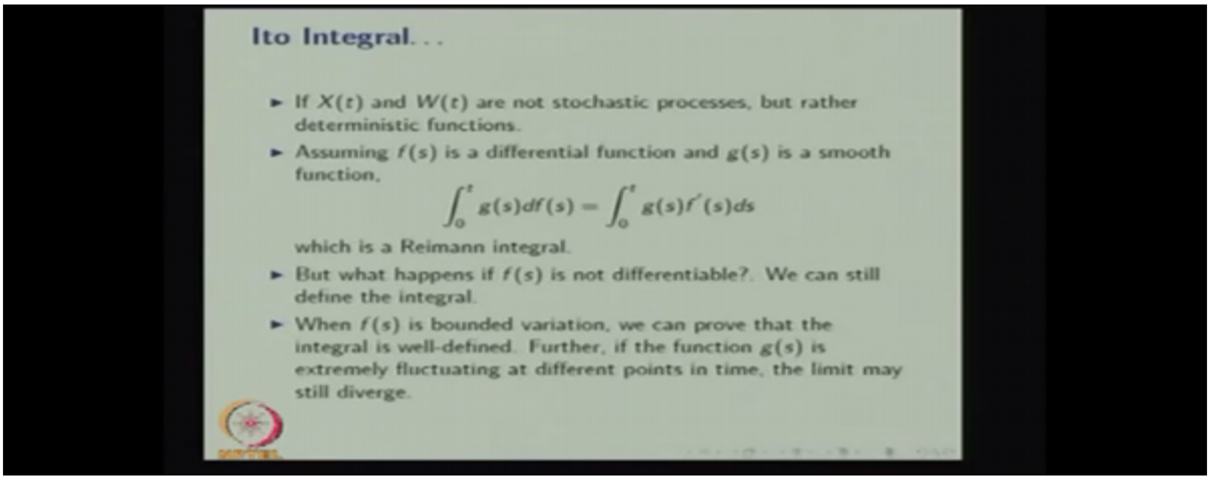
$$X(t)=X(0)+\int_0^t \Delta(u)\,dW(u)+\int_0^t \theta(u)\,du$$

So here we have two types of integration; the one type is the usual Riemann integration, the other one is the Ito integral where X(0) is a non-random and Δ(u) as well as θ(u) are adopted process and the Δ(u) is a mean square integral. If this condition is satisfied then we can say this X(t) is going to be a Ito process. For all T, we are able to write X(t) is of the form the one Ito integral and the Riemann integral and both the integrals are adopted process, that means X(u) is a F(u) measurable as well as θ(u) is F(u) measurable as well as Δ(u) is a mean square Integral, that means that $\int_0^t \Delta(u)\,du$ is a finite one. Then we say it is mean square integral. If these three conditions are satisfied by the X_0Δ(u) θ(u) we can say the X(t) is going to be an Ito process.

Once it is a Ito process, you can write down in a different shape form that is stochastic differential equation because we have dW(u). In the differential form, it will be because X(t) is equal to X(0) plus integration, therefore in a stochastic differential form, it is $dX(t)=\Delta(t)\,dW(t)+\theta(t)\,dt$. So this is a stochastic differential form of Ito process. It means whenever Δ(t) as well as θ(t), both are adopted process and X(0) is non-random as well as Δ(t) mean square integral then the stochastic process written in the stochastic differential form $dX(t)=\Delta(t)\,dW(t)+\theta(t)\,dt$ will be called as a Ito process. So the Ito process is nothing but a stochastic process is of this form as well as satisfying these conditions.

The way use you write the stochastic differential equation $dX(t)=\Delta(t)\,dW(t)+\theta(t)\,dt$ Therefore, if you see the sample path, it is going to be the continuous function. Therefore, all the stochastic process with no jumps are actually a Ito process. Where you see the Ito process, you won't find the jumps. A jump Ito process is nothing but a Ito process in which the moments are discrete rather than continuous So all the stochastic process with no jumps are actually a Ito process because it has the term of dt as well as dW(t) terms for the increment of DX(t) for all T.

Ito integrals as well as the Brownian motion are the examples of Ito process. The way they previously written, I(t) as well as the Brownian motion, both are called the Ito processes.



Now, we are going to discuss the Ito integral. If X(t) and W(t) are not stochastic process, but rather deterministic functions then the situation is different. Assuming f(s) that means we discussed the deterministic situation, we written $\int_0^t g(s)\,df(s)$ where f(s) is a differential function and g(s) is the smooth function. Since it is a differential function you can write df(s) as f'(s)ds which is nothing but the Riemann integral.

Whenever X(t) and W(t) are non-stochastic processes rather than deterministic functions then you can write down this integration g(s)df(s) that is nothing but the Riemann integrals. But in our situation the f(s) is not differential. We can still define the integral, remember that W(t) which is nowhere differentiable as well as unbounded variation. So here we are going to discuss when f(s) is not differentiable as well as when f(s) is bounded variation then still the integral is well-defined.

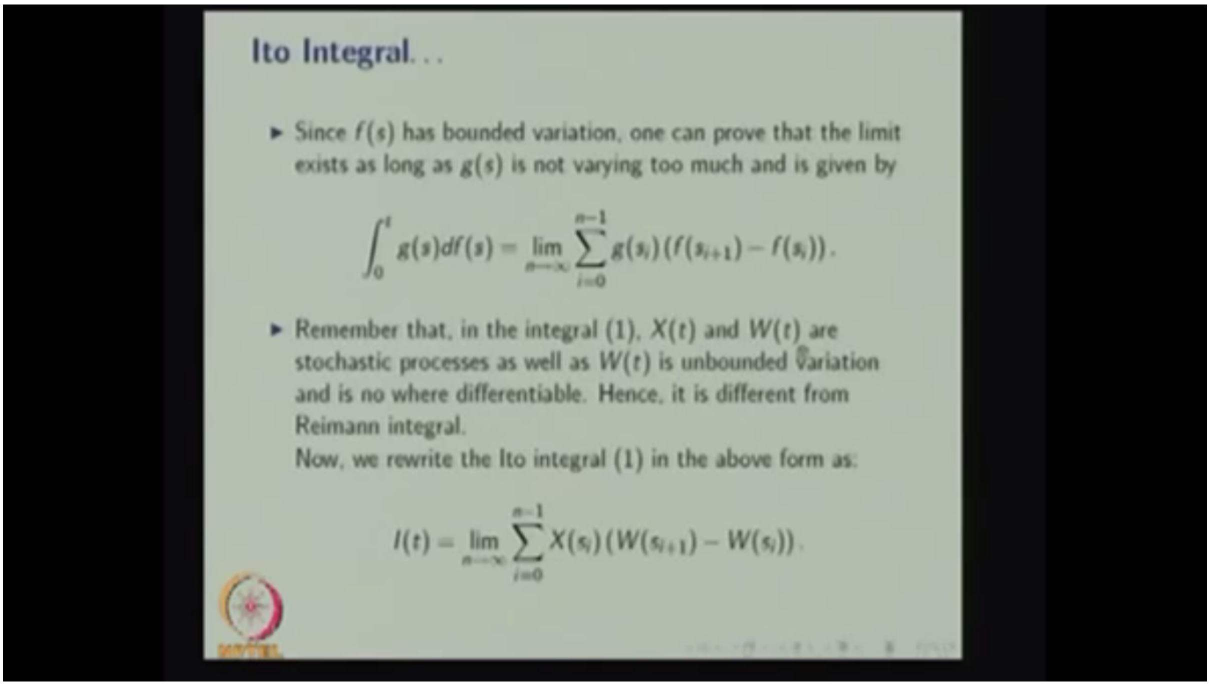
Further if the function g(s) is extremely fluctuating at different points in time, the limits mean still diverge. When f(s) is bounded variation we can prove that the integral is well-defined. Further, if the function g(s) is extremely fluctuating at different points in time the limit may still diverge as s tends to infinity.

Since f(s) is bounded variation, one can prove that the limit exist as long as g(s) is not varying too much and it is given by the $\int_0^t g(s)\,df(s)$ is nothing but

$\lim\; n \to \infty \sum_{i=0}^{n-1} g(s\,i)\left(f(s\,i{+}1) - f(si)\right)$. That means since the integration is with respect to since df(s) so you can find the difference, f(s+i)-f(s_i) then multiply it with g(s_i). That means you are partitioning the interval 0 to t into n parts then as n tends to infinity, then that summation limit, n tends to infinity, will be the integration.

Since f(s) is bounded variation, you can prove that the limit exists. Remember that in the integral 1 that is this is integration, Ito integral I(t) X(u), X(u), X(u) is a random variable for fixed u, W is a random variable for fixed u and W(u) is nowhere differentiable as well a unbounded variation.

Now we have discussed what is the situation if f(s) is a bounded variation, the integration with respect to f(s) as far as f(s) is bounded variation.



Whereas our Ito integral X(t) and W(t) are stochastic processes as well as W(t) is unbounded variation and nowhere differential. So that's the difference between the Riemann integral and the integral which we are discussing now that is Ito integral.

The integrity is X(t), the integration with respect to W(t) where W(t) is an unbounded variation as well as nowhere differential whereas the Riemann integral which we have discussed, the f(s) is differentiable as well as later we discuss f(s) is a bounded variation, whereas the Ito integral W(t) is unbounded variation as well far as W(t) is nowhere differential.

Hence, it is different from the Reimann integral. So with the Reimann integral, the integration with respect to the function which is a bounded variation and it is differential whereas here, the Ito integral, it is unbounded variation as well as nowhere differential. Now you can rewrite the Ito integral 1 in the above form, in the following form.

I(t) that is nothing but the way we have written in the above form, Riemann integral $\int_0^t g(s)\, df(s) \text{ is limit } n \to \infty$, the same way you can write

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} X(Si)(W(Si+1) - W(Si))$$

The way we have written the Riemann integral, in the same way we are writing the Ito integral where X(t) is an adopted process and the W(t) is a Weiner process or Brownian motion and X(t) is adapted to the f(t) and the Ito integral I(t) can be written as $\lim_{n \to \infty} ¿$ the summation form.

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