

Applications

- ▶ Because the geometric Brownian motion is nonnegative, it provides for a more realistic model of stock prices.
- ▶ Suppose that the stock price $S(t)$ at time t is given by

$$S(t) = S(0)e^{H(t)}, t \geq 0$$

where $S(0)$ is the initial price and $H(t) = \sigma W(t) + \mu t$ is a Brownian motion with drift. In this case, $H(t)$ represents a continuously compound rate of return of the stock price over the period of time $[0, t]$.

- ▶ Here, $H(t)$ refer to the logarithmic growth of the stock price, satisfies $H(t) = \log \left(\frac{S(t)}{S(0)} \right)$.
- ▶ Therefore, $H(t)$ has a normal distribution with mean μt and variance $\sigma^2 t$.



As application because the geometric Brownian motion is a non-negative because of the form X_t is equal to $X_0 e^{\sigma W_t + \mu t}$. W_t has a range minus infinity to infinity, the $e^{\sigma W_t + \mu t}$ therefore the range of X_t is 0 to infinity so the geometric Brownian motion is non-negative random variable for X_t . It provides more realistic model for stock prices whereas one cannot use the Brownian motion to model the stock prices. Suppose the stock price S_t at time t is given by $S_t = S_0 e^{H_t}$ where t is greater or equal to 0 where S_0 is initial price and H_t is of the form $\mu t + \sigma W_t$. We are going little generalized one the geometric Brownian motion is not the form $X_t = X_0 e^{W_t}$ whereas here we are considering more general setup $S_t = S_0 e^{H_t}$ where H_t is having a new t term along with the σW_t . You make the μ equal to 0 σ is equal to 1 σ^2 is equal to 1 you will get that standard geometric Brownian motion. [Indiscernible] [00:01:38] whereas here this is the H_t is a Brownian motion with the [Indiscernible] [00:01:45]. In this case the H_t represents the continuously compound rate of return the stock price over the period of time 0 to t . W_t is a standard Brownian motion. The H_t is equal to $\mu t + \sigma W_t$ is the Brownian motion with the [Indiscernible] [00:02:10] and it represents a continuous compound rate of return of the stock price over the period of time 0 to t . Here H_t refers to the logarithmic growth of the stock price because [Indiscernible] [00:02:29] t is equal to $\log \left(\frac{S_t}{S_0} \right)$. $S_t = S_0 e^{H_t}$ so you can divide S_0 so $\frac{S_t}{S_0} = e^{H_t}$ take logarithm both side therefore $\log \left(\frac{S_t}{S_0} \right) = H_t$. Hence it refers to the logarithmic growth of the stock price. Since it is $S_t = S_0 e^{H_t}$ and $H_t = \sigma W_t + \mu t$ therefore H_t you know that the W_t is a normally distributed with the mean 0 variance t standard Brownian motion therefore H_t also normally distributed random variable [Indiscernible] [00:03:27] with mean you can find out the mean of H_t and mean of H_t this mean is 0 therefore mean of H_t is μt find out the variance, variance will be 0 here. Here the variance will be $\sigma^2 t$ and the variance of W_t is t therefore the variance of H_t is $\sigma^2 t$. So from this equation $H_t = \sigma W_t + \mu t$

t by finding mean and variance you can [Indiscernible] [00:03:57] the H of t is a normally distributed with the mean μt and the variance $\sigma^2 t$.

Applications ...

- ▶ Letting $\bar{r} = \mu + \frac{1}{2}\sigma^2$, we get

$$E(S(t)) = S(0)e^{\bar{r}t}$$

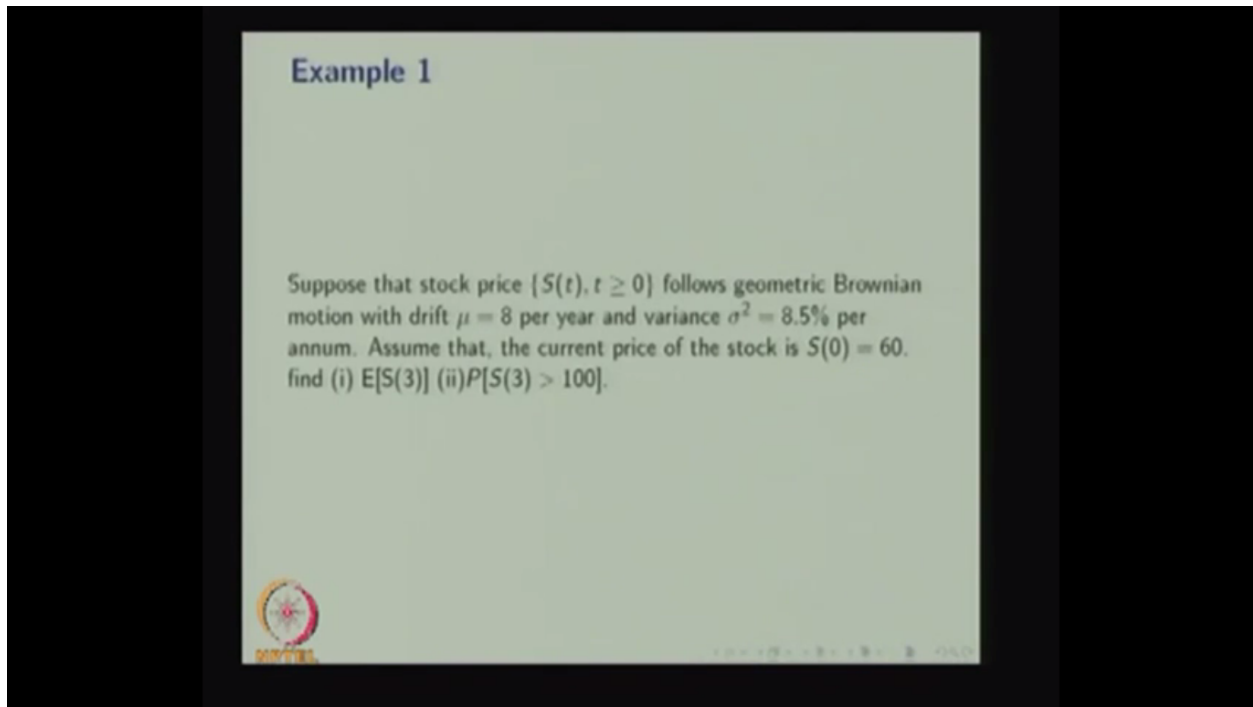
$$\text{Var}(S(t)) = \left(S(0)e^{(\mu + \frac{\sigma^2}{2})t} \right)^2 (e^{\sigma^2 t} - 1)$$

- ▶ Observe that, the expected stock price depends not only on the drift μ of $H(t)$ but also on the volatility σ .
- ▶ Further, it shows that, the expected price grows like a fixed-income security with continuously compounded interest rate \bar{r} . In real scenario, r is much lower than \bar{r} , the real fixed-income interest rate, that is why one invests in stocks. But the stock has variability due to the randomness of the underlying Brownian motion and hence there is risk involved here.

Already we have made these substitution. $\mu + \frac{1}{2}\sigma^2$ [Indiscernible] [00:04:19] hence the expectation of S of t will be $S(0)e^{\bar{r}t}$. Not only that you can find out the variance. The way we have derived mean and variance of geometric Brownian motion we can find out the mean and variance of S of t also. Observe that the expected stock price the expectation of t you can observe that the expected stock price depends not only on the drift μ of the H of t but also the volatility because the expectation of X of t is equal to $X(0)e^{\bar{r}t}$ where \bar{r} is $\mu + \frac{1}{2}\sigma^2$. So this \bar{r} depend on the μ as well as σ^2 . Hence the expected stock price depends not only the drift but also the volatility σ . [Indiscernible] [00:05:23] the drift of H of t but also the volatility σ .

Further it shows that the expected price grows like a fixed income security with the continuously compounded interest rate \bar{r} where S of t is the stock price at time t the expected stock price at time t is $S(0)e^{\bar{r}t}$ so it grows like a fixed income security with the continuously compound interest rate \bar{r} [Indiscernible] [00:06:07]. in real scenario r is much lower than \bar{r} where r is the real fixed income interest rate. That is why one invest in stocks but even though there is a risk attached to it. In the fixed income scenario the interest rate is r there is no risk whereas when you invest in stock the average or expected price growth in the form of S of t $e^{\bar{r}t}$ where \bar{r} is much lower than r but there is a risk. That is the difference between the fixed income scenario with the investing in stocks but the stock has a variability due to the randomness of the underlying Brownian motion and hence there is a risk involved.

So we have taken one application of geometric Brownian motion. Through that we have explained, we have derived mean and variance of stock price through that we are discussing the risk over the investment in stock.



Example 1

Suppose that stock price $\{S(t), t \geq 0\}$ follows geometric Brownian motion with drift $\mu = 8$ per year and variance $\sigma^2 = 8.5\%$ per annum. Assume that, the current price of the stock is $S(0) = 60$. find (i) $E[S(3)]$ (ii) $P[S(3) > 100]$.

This is a very simple example. Suppose the stock price S of t follows a geometric Brownian motion with the drift some value here we make it 8% a year as well as a variance the Sigma square is the 8.5% per annum. Assume that the current price of the stock S naught is 60 the questions are what is the expected stock price at $t = 3$ three years similarly what is the probability that the stock price at the time $t = 3$ will be greater than [Indiscernible] [00:08:31] so since you know the S of t you can find out the probability because S of t is of the form X naught times e power Ht . So you have to use the lognormal concept to find out the probability and the expectation of S of t we have already given the formula for expectation of S of T . you can use that to get the value.