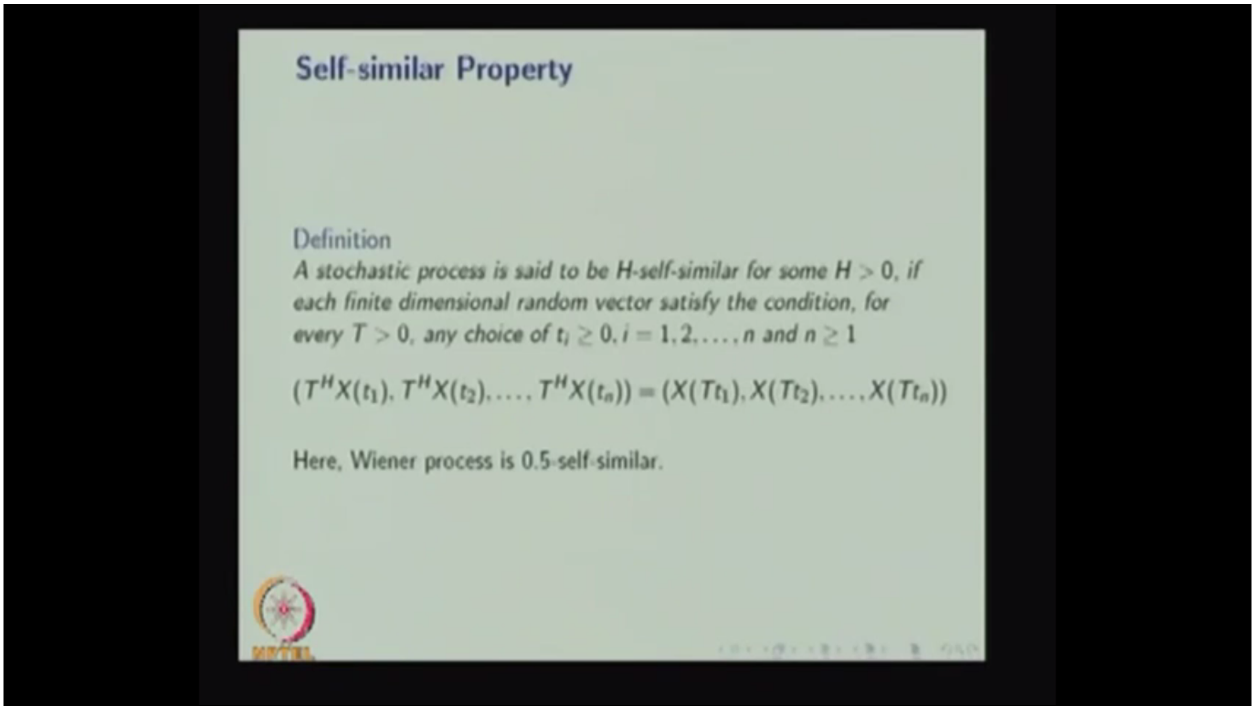
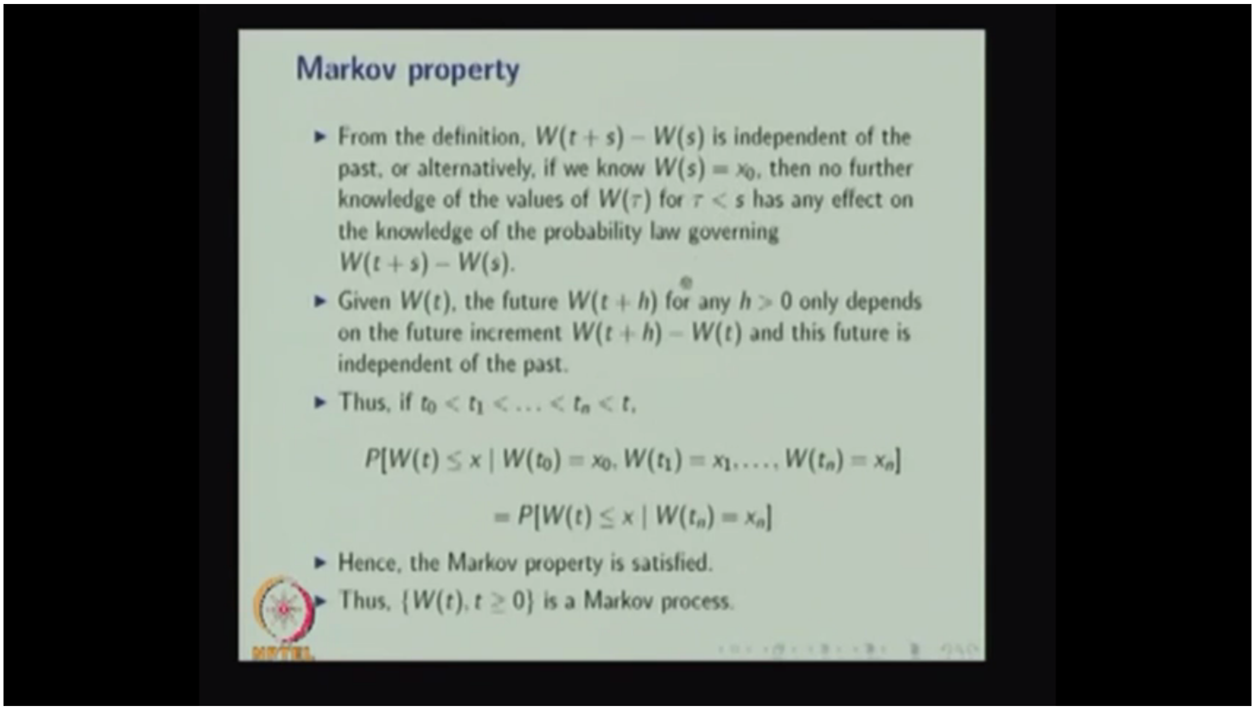


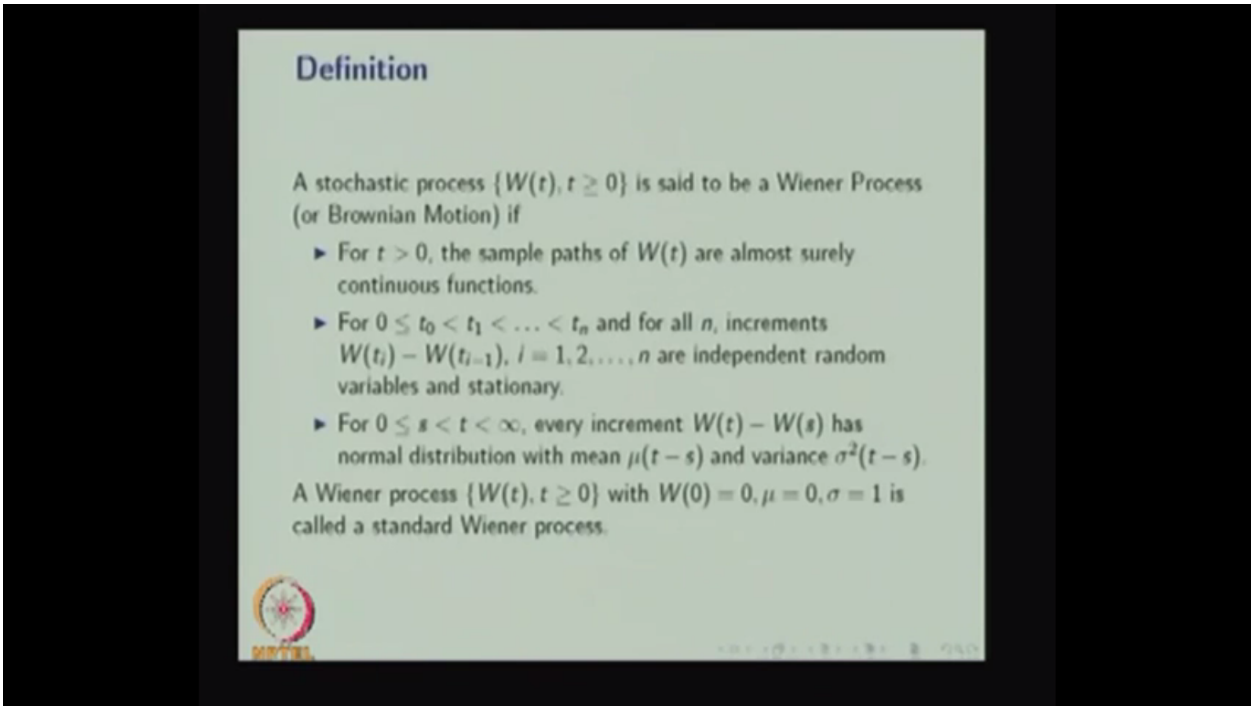
The next property is self-similarity property. Let me give the definition of self-similarity. Then we conclude the Wiener process is a one by two self-similar. What is the definition of self-similar. A stochastic process is said to be H self-similar for some H greater than 0 if each finite dimensional random vector satisfying the condition for every T greater than 1 any choice of Ti's for i is equal to 1 to n the joint distribution for n dimension random variable at the time point say T1, T2, Tn multiplied by T times H for every T and H is the H self-similar for some H greater than 0. If that distribution is same as X of the time point is multiplied by T without H in the whole right hand side. So if the joint distribution T times H1 for the random variable X, T times T2 for the second random variable and so on if this joint distribution is same as the joint distribution of this form then we say it is a H self-similar for some H for every T greater than 0.

One can verify the Wiener process is the 0.5 self-similar. Here I have not given the proof but you can multiply for some T for H is 0.5 you can conclude the Wiener process is the 0.5 self-similar.

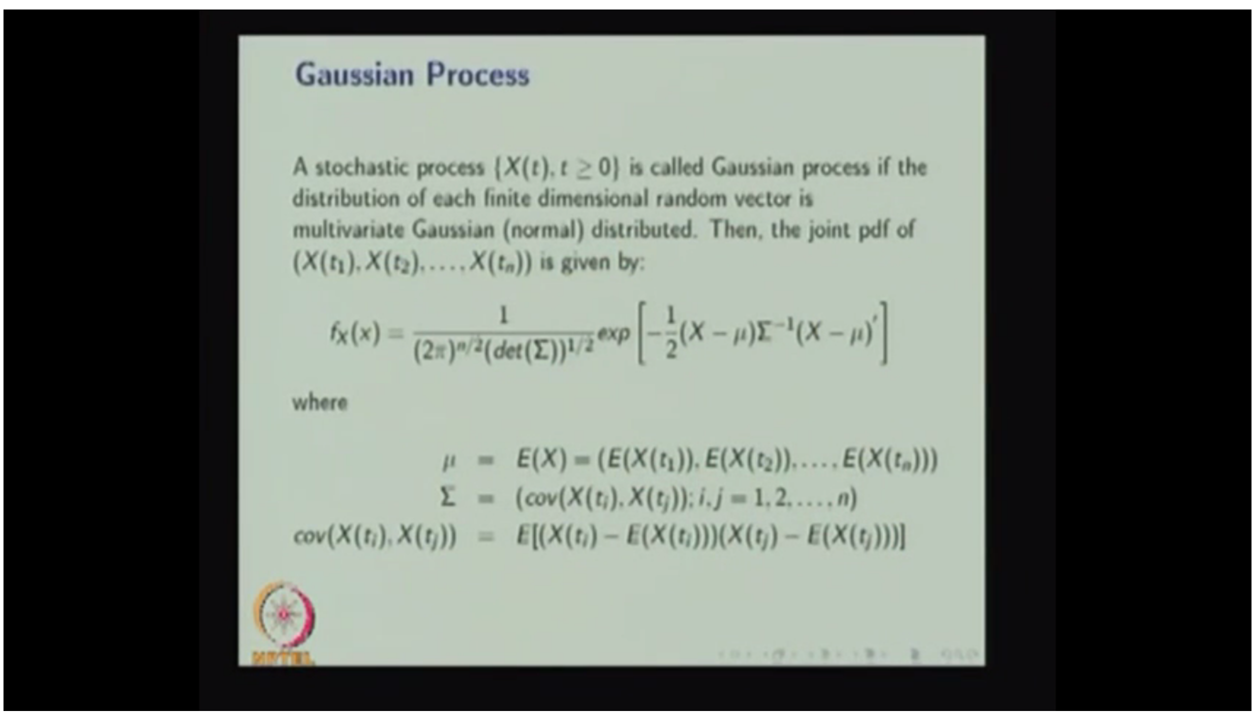


The next property that is very important one that is Markov property. You know the definition of Markov property. So this is the definition of a Markov property. If any stochastic process satisfies the Markov property for arbitrary time point at T naught to Tn which is less than T if this condition is satisfied then the stochastic process will be Markov process.

So here from the definition one can conclude W of t plus s minus W of s is independent of past or alternatively if you know Ws is equal to X naught then no further knowledge of the value W of tau where tau is less than s has any effect on the knowledge of probability law governing Wt plus s minus Ws the whole times k the W of t plus s minus Ws which is independent of the whole past information to s and if you know the information at the s depends only at the time point s not the whole process. From the definition you can make out because the definition says the increments are independent. Therefore the W of t plus s minus Ws is independent of the whole past information from 0 to s. That's what it says.

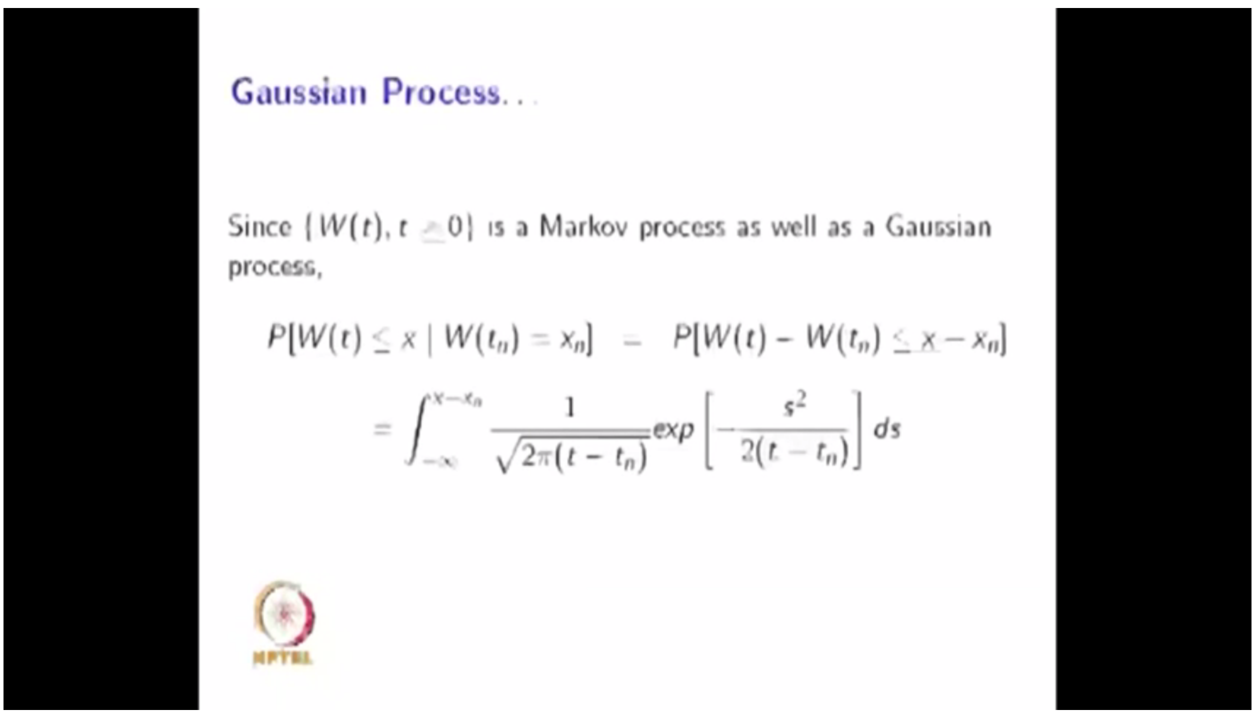


Therefore given Wt the future W of t plus h for any H greater than 0 only depends on the future increment W of t plus H minus Wt and this future is independent of past. Hence this Markov property satisfies since Markov property satisfied for arbitrary time points T naught to Tn therefore this stochastic process is called a Markov process. So hence the Brownian motion is a Markov process.

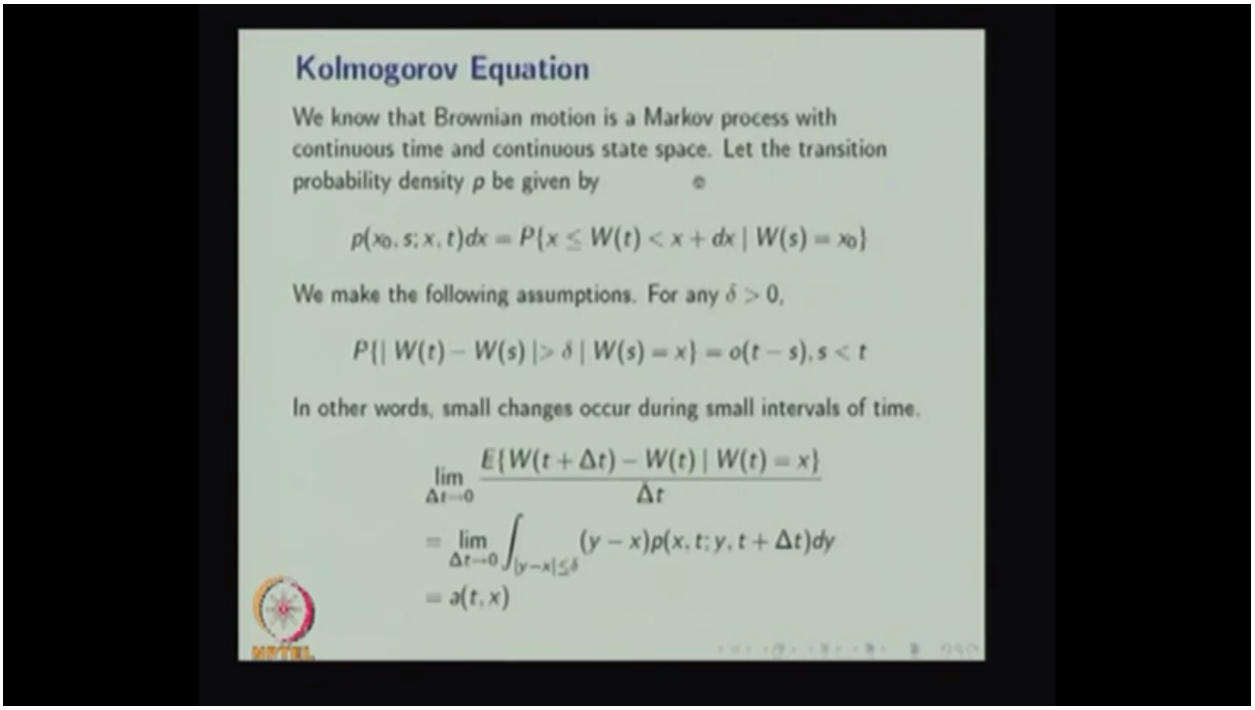


The next one is Gaussian process. First let me define what is Gaussian process then I'm going to relate the Gaussian process with the [Indiscernible] [00:04:57]. A stochastic process is called a Gaussian process if the distribution of each finite dimensional random vector is a multivariate Gaussian distributed. That means if you have a stochastic process and if you take any finite dimensional random vector from that stochastic process if that finite dimensional random vector is a multivariate Gaussian distributed random vector then the underlying stochastic process is a Gaussian process.

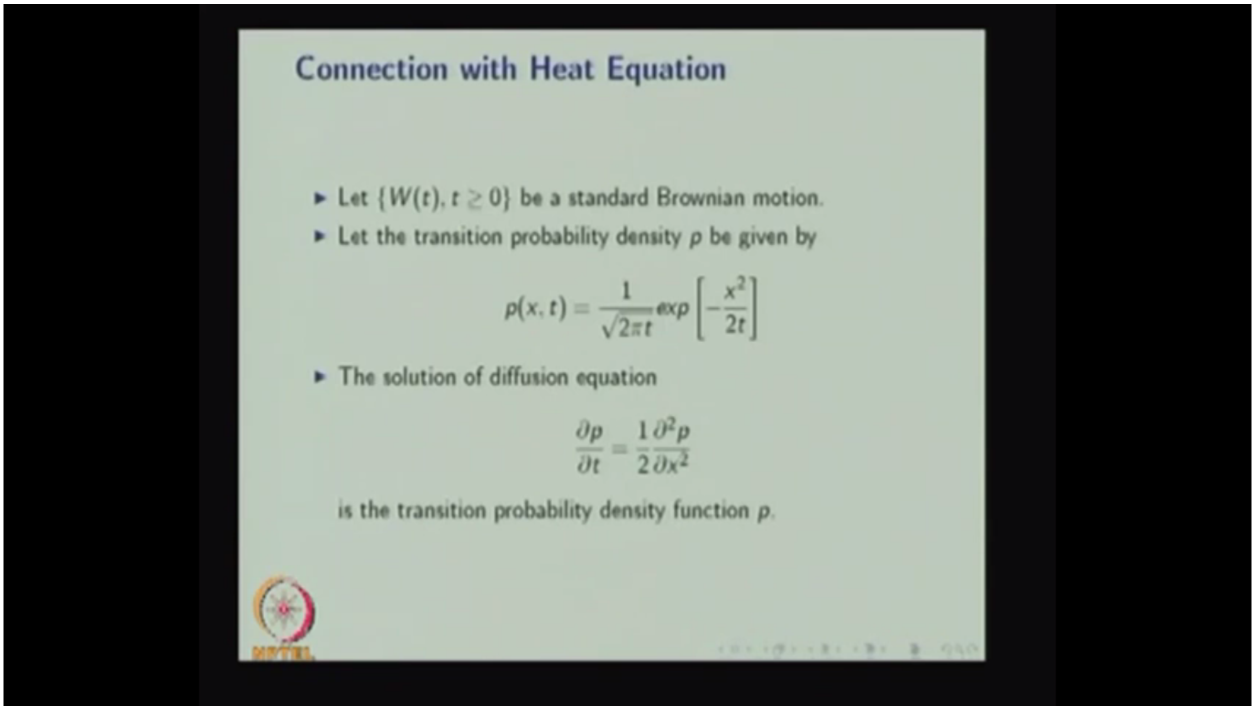
Since for each finite dimensional random vector is a multivariate you can write down the joint probability density function of n dimensional random vector of Gaussian process. That is nothing but this is a joint probability density function that is 1/2 power pie power n by 2 you find out the determinant of the matrix and after that you find out the square root then exponential of this where mu can be written as the vector and elements are nothing but the expectations and this notation sum is the covariance matrix covariance between any two random variables X of ti's with X of tj's where each one is running from 1 to n. Therefore it is the square matrix. And the elements are nothing but the covariance between any two random variables and all the diagonals will be the variance of X of ti's where i is running from 1 to n. And it will be a symmetric matrix because a covariance of X of ti, X of tj is same as covariance of X of tj, X of ti. Therefore this matrix is a symmetric matrix and the diagonal elements are variants of X of ti's. So one can find out the covariance of any two random variable using this formula.



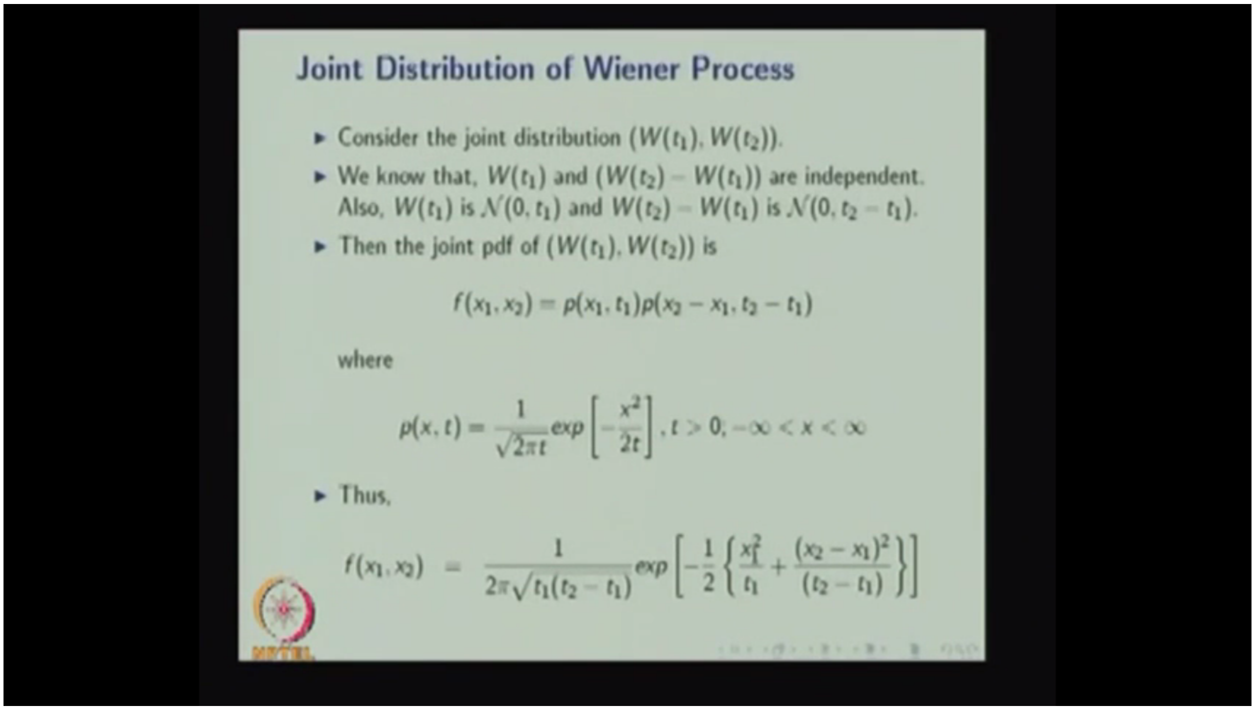
Since the Wt is a Markov process as well as Gaussian process you can write down the conditional CDF. The conditional CDF is same as the difference divided is less than or equal to X minus Xn but since this is normally distributed Wt minus W of Xn is a normally distributed therefore this is nothing but minus infinity to X minus Xn and this is a probability density function of a normally distributed random variable with mean 0 and the variance t minus t whenever we discuss the Brownian motion we are discussing standard Brownian motion with the Wt is equal to W0 is equal to 0 and mu is equal to 0 and Sigma square is 1.



Now we can discuss the Kolmogorov equation for the Brownian motion. We know that the Brownian motion is the Markov process with the continuous time and continuous state space we can write down what is a transition probability density to the probability transition probability density P will be probability that Wt lies between X to x plus Delta x, dx given that W is equal to x naught. We make the following assumptions for any Delta greater than 0 the probability of absolute Wt minus Ws which is greater than Delta given that Ws is equal to X that is the order of t minus s. In other words the small changes occurs during small intervals of time that is the meaning of the above definition.



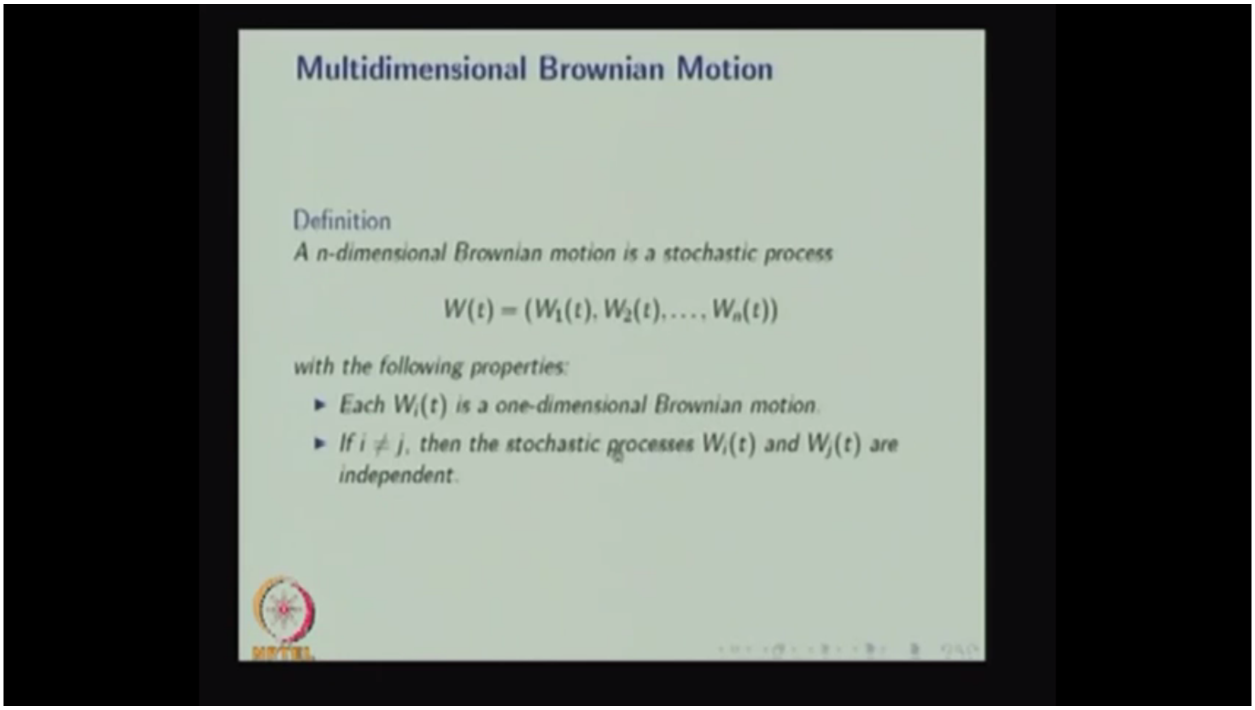
Now we can find out the conditional expectation of Wt plus Delta t minus Wt given Wt is equal to X divided by delta t as limit delta t tends to 0 that is nothing but you can note down as the denoted as the a of t,x this will be a function of t,x that is denoted as the a of t,x. Similarly you can make of the conditional expectation of the whole square given that Wt is equal to X that you can denote it as the b of t,x. In other words the limit of infinitesimal mean of variance of the increment Wt exists and is equal to b of t of t,x which is known as the diffusion coefficient. So a Markov process Wt satisfying the above conditions is known as a diffusion process and the partial differential equation satisfied by its transition probability density function is known as a diffusion equation. The partial differential equation satisfied by its transition probability density function is known as a diffusion equation. So this is the deficient equation this is a PDE or the transition probability density function P and where a and b are earlier defined this equation is also known as a forward Kolmogorov equation and also known as a Fokker-Planck equation. And this equation is possible because of the Wt is a Markov process therefore, and also it's a Gaussian process therefore we land up the transition probability density function P and satisfying the PDE and this PDE is called the Fokker-Planck equation. If you solve PDE which is given here or the standard Brownian motion or the standard means W0 is equal to 0, mu is equal to 0 and Sigma square is 1 in the definition of a Brownian motion then you will get the transition probability density function P is a 1 divided by square root of 2 times pie t exponential of minus X square by 2 times t and this is the probability density function of a standard normal distributed random variable with the mean 0 and the variance t. and the corresponding diffusion equation is a dou P by dou T is equal to 1 by 2 dou square P by dou X square.



Now we are going to discuss the joint distribution of a Wiener process. The way we discuss that Gaussian process as the Gaussian process every finite dimensional random vector is a multivariate random, multivariate normally distributed random variable therefore you can find out the joint distribution of W of t1 with W of t2. We know that a W of t1 and W of t2 minus W of t1 are independent. Here we made an t1 is less than t and also we know that a W of t1 is normally distributed in the mean 0 variance t1 and this difference is also normally distributed with the mean 0 and the variance t2 minus t1 and both are independent. Our interest is to find out the joint distribution of W of t1 with W of t2 but for that first we find out the joint distribution of a W of t1 with the W of t2 minus W of t1 then use the function of the random variables tool then you can find out the joint distribution of these two. So first you - so that is a way here I have not given the derivation. So finally you will get the joint distribution of joint probability density function of W of t1 with the W of t2 is in this form where the probability density function is going to be the normally distributed random variable. Hence, the joint distribution will be 1 divided by square root of 1 divided by 2 pie times the square root of t1 times t2 minus t1 exponential of this expression. Note that W of t1 and W of t2 are not independent whereas W of t1 with the W of t2 minus W of t1 are independent random variation. So using that we are finding the joint distribution of a W of t1 with the W of t2.

Once you know the joint distribution for any two random variables the same way you can find out the joint distribution of any n random variables in the Wiener process in the same way. I have not given the derivation here and we can find out the joint distribution joint probability density function of the n random variables also. And we need covariance matrix and expectation so the expectation vector that is mean therefore all the means are 0 whereas the covariance already we got the covariance of any two random variables of W of t1 with the W of ti's with W of tj's it will be a symmetric matrix and the diagrams are nothing but the variance of W.

We can go for the multi-dimensional Brownian motion. We can have a W1 is a Brownian motion. W2 is another Brownian motion so you can collect it as a make it as another Wt and each W is a one-dimensional Brownian motion and then you can go for the stochastic process are independent therefore you will have a n-dimensional Brownian motion also.



Here is the reference. So in this lecture we have discussed the definition of Brownian motion and also we discussed the derivation of Brownian motion and we have discussed important properties of Brownian motion starting from stationary increment increments or independent Markov property, Martingale property and also finally we discussed the the multi-dimensional Brownian motion.

References

- S Karlin and H M Taylor, "A First Course in Stochastic Processes", 2nd edition, Academic Press, New York, 1975.
- J Medhi, "Stochastic Processes", 3rd edition, New Age International Publishers, 2009.