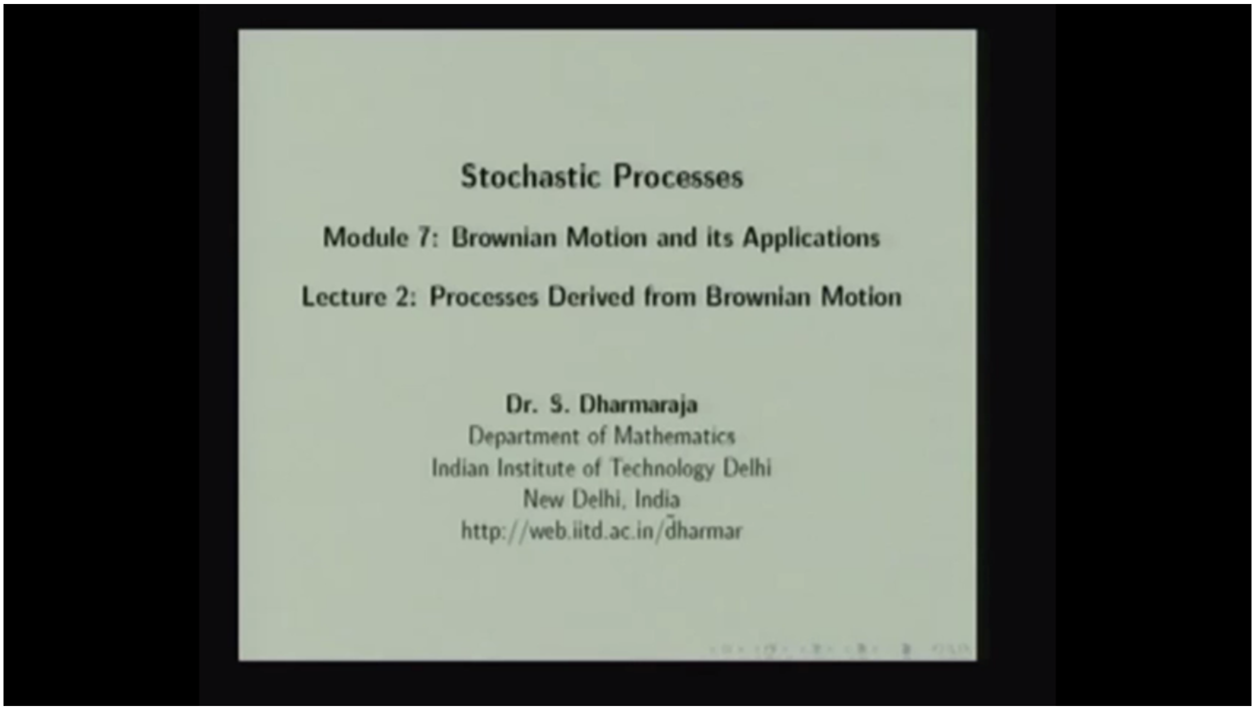

Video Course on

Stochastic Processes -1

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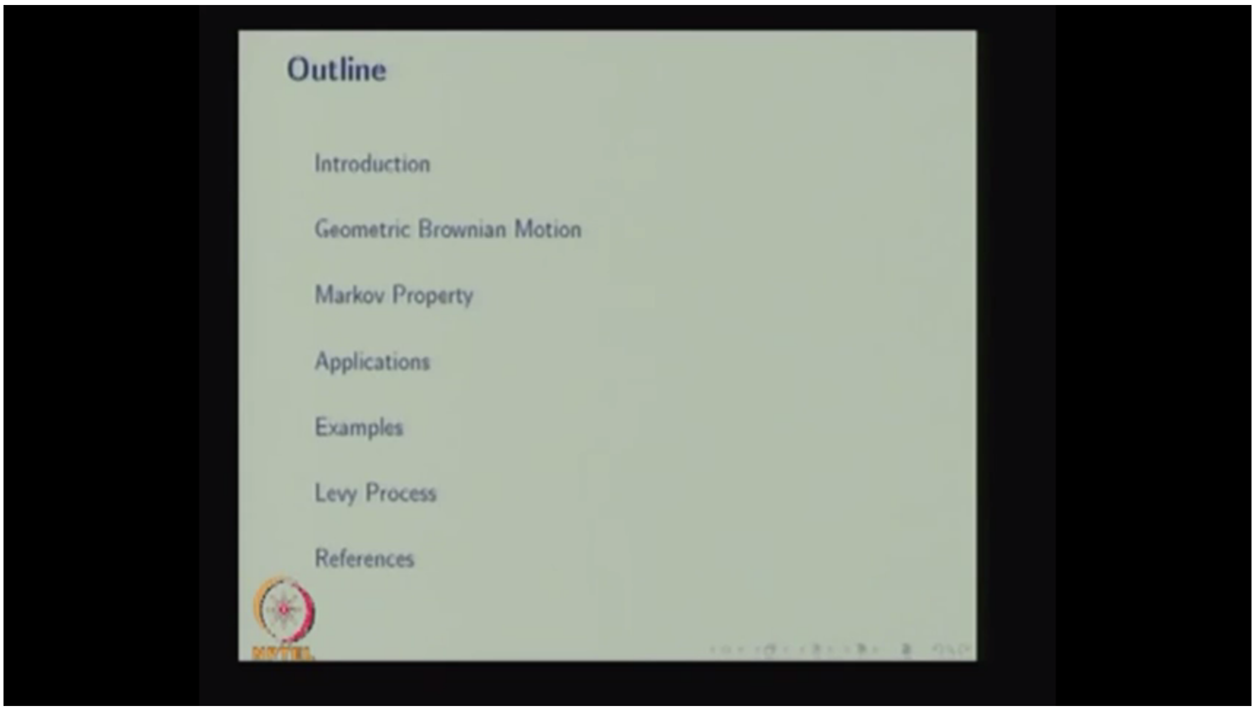
Module #7: Brownian Motion and its Application

Lecture#2

Processes Derived from Brownian Motion

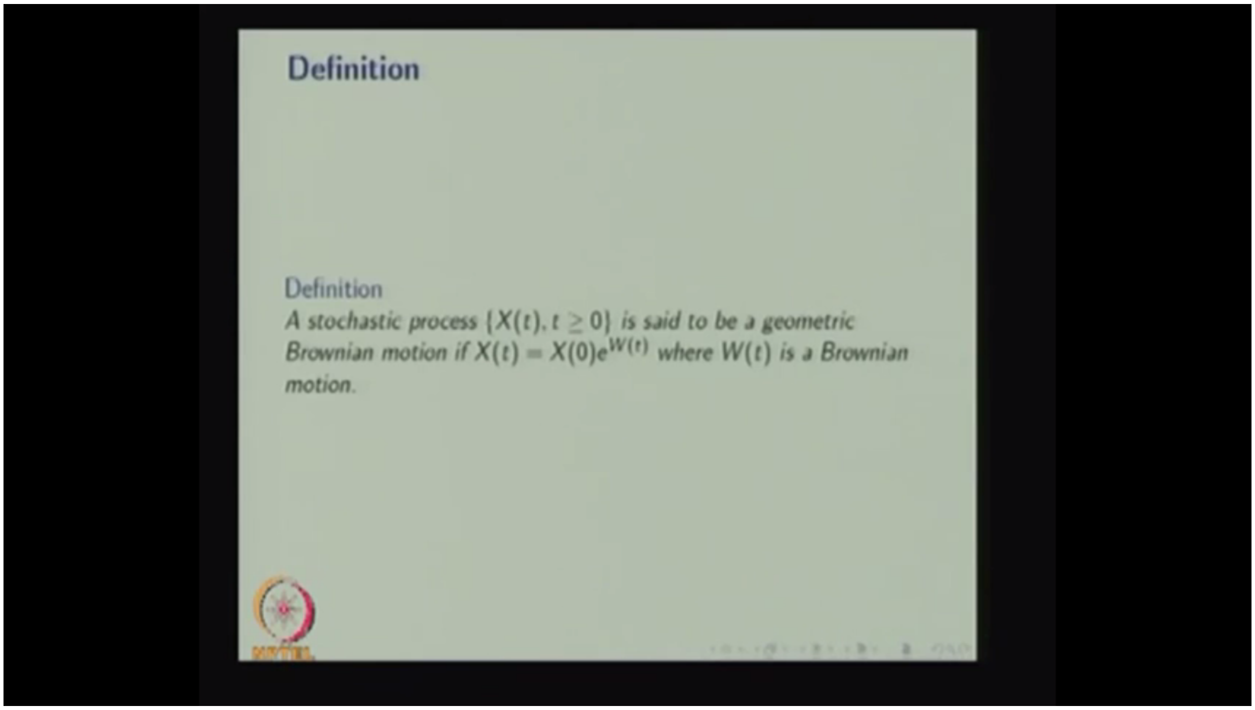
This is a stochastic processes model 7, Brownian motion and its applications. Lecture 2; processes derived from Brownian motion.

In the lecture one we have discussed the definition and properties of Brownian motions we started with the random walk. Then the sample path of a random walk. Then we have given few properties of random walk followed by that we have made the derivation of Brownian motion through the random walk. Then we have discussed the sample path of a Brownian motion. Followed by that we have discussed a few important properties such as strict sense stationary increment, independent increment property, nowwhere differentiable property, self-similar property, Markov property, between Gaussian process and the Brownian motion. Also we have discussed the Kolmogorov equation and we have given the connection with the heat equation. Then we have discussed the joint distribution of Poisson process also and finally we have discussed the Martingale property of Brownian motion. So in that lecture 1 of module 7 was completed.

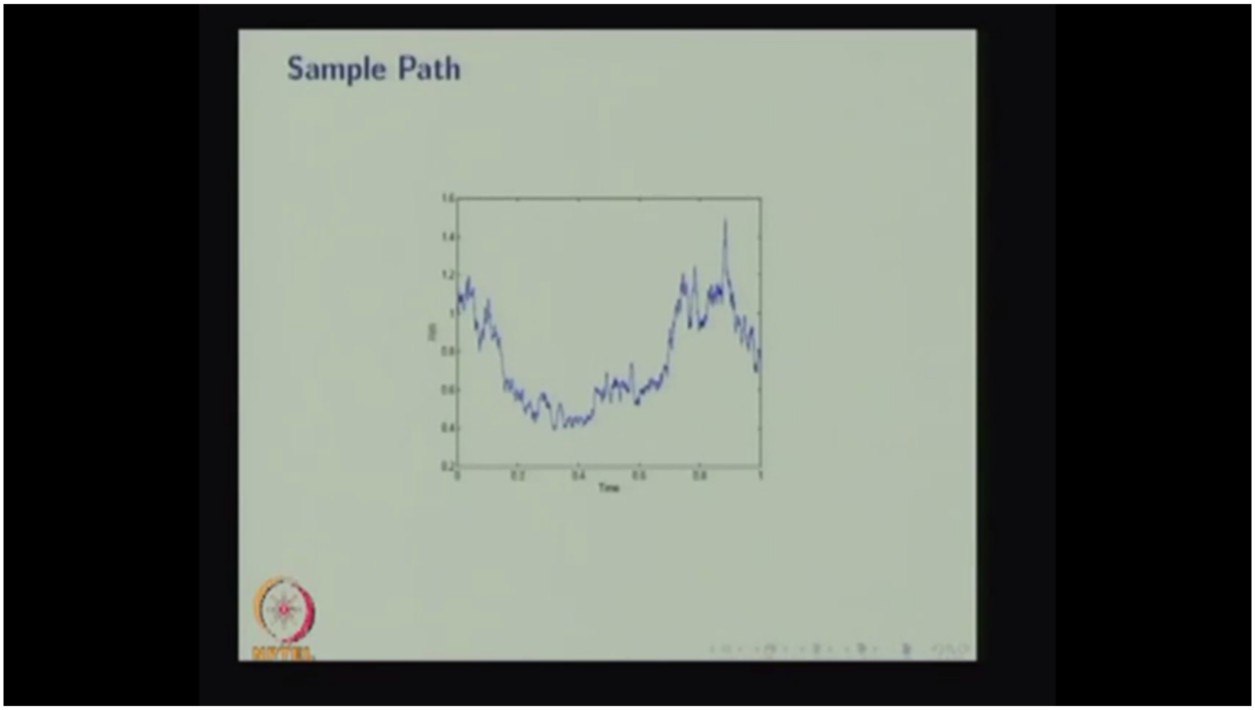


In the lecture 2 we are going to cover the definition of a geometric Brownian motion. Then few properties also going to be discussed followed by that we are going to discuss the applications of geometric Brownian motion and also few examples of geometric Brownian motion.

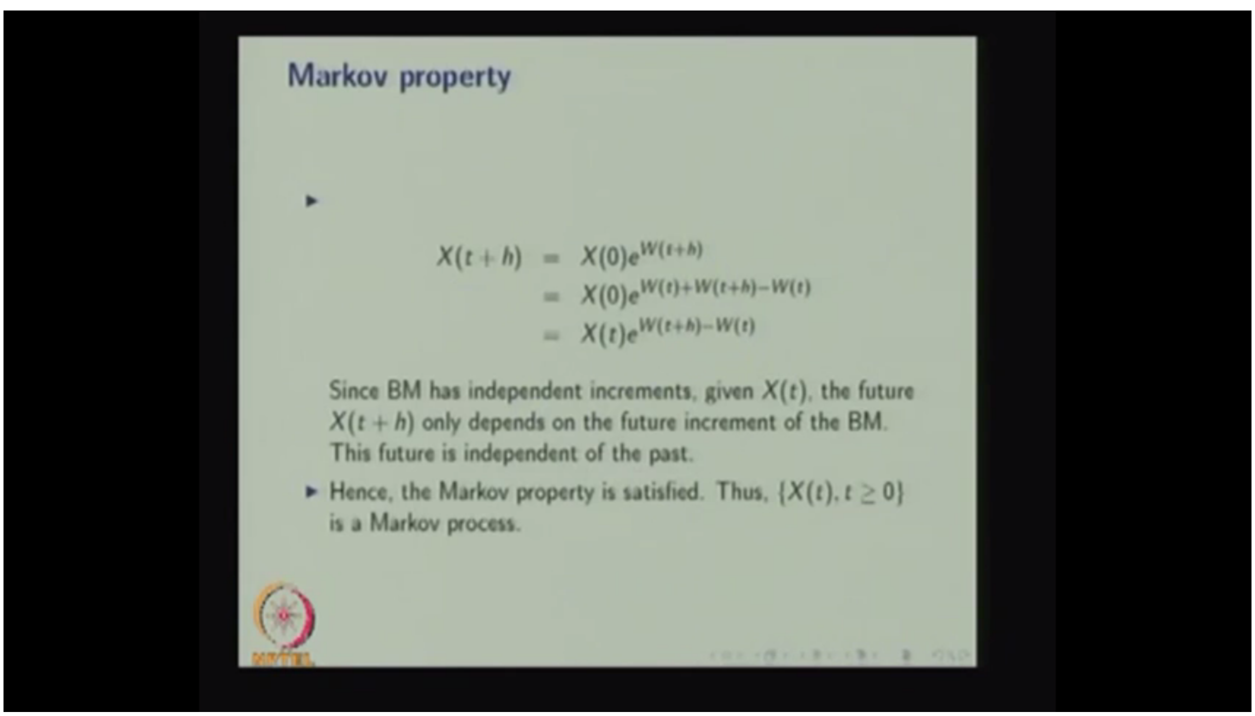
Finally the process derived from Brownian motion that's a levy process also going to be discussed.



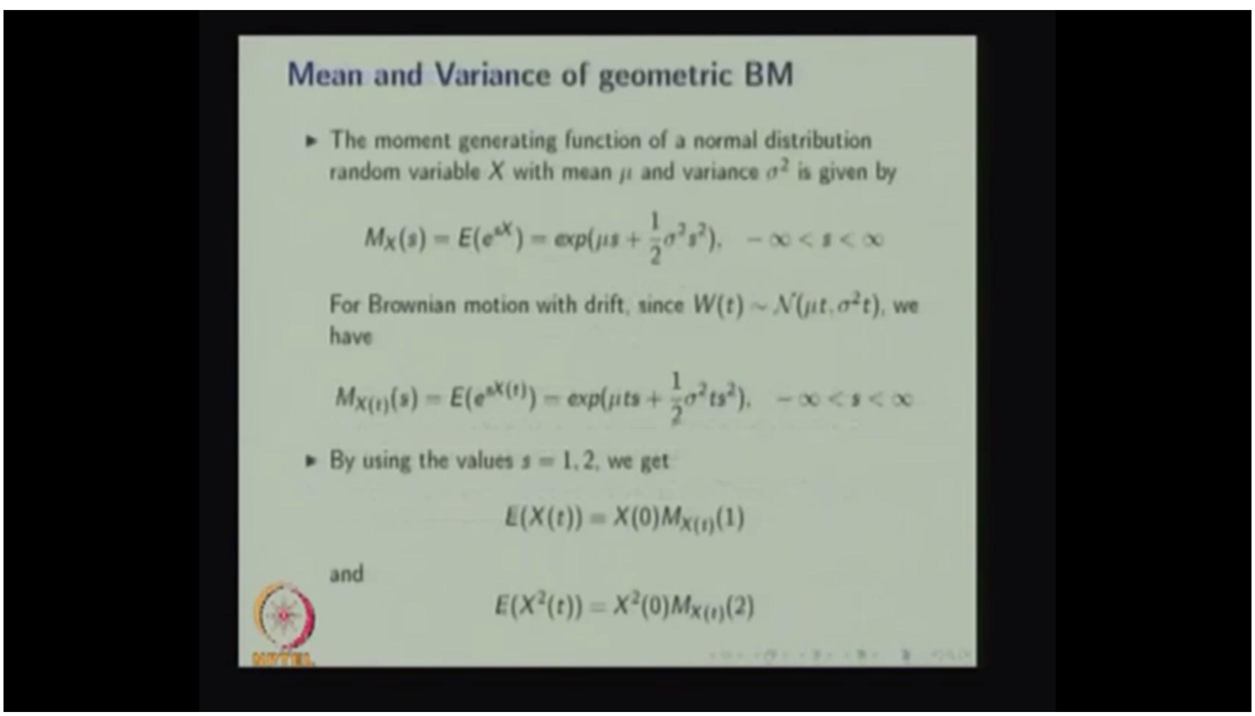
The definition of geometric Brownian motion. A stochastic process Xt is said to be a stochastic process X of t where t is varies from 0 to infinity is said to be geometric Brownian motion if X of t is of the form X naught exponential of W where Wt is a Brownian motion. That means that whenever you have Brownian motion then you make another stochastic process as a function of Brownian motion but it's of the form X of t is equal to X of 0 e power Wt then the X of t is stochastic process is said to be geometric Brownian motion. You know the range of Wt that is minus 3 infinity to infinity since X of t is of the form X of 0 e power Wt therefore the range of Xt will be 0 to infinity. The range of Xt is 0 to infinity. Therefore you can use this as a model for the stock price at any time t like that you can go for modeling any pricing of any security or derivatives at time t. So the Xt can be directly used in the application of finance.



You can see the sample path over the time the Xt here the range is from 0 to infinity so you can see the continuous functions. You know that the sample path of Wt is a continuous function and the Xt that is a geometric Brownian motion this is also a continuous function. [Indiscernible] [00:05:01] W of t.



Now we are going to discuss the very important property that is the Markov property of the geometric Brownian motion. The geometric Brownian motion is of the form X of t is equal to X naught e power Wt therefore for any H greater than 0 you can write X of t plus HbA1c is same as X naught e power in W of t plus h. I'm going to verify the Markov property therefore what I am going to do I'm going to add the Wt and subtract Wt in the exponential. Therefore the next step will be X naught e power Wt + W of t minus h minus Wt. Already we we know that X of t is equal to X naught E power Wt therefore X naught e power Wt can be replaced by X of t. Hence, X of t plus 1 is same as X of t multiplied by e power Wt + h. That means the stochastic process at the time point t plus h is same as at the time point t multiplied by exponential of the increment between the time points t to t plus h in W. We already know that the Brownian motion or Wiener process satisfies the Markov property and also the increments are independent along with increments are stationary here we are going to use independent increments are independent. That means since the Brownian motion has independent increments given Xt the future X of t plus h only depends on the future implements of Brownian motion given Xt the X of t plus 1 t plus h it depends only on W of t plus h minus W of t but since W is the Brownian motion, the Brownian motion increments are independent therefore W of t plus h minus Wt is independent of 0 to W. So this feature is independent of the the positive intergers. Here the positive integers is from 0 to small time. Hence, the Markov property satisfies because the future depends only on the present not the past. Therefore the Markov property is satisfied hence then W the Xt is the [Indiscernible] [00:08:16] So this is the above result is valid for all h greater than 0 therefore the Markov property is satisfied instance the future depends only on the present not on the past or it is independent of the past. Hence the Markov property is satisfied by Xt hence the Xt is a Markov process. So the Brownian motion is also a Markov process geometric Brownian motion is a Markov process also geometric Brownian motion is also a [Indiscernible] [00:08:56].



we are going to find out what is the standard [Indiscernible] [00:09:05] Brownian. We can use the moment generating function also. Here the moment generating function of a normal distribution random variable X with the mean mu the variance is Sigma square is given by for any random variables which is normally distributed with the mean mu and variance [Indiscernible] [00:09:30] the moment generating function it is M x of s as a function of s that is nothing but expectation of e power s times x. So this expectation is exist because the random variable X is normally distributed so the moment generating function exist because of the moments of all order is exists. So that is same as a you can do it separately this calculation. So here I'm using the result of moment generating function of normal distribution that is same as exponential of because it's a function of [Indiscernible] [00:10:13] that's why it is mu times s plus half Sigma square s square where s can take the value from minus infinity to infinity.

Why we are using the moment generating function because here the geometric Brownian motion and the Brownian motion is the connected in the form of X of t is X of 0 e power Wt where Wt is a normally distributed with the mean 0 and the variance t. Hence I'm using the moment generating function of the X of t as a function of s that is expectation of e power X times Xt. That is same as since Wt is normally distributed with the mu times t Sigma square t for a standard normal distribution the mu is 0 and the variance is 1. So here you can make out that is same as exponential of the mu ts plus half Sigma square t s square by replacing – by using the same the above logic here. [Indiscernible] [00:11:38] of this problem.

So sorry, there is a mistake. It is moment generating function of Wt and e of X times Wt because X is normally distributed here it should not be exchanged with Wt. Here also Wt therefore moment generating function of Wt is same as expectation of e power s times Wt [Indiscernible] [00:12:09] now using s equal to 1 and 2 now we are finding the mean and the variance of geometric Brownian motion because the mean of Xt is X naught times the moment generating function of X of t at 1. So we are using the moment generating function therefore we can get the mean of the geometric Brownian motion. Similarly finding the second order moment we can get the variance of geometric Brownian motion.

After simplification that means first we have to use what is the moment generating function of Wt and we have to use the form Xt is equal to X naught e power Wt using that you can get that and after doing some simplification we get mean and variance of geometric Brownian motion that is a expectation of Xt is at X naught times e power mu plus Sigma square by two times t and first we can get the expectation of second order moment [Indiscernible] [00:13:22] and the variance is equal to expectation of [Indiscernible] [00:13:24] minus expectation of [Indiscernible] [00:13:27] then you can get the variance of Xt. So the variance of Xt is of the form X naught whole square e power 2 mu plus Sigma square t multiplied by e power Sigma square t minus 1. By substituting mu plus Sigma square by 2 as a power cap upon bar you will get expectation of X of t is equal to X naught times e power [Indiscernible] [00:14:04] t. More generally you can go for expectation of Xt divided X of s that is same as e power power bar t – s. So in this way we are finding mean and variance of geometric Brownian motion using the moment generating motion of normal distributed randomly.