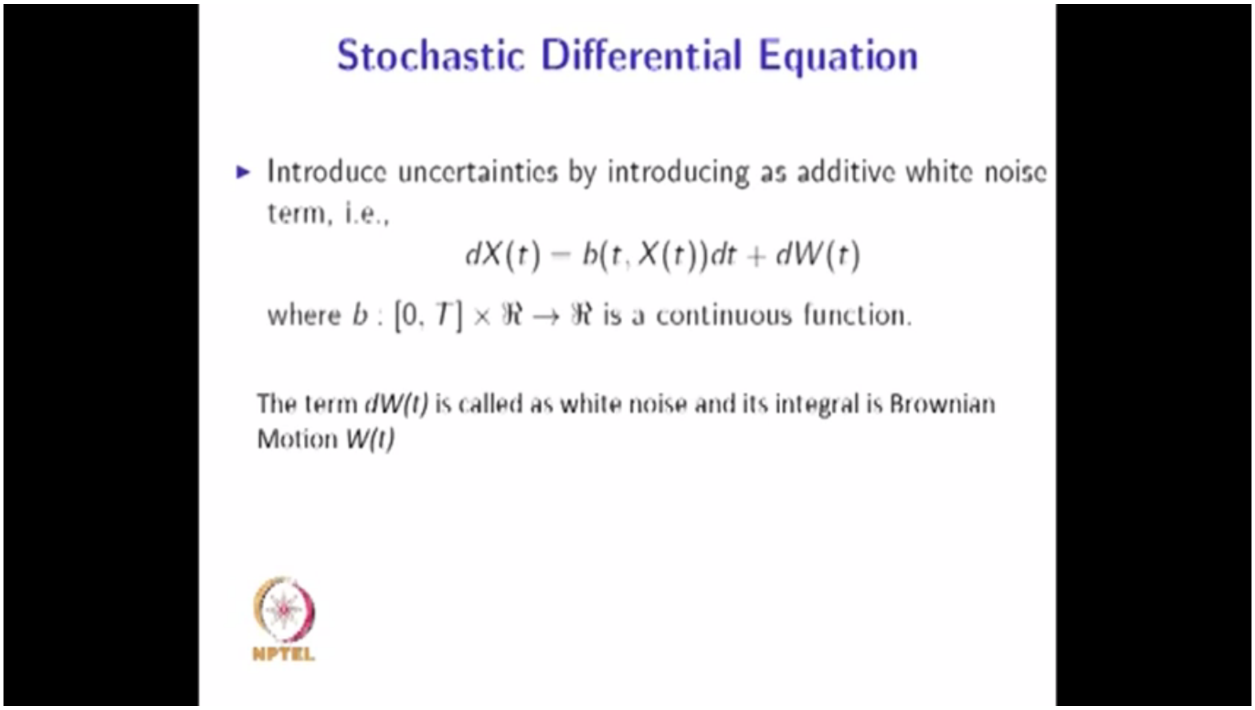
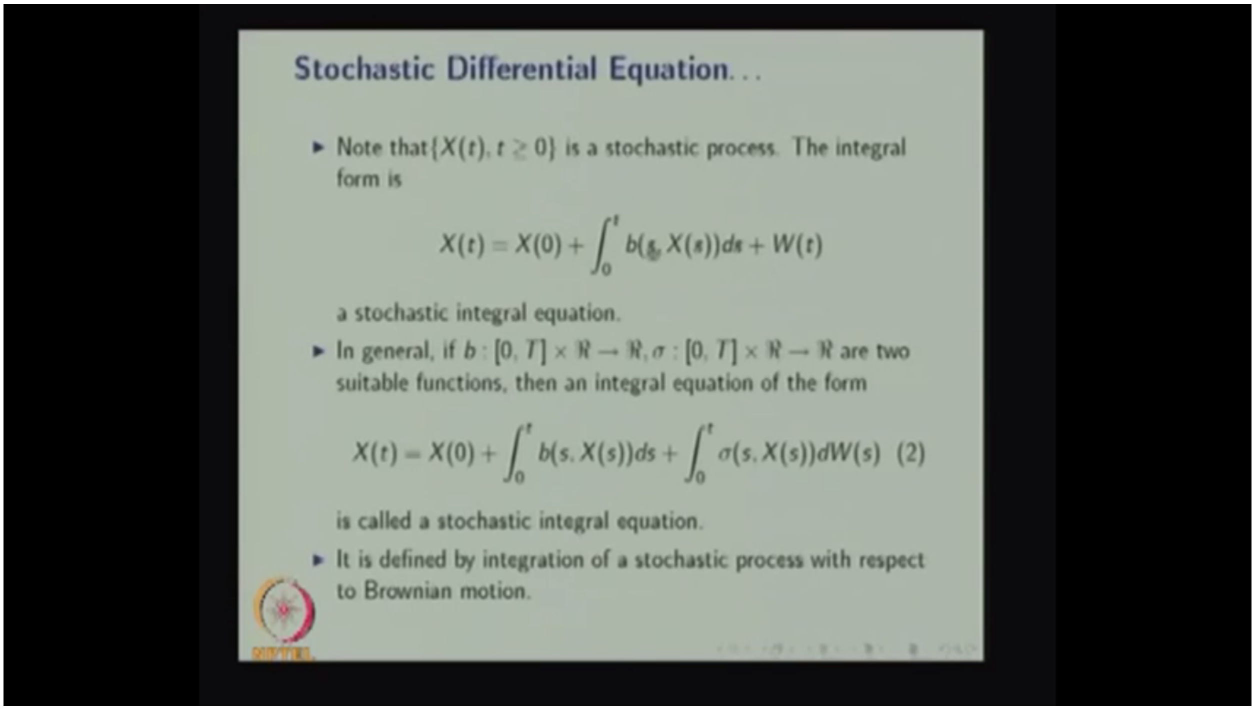
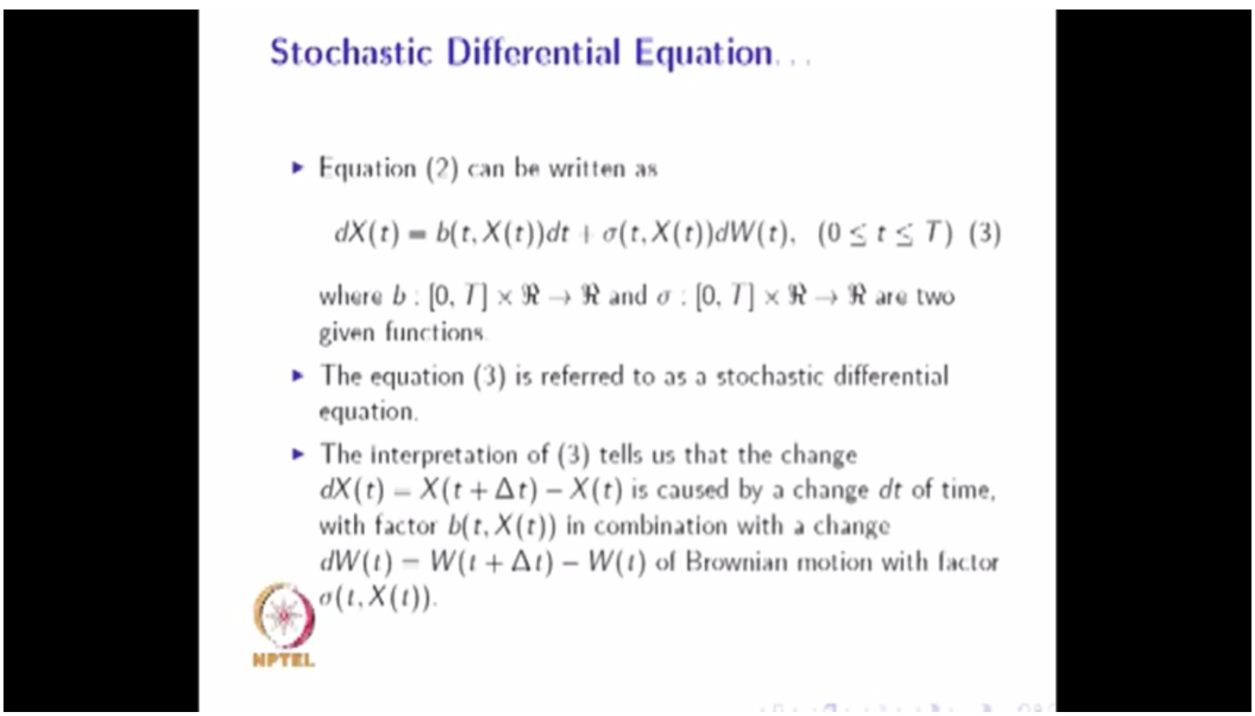


Now we are moving into the stochastic differential equations. Introduce uncertainties by introducing as additive white nice term that is a dXt is equal to b of t,Xt dt plus dWt where b is a real valued continuous function from 0,T cross R. The term dWt is called as a white noise and its integral is Brownian motion Wt. Here the above equation is also known as stochastic differential equation or SDE the meaning of which would be more clear after the introduction of a stochastic integral concept.

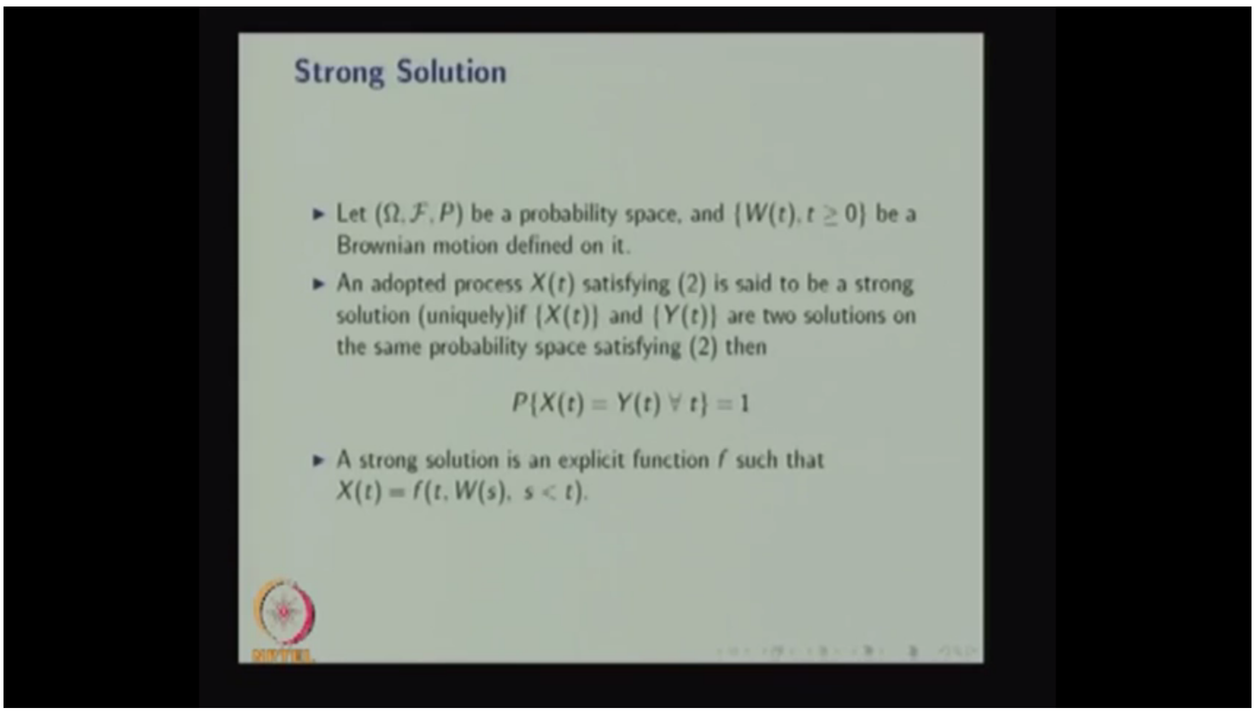


Note that X of t is a stochastic process. The integral form of the differential equation is the X of stochastic differential equation is X of t is equal to X of 0 plus integration 0 to t of b of s of X of s ds plus Wt is a stochastic integral equation.

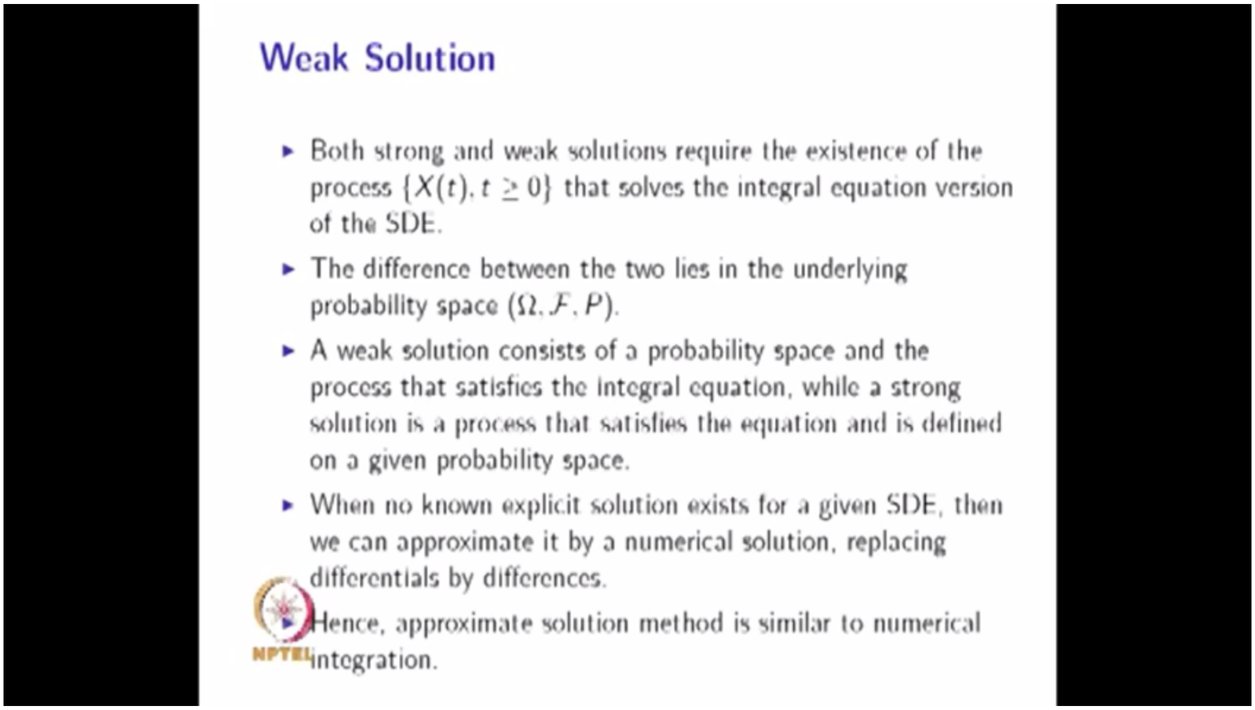


In general if b and Sigma are the two suitable functions then the integral equation of the form X of t is equal to X of 0 plus integration 0 to t function b division with respect to s plus 0 to t Sigma s of X of s dWs. In the equation two, the first equation – the first integral is different from the second integral. The second integral is integration with respect to the Brownian motion sample path Ws. This integral equation is defined by the integration of stochastic process with respect to the Brownian motion. So the equation two that is nothing but this equation, equation two can be written as dXt is equal to B of t, Xtdt plus sigma t of Xt dWt where t is lies between 0 to T where b and Sigma are two given functions. This the equation 3 is referred to as a stochastic differential equation. The interpretation of equation 3 tells us that the change that is d of Xt that is nothing but the X of t plus Delta t minus Xt is caused by a change dT of time with the factor b of a t, Xt in combination with the change dWt that is nothing but W of t plus delta t minus Wt of Brownian motion with the factor Sigma of t,Xt.

The Brownian motion is adopted to the natural filtration. So the unknown is in the Sigma as well as b and the increment of Brownian motion therefore this equation is called stochastic differential equation.

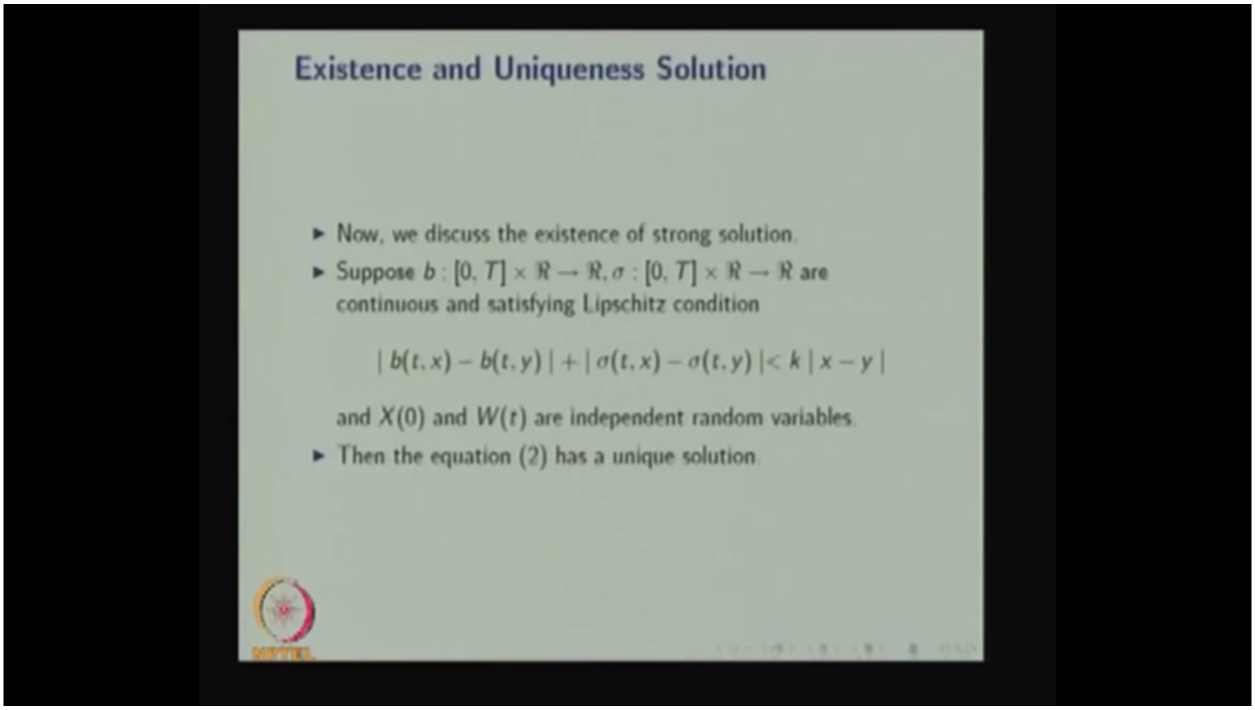


Now we are going to discuss there are two types of solutions for the stochastic differential equation. The first type is called a strong solution. The second type is called weak solution. So we are going to discuss the strong solution first. Let us Sigma sorry let Omega FP be the probability space and Wt be a Brownian motion defined on it. The adopted process X of t satisfying the equation two that's a stochastic differential equation is said to be a strong solution uniquely if X of t and the Wt are the two solutions on the same probability space satisfying the stochastic differential equation two then the probability of X of t is equal to Y of t for all t that will be 1 then Xt is called a strong solution and it is also a unique solution. That means if you have another solution Y of t then probability of X of t is equal to Y of t for all t will be 1. In general a strong solution is a explicit function F such that X of t is a function of t, Ws where s is less than t. One can write the solution in that explicit function F of t with the Brownian motion. Then this solution is called the strong solution.



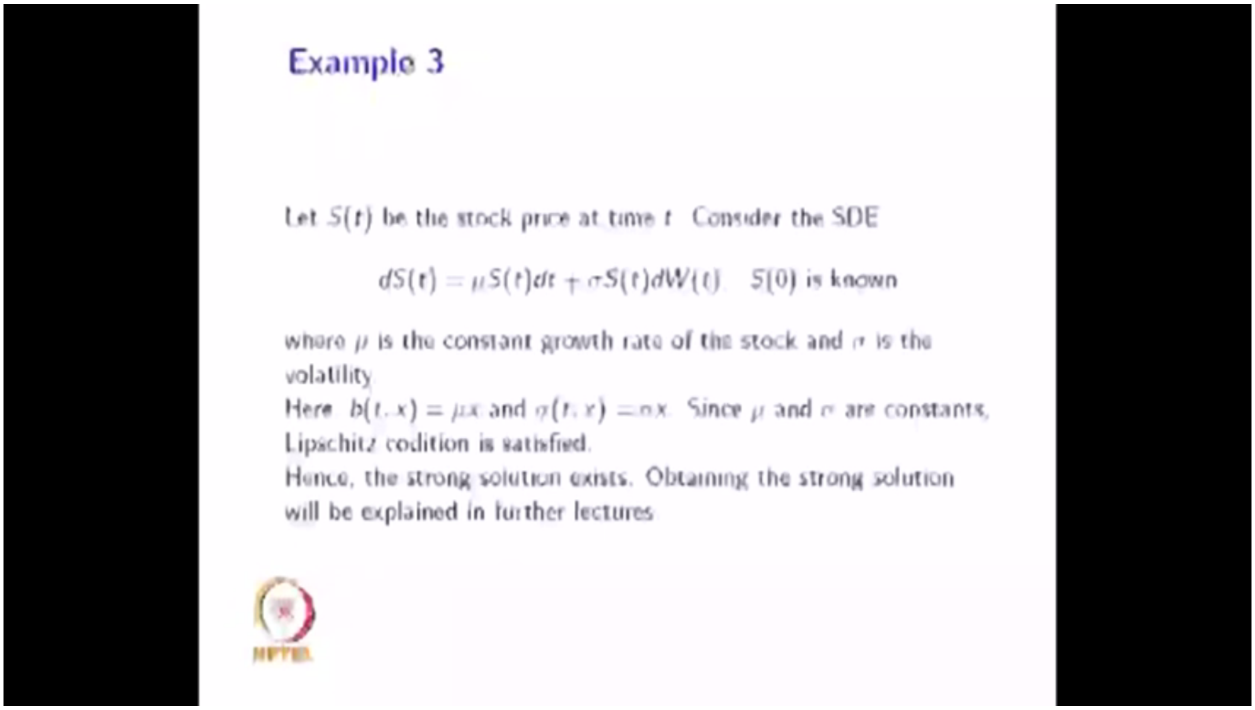
Now we are going to discuss what is the weak solution of stochastic differential equation. Weak solution both strong and weak solutions require the existence of the process Xt that solve the integral equation version of the SDE. The difference between the two lies in the underlying probability space. You have each solution consists of a probability space and the process that satisfies the integral equation. While a strong solution is the process that satisfies the equation and is defined on a given probability space. When no explicit solution exists for a given SDE then we can approximate it by the numerical solution replacing differentials by differences. Hence, approximate solution method is similar to the numerical integration.

So with this we have discussed the strong solution and weak solutions of stochastic differential equations.

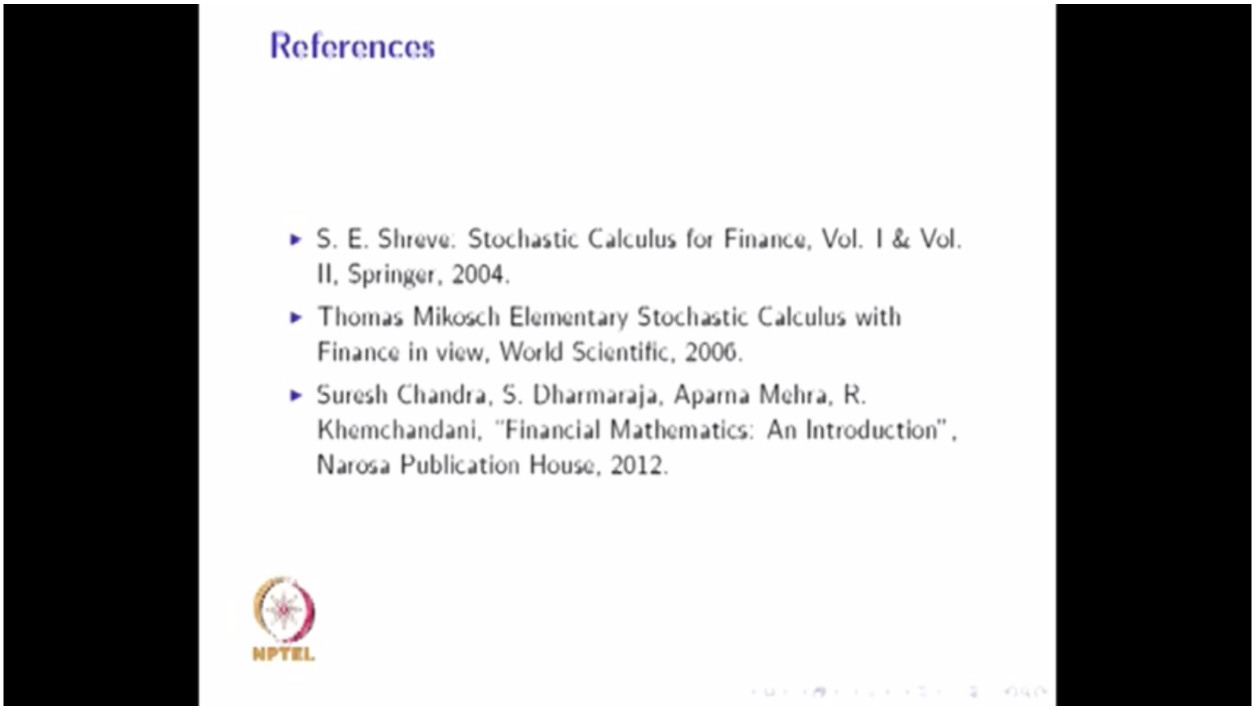


In this course we are interested to find the strong solution not the weak solution. When the above results hold good we say that the quadratic variation accumulates accumulated by the Brownian motion over the interval 0,T is T in mean square and this. Equation 2 can be written as dxt is equal to b of t,xt dt plus Sigma t of xt dwt where t is lies between 0 to T where b and Sigma are two given functions.

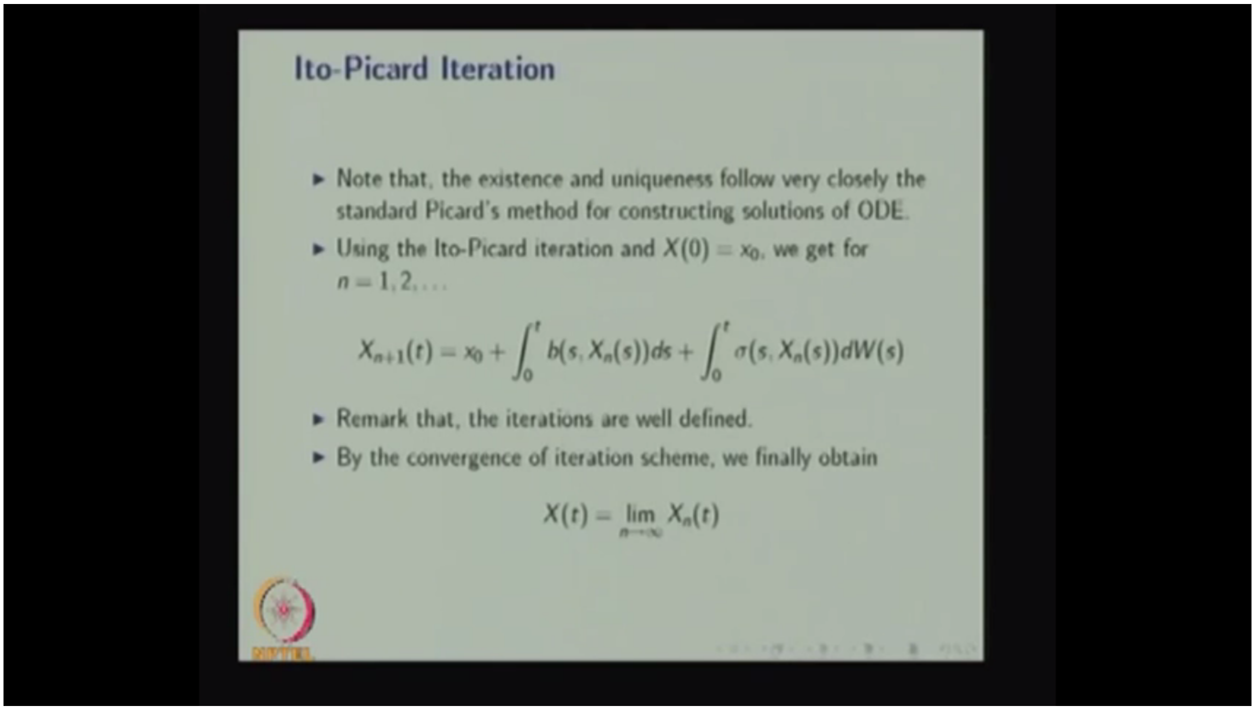
Now we discuss the simple examples for the stochastic differential equation. Consider this stochastic differential equation dxt is equal to xt dwt with x of 0 is equal to 1. here b of t,x is equal to 0 and Sigma for t,x is equal to x. You can verify the Lipscchitz condition for this b is equal to 0 and Sigma is equal to x hence the strong solution exists. Obtaining the strong solution will be explained in the further lectures.



We will see one more example for the stochastic differential equation. Here s of t be the stock price at time t and the corresponding stochastic differential equation for this example is a d of st is equal to mu times s of tdt plus Sigma of s of tdwt with s of 0 is known. Here mu is a constant growth rate of the stock and Sigma is the volatility when you compare it to standard as stochastic differential equation we get a b of t, x is equal to mu of x and Sigma t,x is same as Sigma x. Since mu and Sigma are constants the Lipscchitz condition is satisfied. Hence the strong solution exists and this example also how to find the solution that will be discussed in the further lectures.



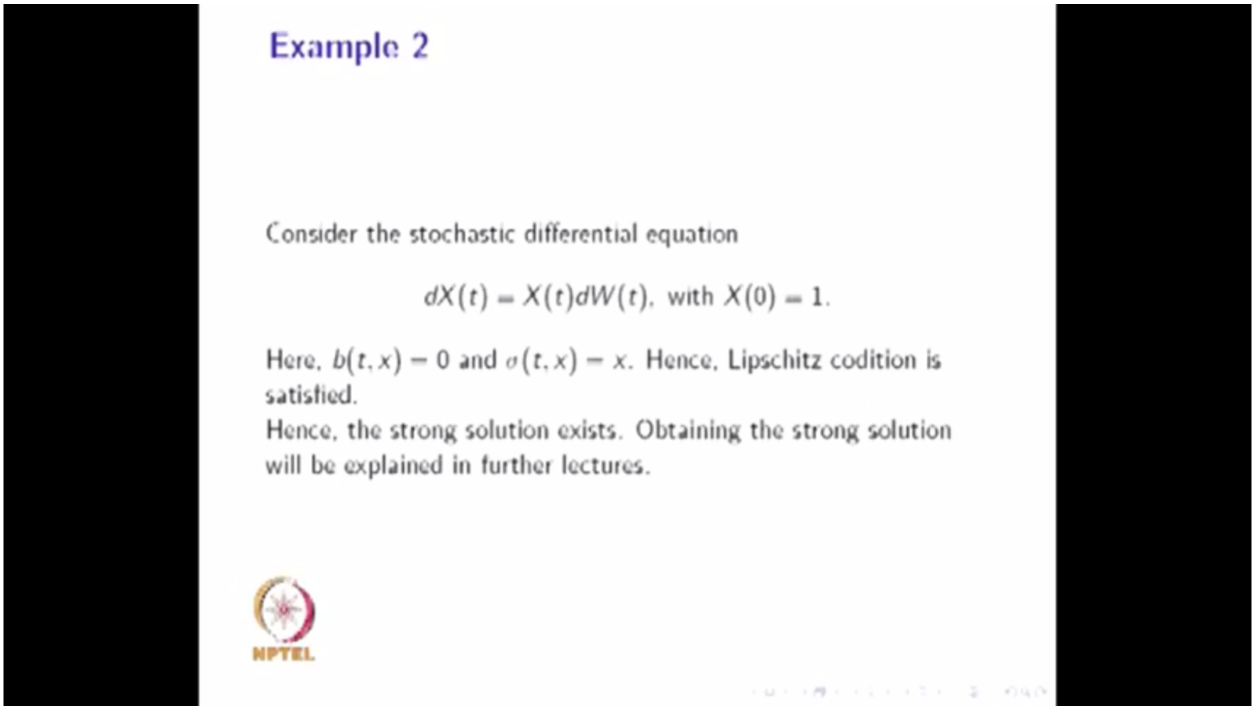
Now we are going to discuss the existence and uniqueness solution that is basically a strong solution. Now we discuss the existence of strong solution. Suppose b is a continuous function. Similarly Sigma is the continuous function. Satisfying the Lipscchitz condition the absolute of difference of b of t,x minus b of t,y plus in the absolute Sigma t,x minus Sigma t,y if this summation is less then k times absolute of x minus y where k is a positive constraint and also the initial distribution x naught and wt are independent random variables then we can say the solution is going to exist that will be unique also. So whenever the Lipscchitz conditions satisfied with the two continuous function b and Sigma for a positive constant k along with x of 0 and w of t are independent random variables. If both the conditions are satisfied by any stochastic differential equations then we can conclude it has the unique and it has the existence of strong solution as well as it will be unique.



This is similar to the existence and uniqueness solution of a ODE the only difference is it does not have the term and the Sigma term so it has only the first term which is less than a times absolute of x minus y that's Lipscchitz condition for ODE. So here also the same thing along with the continuous function Sigma. If this condition is satisfied along with this condition x0 and wt are independent random variables then the given SDE as unique solution have the existence of a strong solution and that will give unique. Note that the existence and uniqueness follow very closely the standard Picard's method for constructing solutions of ODE. You know the Picard iteration for a ODE, Ordinary Differential Equation and this iteration is called Ito-Picard iteration so using Ito-Picard iteration X naught is equal to x0 we get for n is equal to 1, 2, 3. Xn plus 1 of t will be X naught plus the integration plus the another integration.

That means with the initial value X naught you can find for n is equal to 1 you can find for n is equal to 0 you will find the x1 of t first using X naught. Then for n is equal to 1 you will get x of 2 x suffix 2 of t and recursively you can get the xn plus 1 of t for every n as n tends to infinity you can get the x of t.

So remark that the iterations are well defined because it satisfies the Lipscchitz conditions as well as X naught and the wt are independent random variables the solution is going to be exist as well as it will be unique and these iterations are well defined by the convergence of iteration scheme we finally obtain x of t is the limit n tends to infinity xn of t. for every n it is a random variable so this random variable converges to the random variable x of t. So this we are showing through the Ito-Picard iteration. This Ito-Picard iteration is similar to the Picard iteration of ordinary differential equation.



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Here is the list of books for the reference.

In this lecture we have discussed stochastic differential equation. For that we have discussed the the variations of a real valued function starting with the first order variation. Pth order variation. Then followed by that we have discussed the variations of Brownian motion starting with the first order variation, quadratic variation and Pth order variation also. Then we have discussed that stochastic differential equation by adding white noise term in the ordinary differential equation. Then we have discussed the equivalent as stochastic integral equations and also we have discussed the strong and weak solutions and finally we have given existence of – existence as well as a uniqueness of a strong solution. And finally we have discussed Ito-Picard alternation methods.