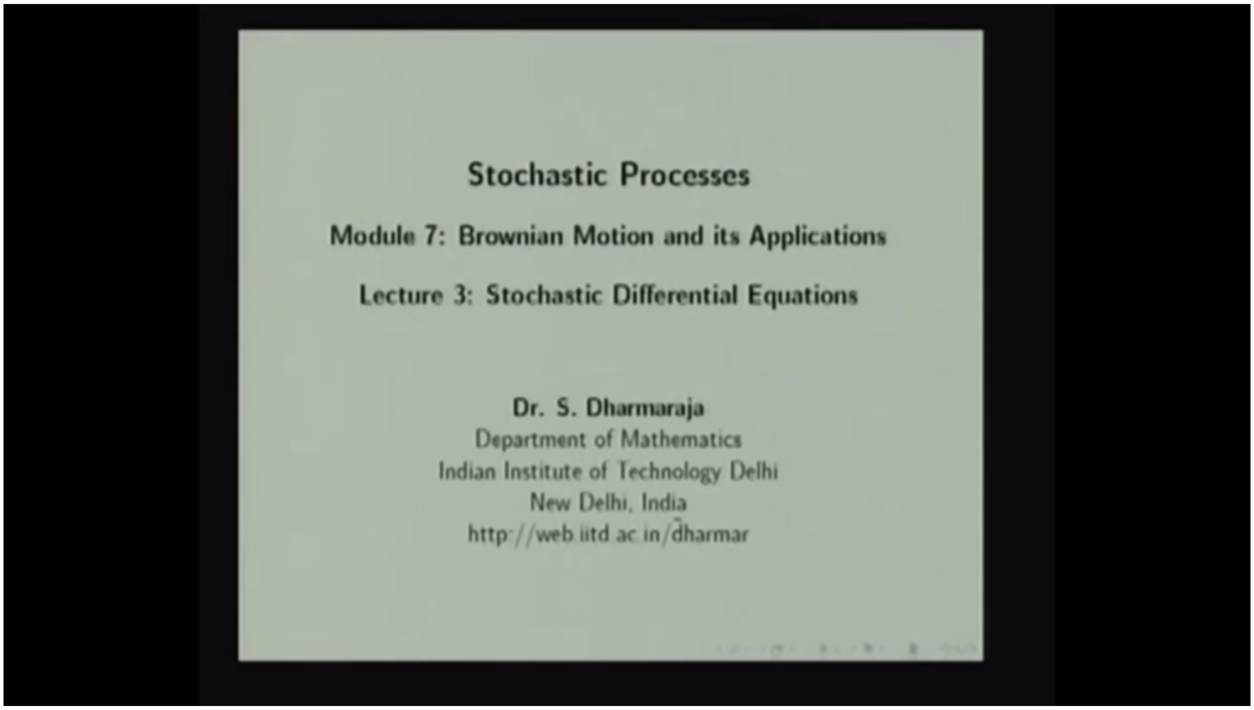

Video Course on

Stochastic Processes -1

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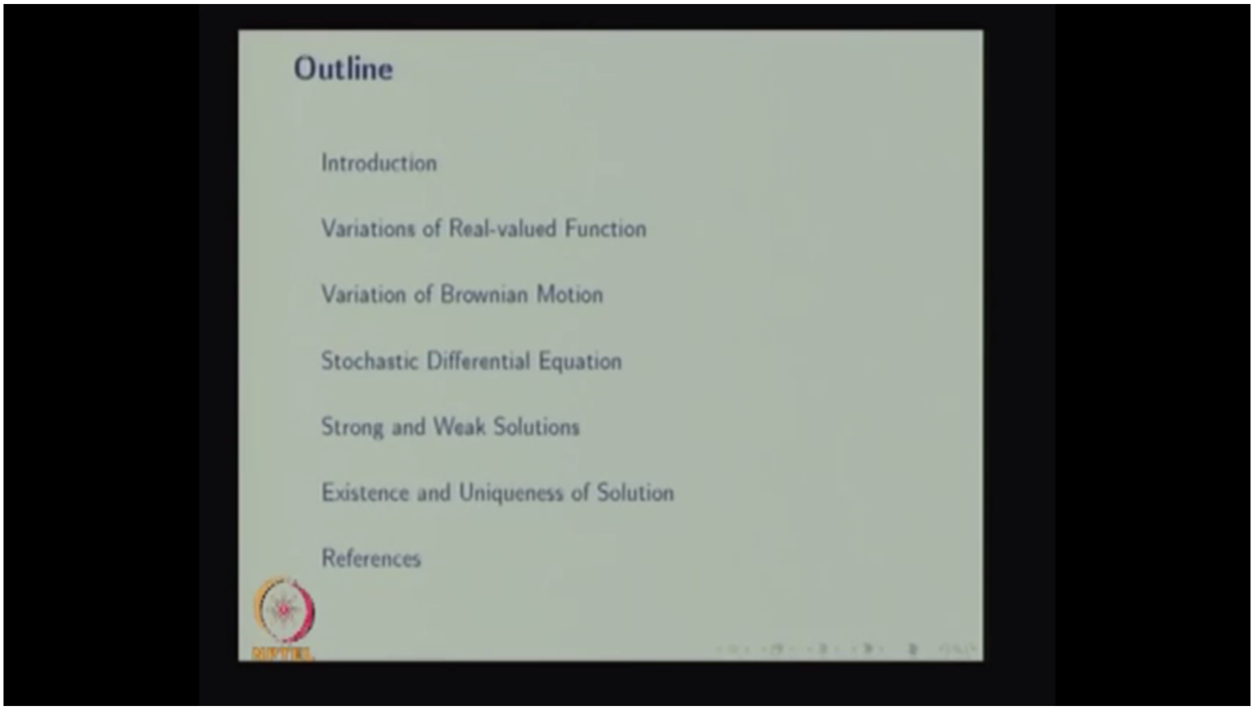
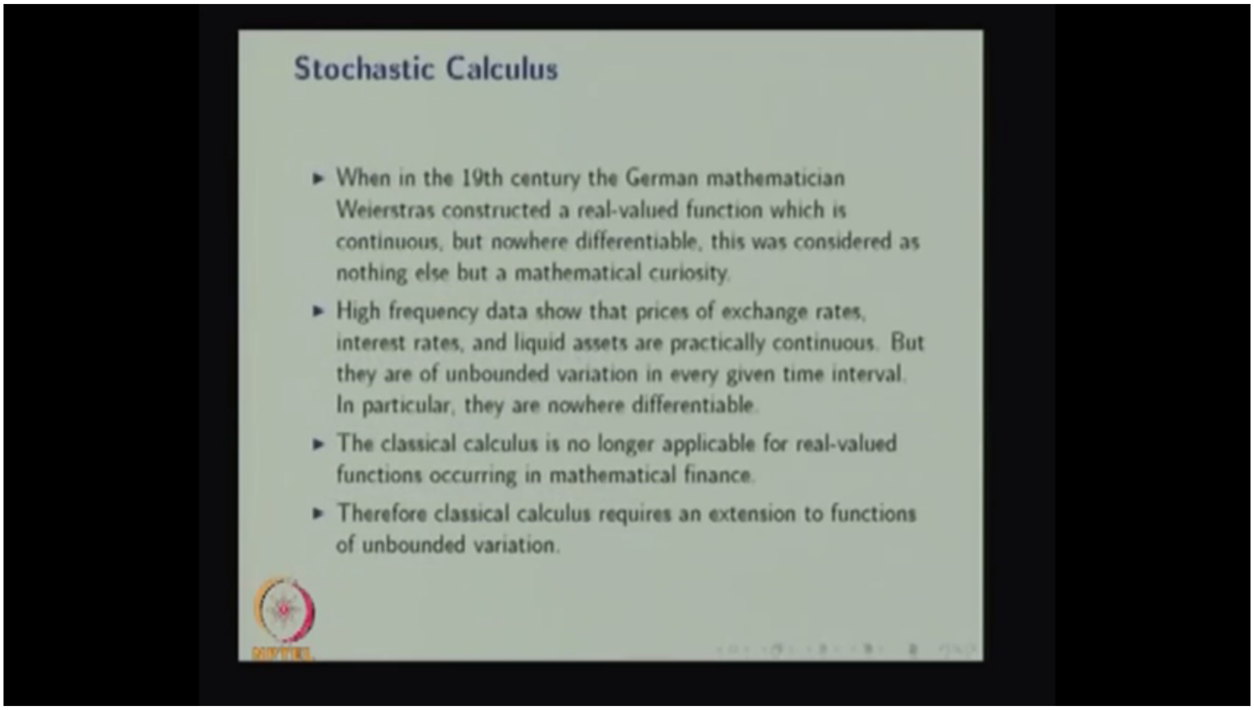
Module #7: Brownian Motion and its Application

Lecture#3

Stochastic Difference Equations



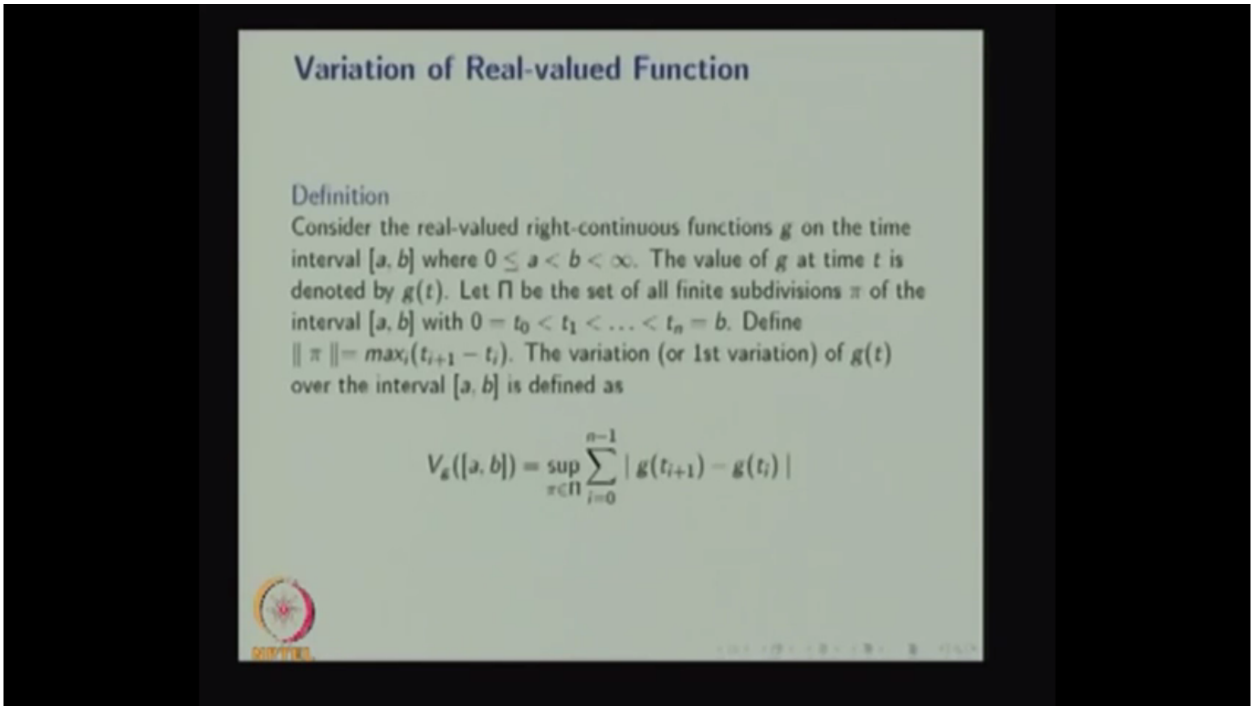
This is a stochastic processes module 7, Brownian motion and its applications. This is lecture 3 stochastic differential equations. In the lecture 1 we have discussed the Brownian motion definition and its properties and in the lecture 2 we have discussed process derived from Brownian motions. In particular we have discussed the geometric Brownian motion and then the levy process and a few application also. In this lecture we are going to discuss stochastic differential equations. We are going to start with the motivation behind the stochastic differential equation. Then we are going to discuss the variations of real valued functions followed by the Brownian motions. Then we are going to give the definition of a stochastic differential equation. Then we are going to discuss the strong and weak solutions. And also we are going to discuss the existence and uniqueness of solution. The motivation behind the forecasting differential equation when in the 19th century the German mathematician Karl Weierstrass constructed a real-valued function which is continuous but nowhere differentiable. This was considered as nothing else but a mathematical curiosity. High frequency data showed that prices of exchange rates, interest rates, and the liquid assets are practically continuous but they are of unbounded variation in every given time interval. In particular they are nowhere differential.



The classical calculus is no longer applicable for real valued functions occurring in mathematical finance. Therefore, the classical calculus occurs an extension to functions of unbounded variation.

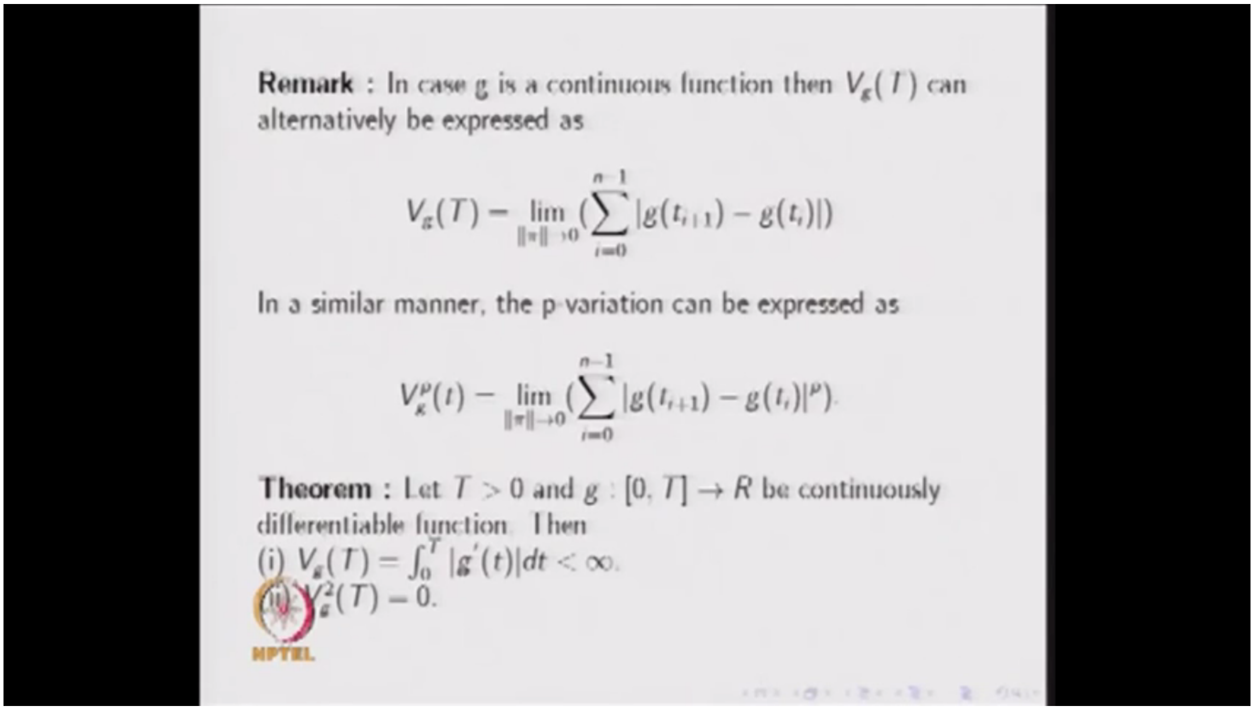
Stochastic calculus is a necessary extension of real analysis to cope up with the functions of unbounded variation. The Brownian motion which is a stochastic process is the sample path is a continuous but the sample path is nowhere differentiable and also it is a unbounded variation. Some important concepts of the stochastic calculus stochastic differential equations, Ito integral, and the Ito's formula will be discussed in this model. Ito integral is nothing but stochastic integral equations and also we are going to discuss the Ito formula to solve the stochastic integral equation or stochastic differential equation.

So in this model we are going to discuss the stochastic differential equations. In the next lecture we are going to discuss the stochastic integral equations and in the lecture 5 we are going to discuss the Ito formulas and some important stochastic differential equations and their solutions.



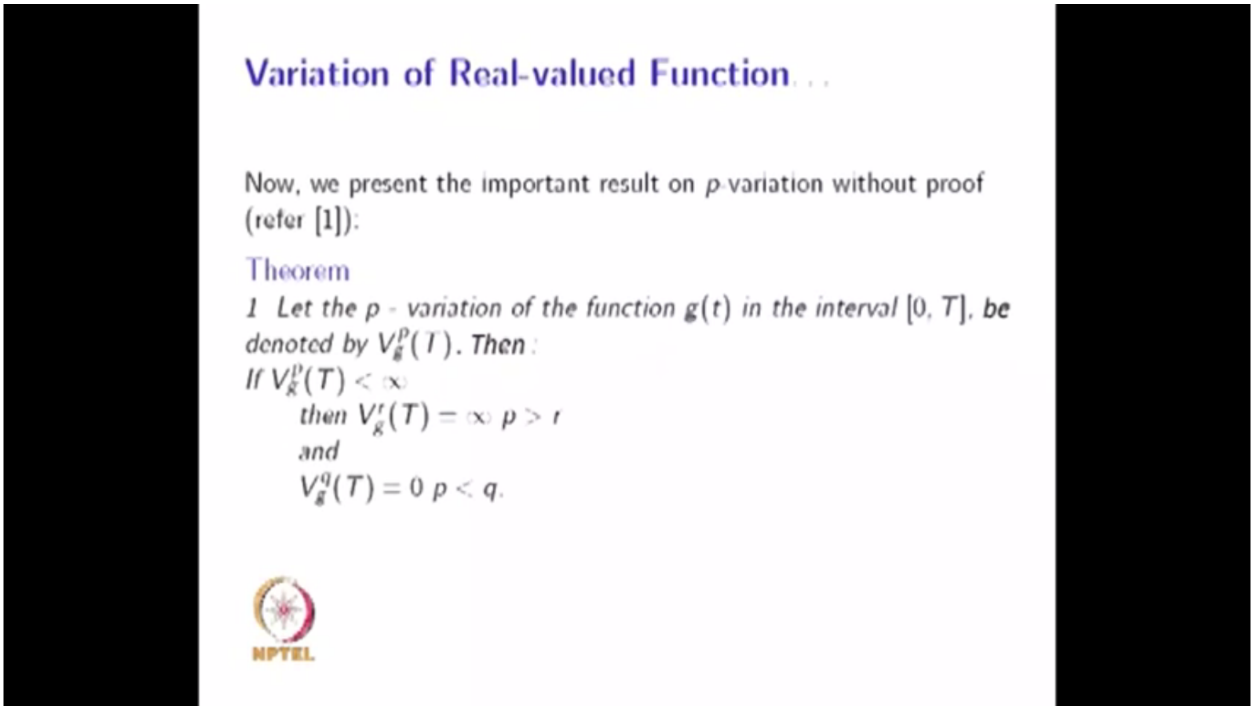
First we are going to discuss the variations of real valued function. Our interest is to study the variation of a Brownian motion but before that we will discuss the variation of real valued function then followed by that we are going to discuss the variations of Brownian motion. The first variation is a defined as follows. Consider the real valued right continuous functions g on the time interval a to b. The value of g at time t is denoted by g of t. Let Phi be the set of all finite subdivisions pi of the interval a to b with 0 equal to t naught which is less than t1 which is less than and so on which is less than tn where tn is equal to b. Define the norm of Pi that is a maximum of i the length of the interval that is ti plus 1 minus ta. The variation or the first variation the first order variation of g of t over the interval a to b is defined as the variation of g in the interval or over the interval a to b is nothing but supremum of a Pi belonging to Pi the summation running from i0 to n minus 1 the absolute of g of t plus i t plus t of i plus 1 minus g of ti. So the modulus of this difference of the value evaluated at Ti plus 1 and Ti the difference of the function g take absolute, then find the summation then find the supremum. That will be called it as a variation of a g of t.

The function gt is of a finite variation if for every t if Vg of t that is nothing but the Vg of between the interval 0 to t is a finite. Whenever the interval 0 to t the first variation of the function g is a finite one then we save the function gt is a finite variation for every t. For all t if Vg of t is bounded by a constant K which is independent of t then we say the function g of t is of bounded variation. This is for the first order variation for the any real valued function g.



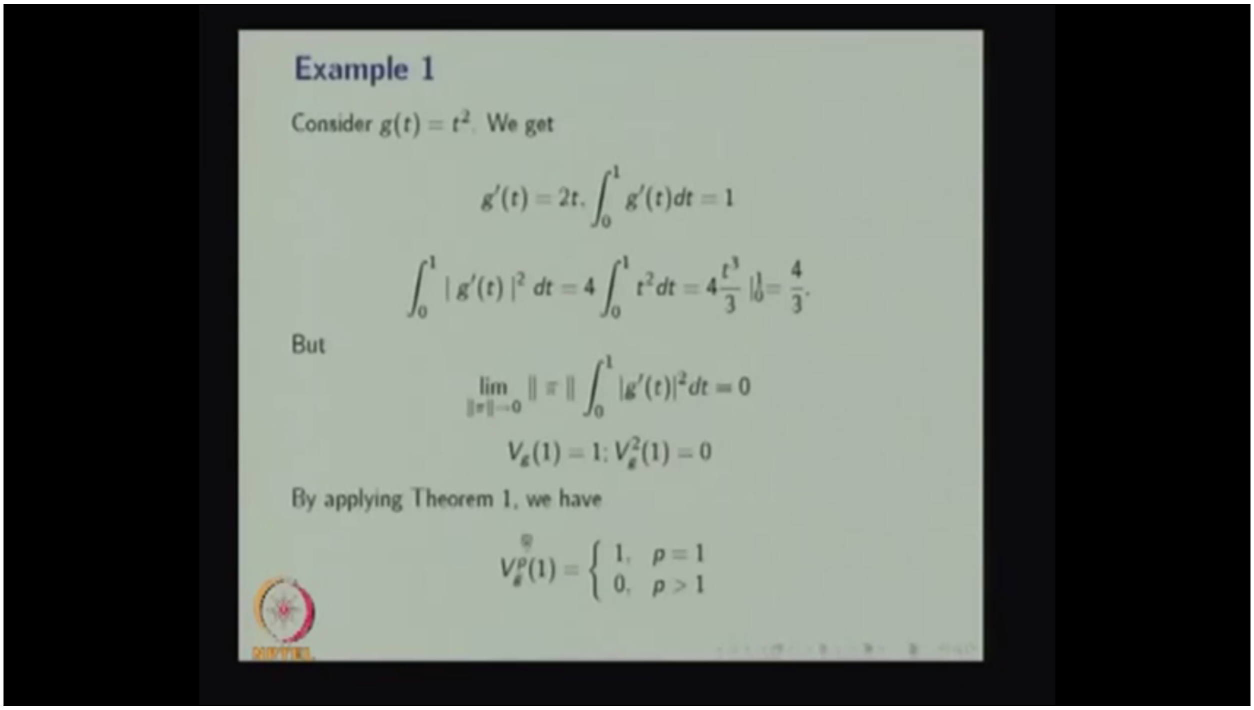
Alternatively a function g is said to have a bounded variation often closed interval. It's total variation is finite. In the same way we are going to define the P variation or Pth order variation of real valued function. The same interval 0 to t that is nothing but the only difference is absolute power. So if P is equal to 1 then it is a first order variation. If it is P is equal to 2 then it is second order variation and so on. Remark in case g is a continuous function then Vg of T can be alternatively expressed as a Vg of T is a limit norm Pi tends to 0 summation i is equal to 0 to n minus 1 absolute of g of t plus t of i plus 1 minus g of ti where here Pi is a arbitrary partition of the interval 0 to T and the norm Pi is a maximum offer t K plus 1 minus TK where K is lies between 0 to n minus 1.

In a similar way the P variation can be expressed as written here the Vg of P variation of T that is written as a limit norm pi tends to 0 the similar expression the absolute power p. If g is a continuous function then supreme is replaced by limit.



Later on this video lecture you will see how to calculate the second order variation which is known as the quadratic variation for Brownian motion. In finding the quadratic variation of a Brownian motion g of t will be replaced by w of t and the limit will be taken in the sense of limit of sequence of random variables. Now we have this theorem if g is continuously differentiable function from the interval 0 to T then the first order variation is integral of modulus of g dash of t with the limits 0 to T and second order variation is 0. Now we present the important result on P variation without proof as a theorem.

Theorem one, let the P variation of the function g of t in the interval 0 to T where T is a positive real number denoted by V suffix G superscript p of t if the pth variation is a finite then all the earlier order variation is going to be infinite and all the further order qth order variation from the P that will be 0 whenever the pth order variation is a finite then all the earlier order variation from the pth order P that will be infinity that means it is unbounded and for variation of qth order will be 0 where p is less than q. So this is a very important result and using this result we are going to discuss that variations of Brownian motion also.



Examples for these can be found in the problem sheet. As an example to understand the application of the previous theorem and the results we have these consider the function g of t that is t square. So polynomial of order two. So you can find the derivative and you can find the derivative absolute whole square therefore if you find Pi tends to 0 0 of pi times this one is equal to 0. Therefore you will get and the first order variation is 1 and the second order variation will be 0 in the interval 0 to 1 or at the time at the t equal to 1. By applying theorem one we get whenever P is equal to 1 it is a value is 1 and for all the further order that will be 0. This is for the function which is a polynomial of degree two. Therefore, you are getting P is equal to 1 the first order variation is equal to 1 and further orders second, third, fourth and so on the variation will be 0 for the second order degree polynomial.