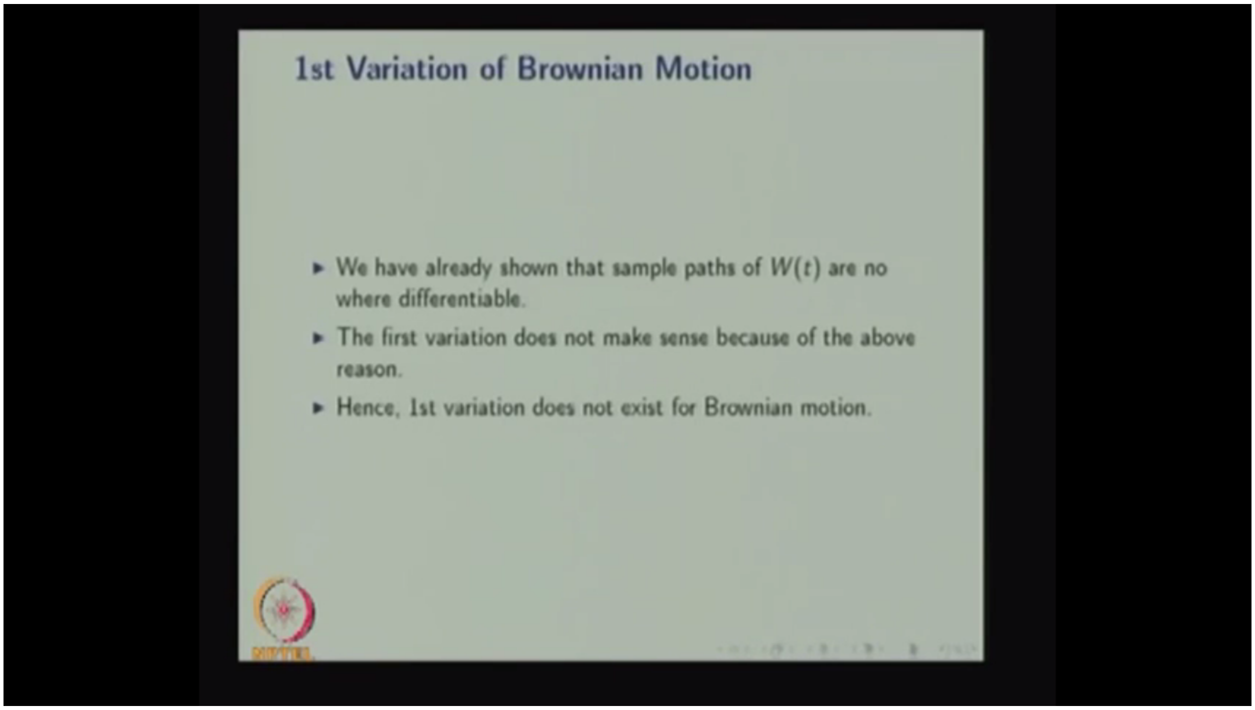
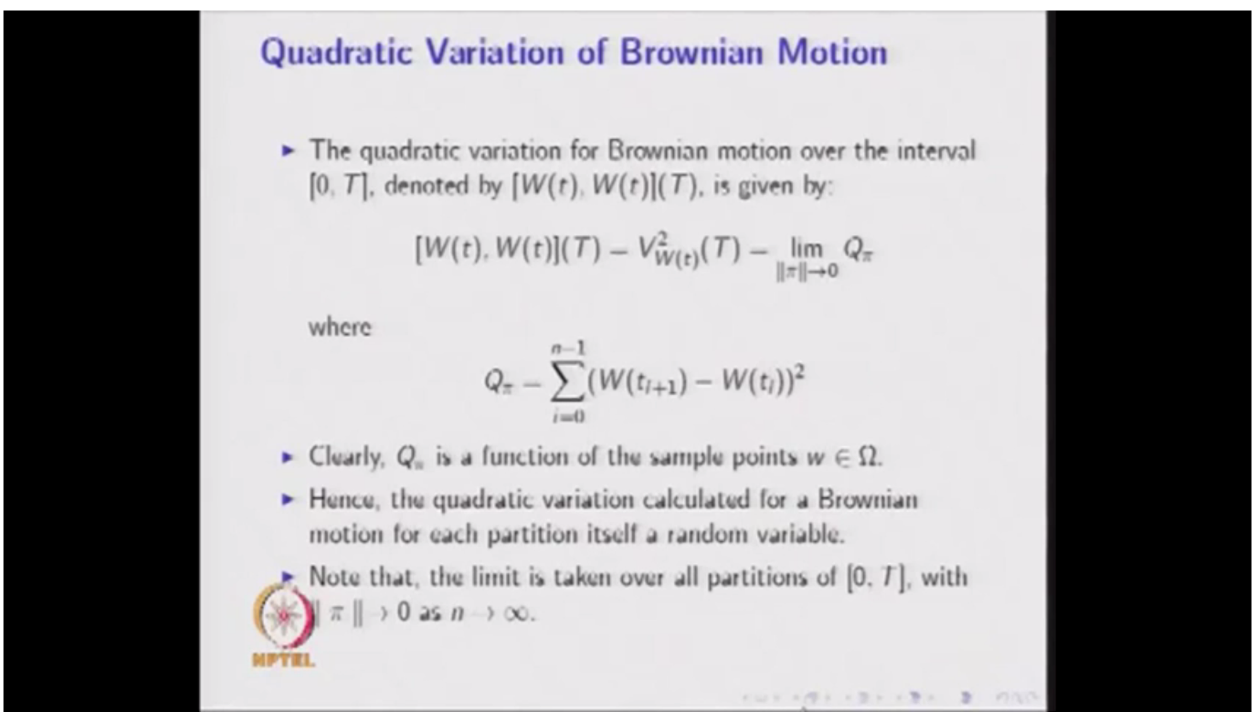
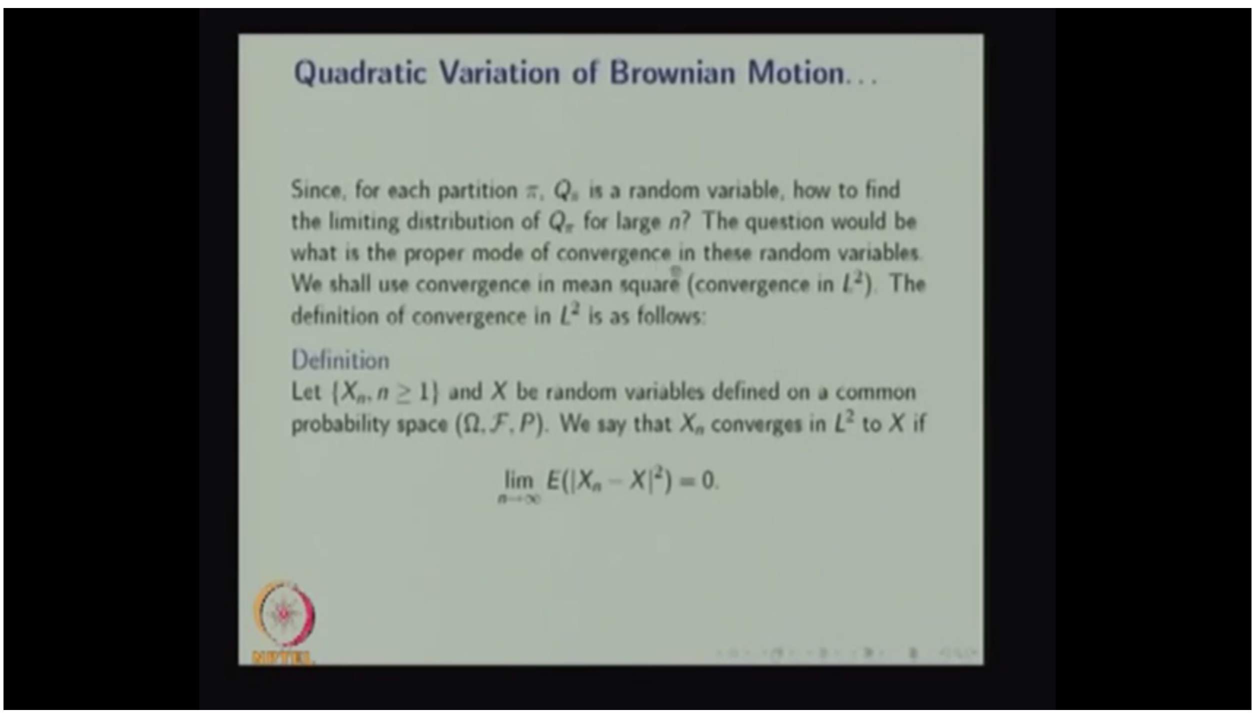


Now we are going to discuss the first variation of Brownian motion. We have already shown that the sample path of Wt are nowhere differential. Therefore the first order variation does not make sense because of the above reason because the derivative it is nowhere differential therefore you cannot get the first order variation. Hence, the first order variation of the Brownian motion does not exist.



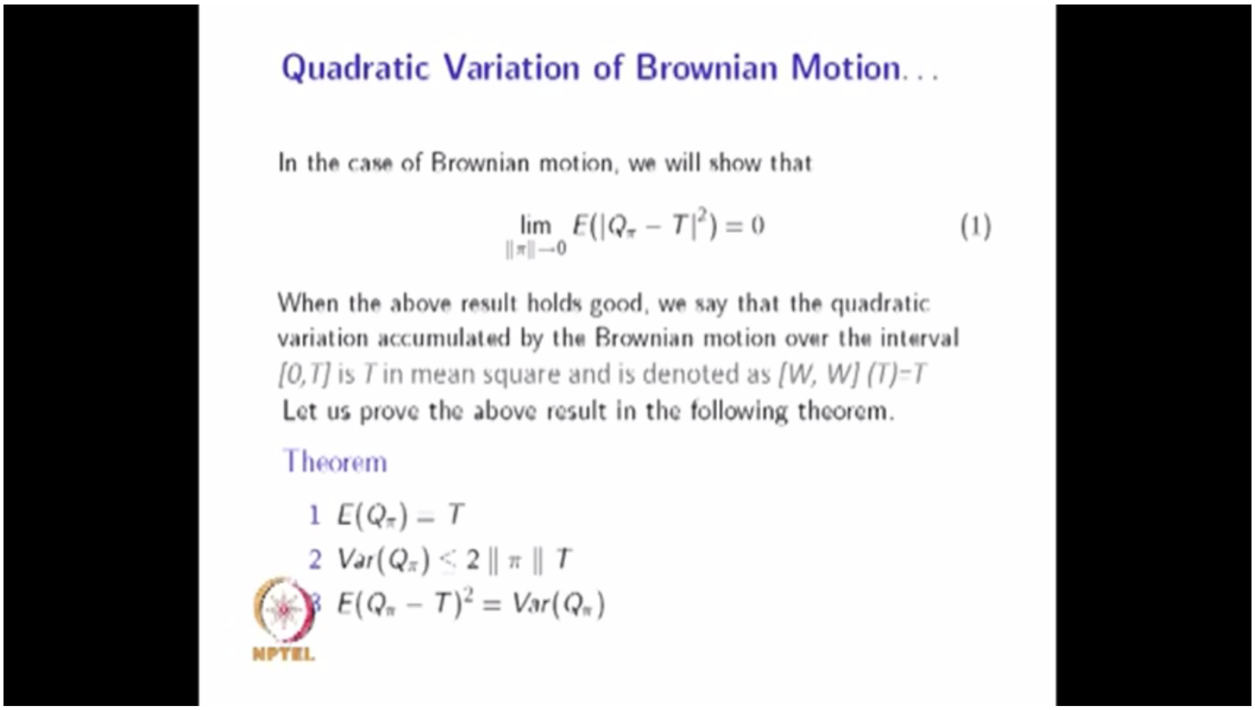
Now we are moving into the quadratic variation of Brownian motion. The quadratic variation of Brownian motion over the interval 0 to t where t is a positive real number denoted by the notation Wt, Wt of t that is given by V suffix t is wrong notation it's a V suffix Wt superscript 2 of t that is nothing but limit Pi tends to 0 of Qpi where Qpi is defined summation i is equal to 0 to n minus 1 the difference of W's and the time pointer ti to Ti plus 1 the whole square; clearly because you are making a difference of W's so the Qpi is a function of a sample points of W belonging to Omega, and also hence the quadratic variation calculated for the Brownian motion for each part is in itself a random variable because this is a random variable, the difference is a random variable, the summation will be sum of random variables is a random variable therefore the Qpi is a random variable and you are finding limit Pi tends to norm of Pi tends to 0 of Qpi. That is nothing but note that this limit taken over all partition of 0 to pi with norm of Pi tends to 0 as n tends to 0 as n tends to infinity. Norm of Pi is defined as the maximum of i of the length of the interval Ti plus 1 minus Ta therefore norm of Pi tends to 0 means you are finding the limit is taken over all partitions of 0 to Pi, 0 to t.

So we have to find out what is a limit norm Pi tends to 0 of this random variable. For every n this will be a random variable so you have to find out the limit taken over all partitions of 0 to T with norm of Pi tends to 0 as n tends to infinity.

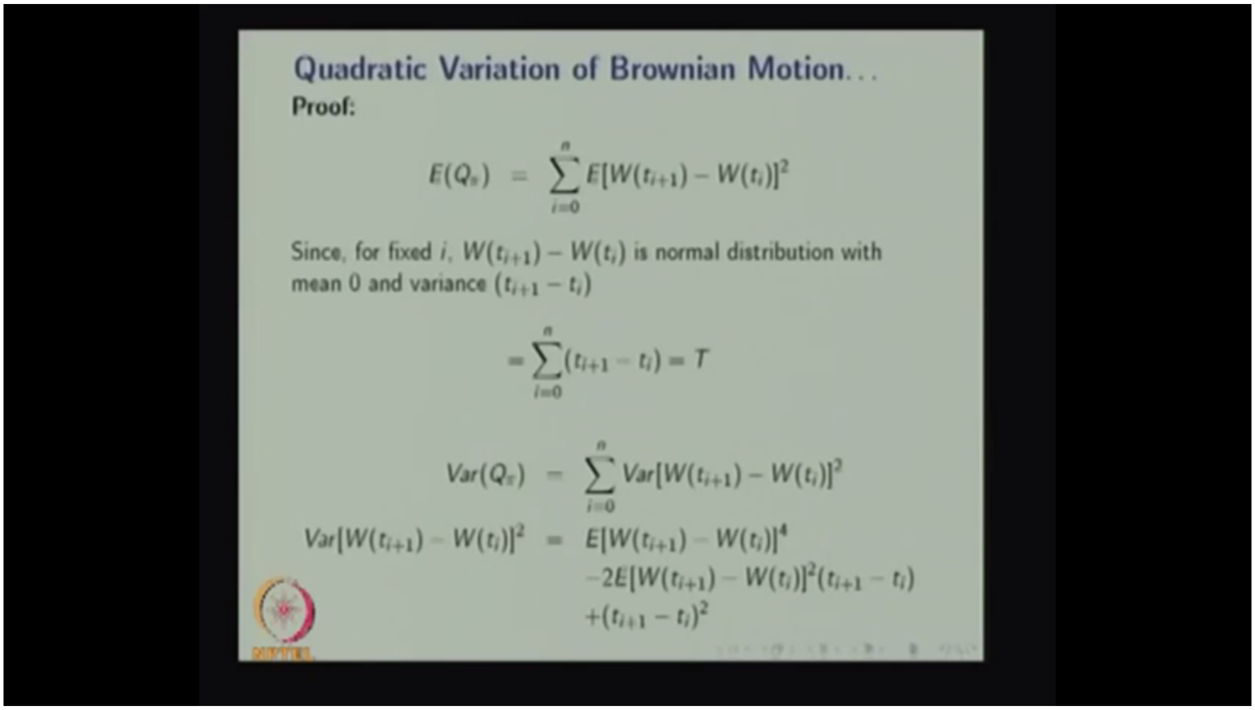


Since for each Pi for each partition Pi the Qpi is a random variable how to find the limiting distribution of Qpi for a large n? The question would be what is the proper mode of convergence in this random variables? We shall use the convergence in mean square that is a convergence in l2 to find the limit of norm Pi tends 0 Q of Pi as n tends to infinity. So for that we are going define the convergence in l2. Let Pi n, let X of n, n is greater than or equal to 1 and X random variables defined on a common probability space Omega F, P. We say that Xn converges in l2 to the random variable X if limit n tends to infinity expectation of the absolute of Xn minus X whole square is equal to 0. So if this condition is satisfied this is a sequence of random variable and this is a random variable both are defined in the same probability space Omega F, P then we say the sequence Xn convergence to the random variable X in l2.

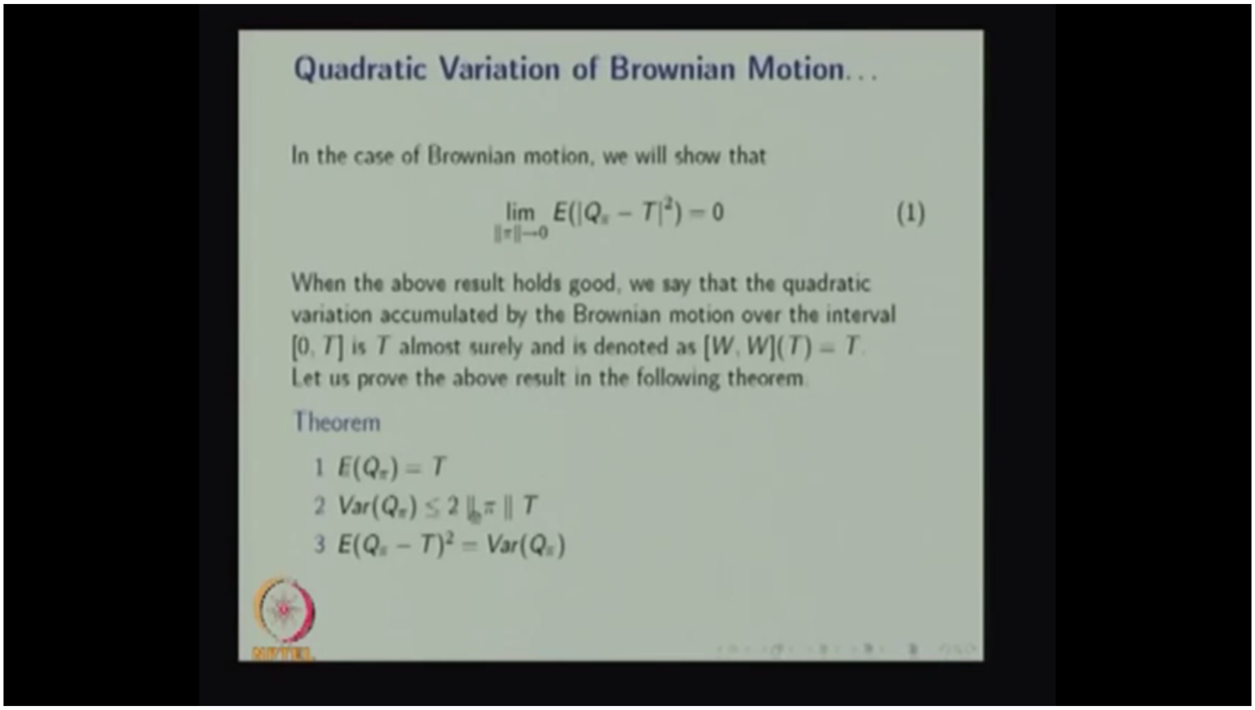
So this the same approach we are going to use to find out that limiting distribution of the random variables Q of Pi for a large n as n tends to infinity.



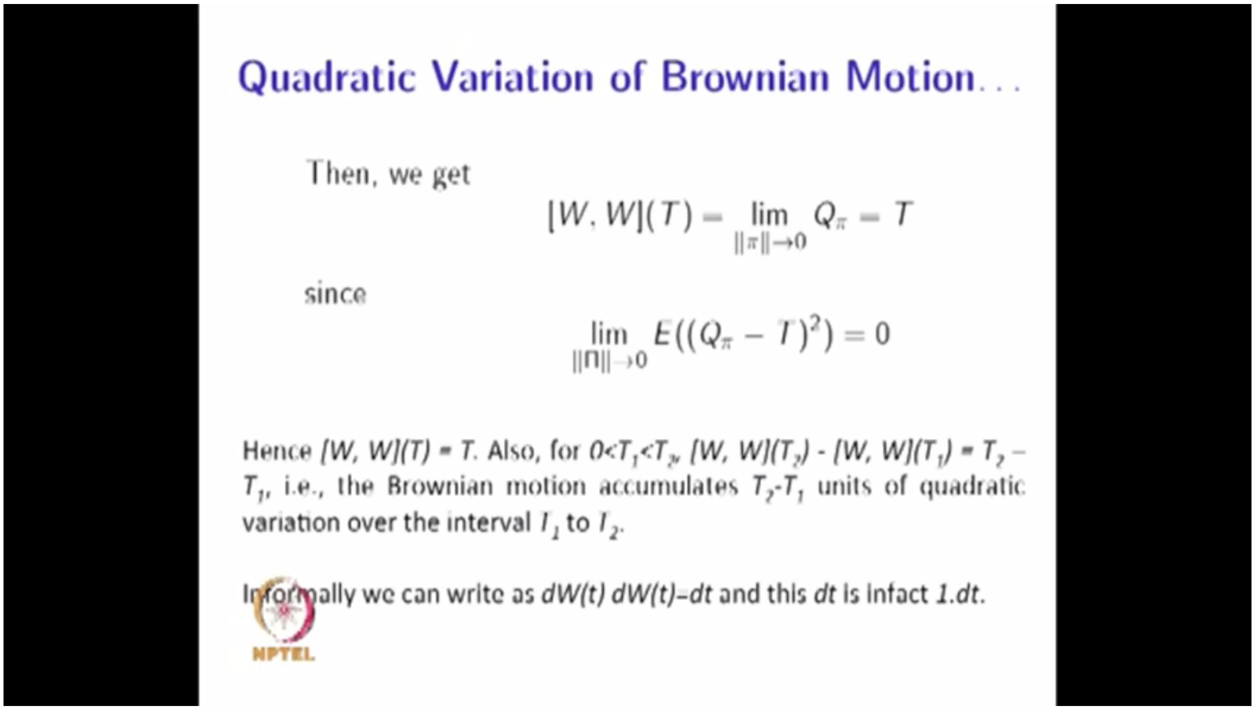
In this case of Brownian motion we will show that the limit norm Pi tends to 0 expectation of absolute of PiQ sorry Qpi minus t whole square is equal to 0. That means the sequence of random variable Q of Pi as n tends to infinity converges to the random variable which is a constant to T in l2. If this condition is since this condition is satisfied. When the above results hold good we say that the quadratic variation accumulates accumulated by the Brownian motion over the interval 0,T is T in mean square and it is denoted W of W of the interval 0 to T that is T. So to prove the sequence of random variable QPi converges to the random variable T as n tends to infinity in l2 we prove it in a 3 stage. The first stage we find out we will prove that expectation of Q of Pi is equal to T. Then we will prove the variance of Q of Pi that is less than or equal to two times norm of Pi of T. Therefore you can prove the final result expectation of QPi minus T whole square that is nothing but variance of QPi because the expectation of Qpi is T therefore expectation of QPi minus T whole square is variance of QPi so as n tends to infinity like Qpi will converges to the random variable T in l2.



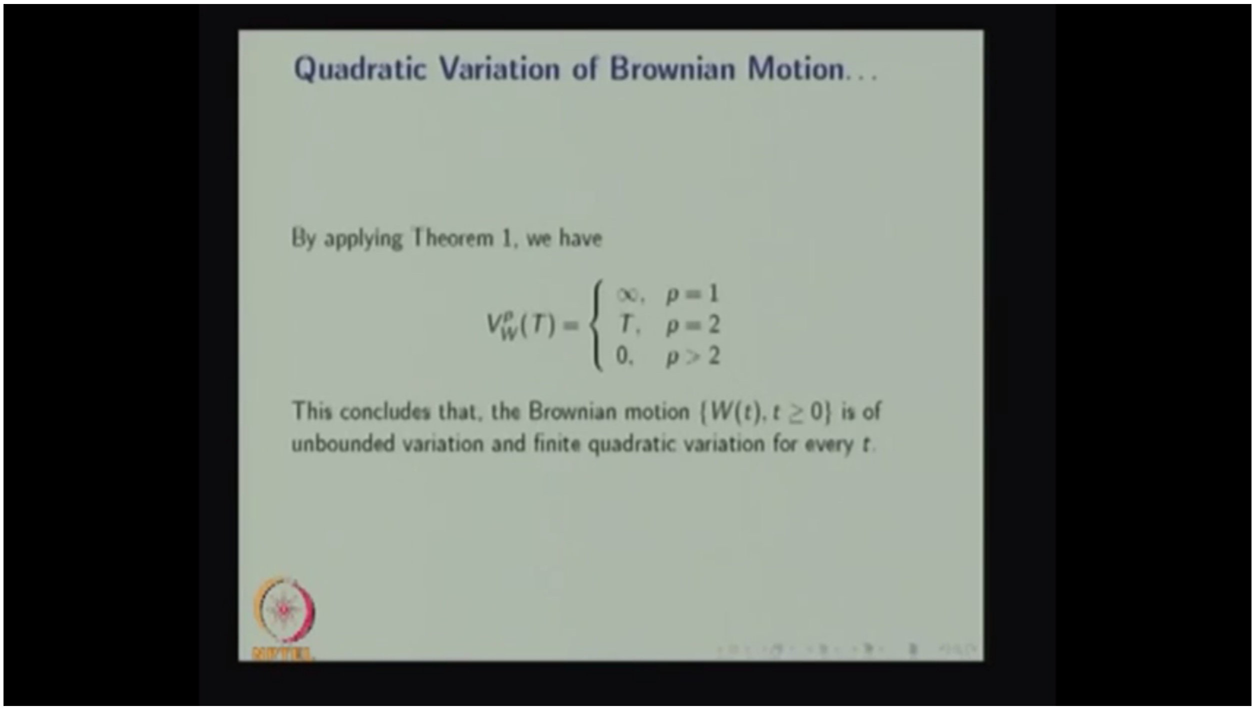
The proof as follows first we will find out the expectation of QPi that is nothing but the summation of i is equal to 0 to n minus 1 expectation of W of a ti plus 1 minus W of ti a the whole square. Since for fixed i the difference of the Ws' is normally distributed random variable with the mean 0 and the variance is nothing but the length of the interval. Therefore, the expectation of the difference of random variable whole square is nothing but the variance therefore our fixed i that is nothing but the T of i plus 1 minus ti. The summation is varies from i is equal to 0 to n minus 1 therefore you will get T. So the first part is proved that is expectation of QPi is equal to T. Now we will find out the variance of QPi that is less than or equal to we have to prove the second part variance of QPi is less than or equal to two times norm of Pi multiplied by t. third part is quite easy.



So the variance of QPi is nothing but summation i is equal to 0 to n minus 1 variance of the difference of random variable whole square but variance of difference of the random variable whole square that is nothing but the expectation of difference of random variable whole power 4 minus 2 times expectation of the difference of the random variable whole square multiplied by Ti plus 1 minus Ta plus a Ti plus 1 minus Ta whole square for fixed i. Using the fourth order moment of normally distributed random variable with the mean 0 and the variance Ti plus 1 minus t minus 1, the first term in the right hand side expectation of difference of the random variable power 4 that is the fourth order moment about the fourth order momenta that is nothing but at 3 times Ti plus 1 minus Ti whole square. Therefore, the right hand side variance of the difference of the random variable whole square that is nothing but three times the difference the time difference whole square minus 2 times the time difference whole square plus time difference whole square. So therefore this is nothing but 2 times time difference whole square.



The 2 times a time difference whole square is nothing but that is less than or equal to 2 times norm of Pi multiplied by the time difference. Therefore, the variance of QPi is less than or equal to two times norm of Pi times T. Therefore since you know that the expectation of QPi is equal to T therefore expectation of norm of QPi minus T whole square that is nothing but variance of Qpi. Therefore the second order variation of the Brownian motion Wt between the powers interval 0 to T is nothing but limiter norm of Pi tends to 0, QPi that is same as T. Since limit norm Pi tends to 0 the expectation of QPi minus T whole square is equal to 0. So the conclusion is the second order or the quadratic variation of a Brownian motion is T between the interval 0 to capital T. This means it accumulates a unit quadratic variation per unit. Also for 0 less than T1 less than T2 the quadratic variation till T2 the quadratic variation, sorry, the quadratic variation till T2 minus quadratic variation till T1 that is same as T2 minus T1. That is the Brownian motion accumulates T2 minus T1 units of quadratic variation over the interval T1 to T2. Since this is true for every interval we refer that the Brownian motion accumulates quadratic variation at the rate 1 per unit. This last statement we write informally as dWt dWt is equal to dt and this dt is in fact 1 times dt. In other words the above phenomena can be represented in a differential form as a differential of Wt multiplied with a differential of Wt. This is a quadratic relation that is nothing but the differential of t, that is a meaning of the Brownian motion accumulates unit quadratic variation per unit time.



Now we are applying the same theorem to discuss for the variations of the real valued function we have the pth order variance of Brownian motion between the interval 0 to T that value will be does not exist for the first order variation therefore P is equal to 1 it is infinity. For the quadratic variation it is a T. It's a bounded variation of that a quadratic variation whereas the first order is unbounded variation. For P greater than 2 it is 0. So the example we have taken is G of t is equal to t Square for that the first order variance is finite and further variations are 0 whereas for the Brownian motion the first order variation is a infinity that is unbounded variation and the second order variation is a finite value that is T and the further variations are 0. This concludes that the Brownian motion is of unbounded variation because P is equal to 1 the variation is infinity and the finite quadratic variation because P is equal to 2 value will be T for every t.