Video Course on
Stochastic Processes

Doob's Martingale Process, Sub martingale and Super Martingale

by

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Now we are moving into the concept called Doob's Martingale Process.

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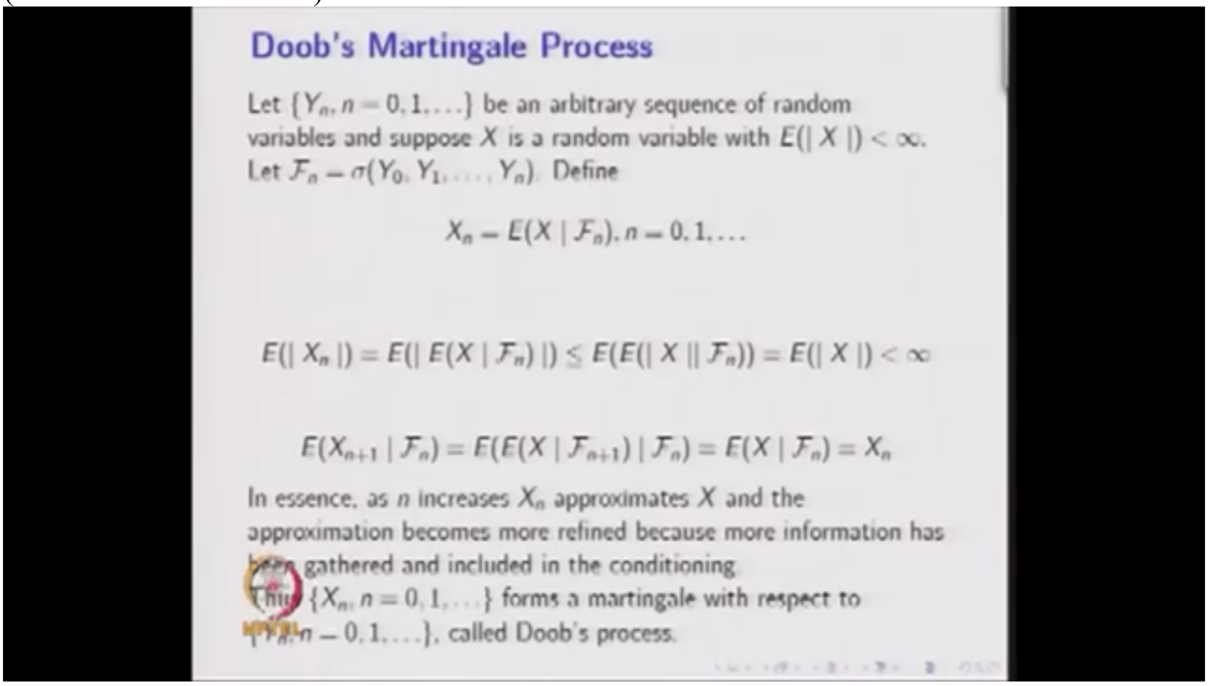

When we say the given stochastic process is going to be a Doob's Martingale Process or Doob's Process? See the definition. Let Y_n be an arbitrary -- arbitrary sequence of random variables and suppose X is a random variable with expectation in absolute finite. Define X_n is the conditional expectation of X_n given Y_0, Y_1 and so on till Y_n for every X_n where n is running from 0, 1, 2 and so on.

Expectation of expectation of X given F_{n+1} given F_n, we know that F_n is contained in F_{n+1} and using the property, this becomes expectation of X given F_n. That is same as X suffix n. In essence, as n increases, X_n approximates X and the approximation becomes more refined because more information has been gathered and included in the conditioning. Thus X_n, n is equal to 0, 1 and so on forms a martingale with respect to Y suffix n where Y_n is equal to 0, Y_n for n is equal to 0, 1, 2 called the Doob's Process.

Now we define Submartingale and Supermartingale.

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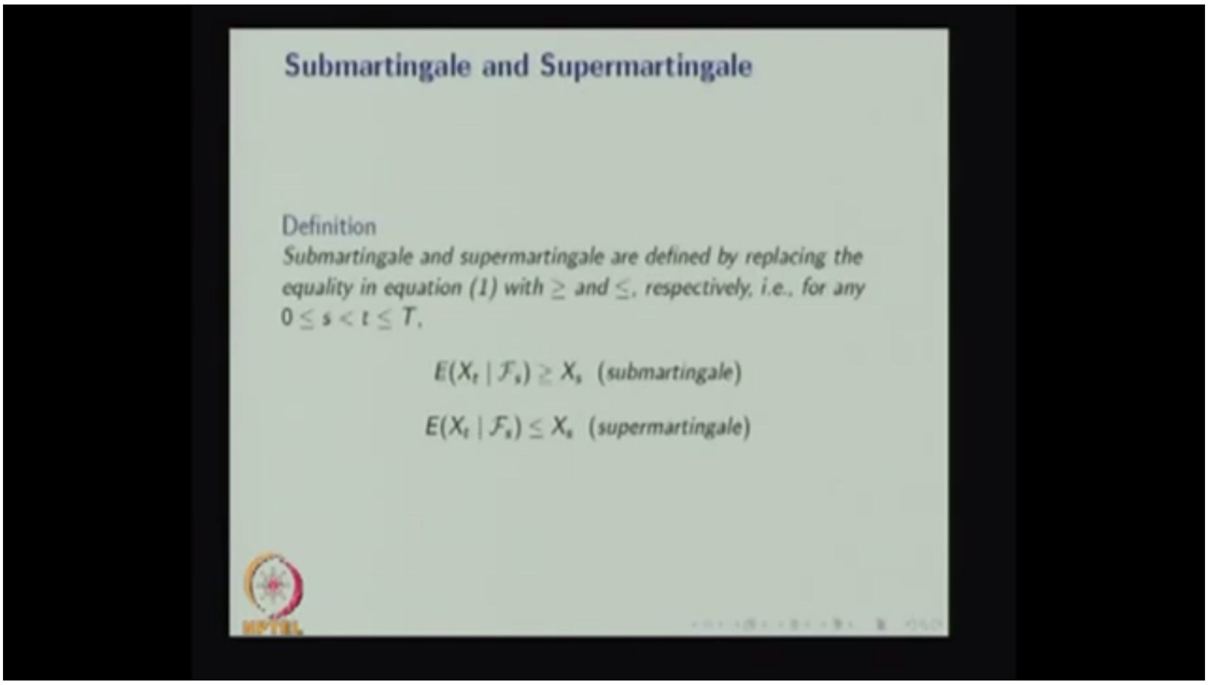

Submartingale and Supermartingale by replacing the equality in the equation (1), the equation (1) in the definition of martingale whether it is a discrete time or the continuous time, if you replace equality sign with the greater than or equal to or less than or equal to respectively for any 0 less than or equal to s less than t less than or equal to capital T, the third condition.

The first two conditions are expectation exists and then X_t is F_t measurable or X_n is F_n measurable. That's the two conditions. So there is no change in those two conditions; only the change in the third condition that is by replacing the inequality in equation (1) with the greater than or equal to, less than or equal to, then the corresponding stochastic process will be call it as submartingale whenever it is greater than or equal to sign or it is supermartingale if conditional expectation is less than or equal to X_s.

If it is equal to for all this interval, in that case, it is a martingale. If this property is satisfied greater than or equal to s where s is less than t and t is less than or equal to capital T, then this stochastic process is called a submartingale and less than or equal to condition is satisfied, then the stochastic process is called a supermartingale.

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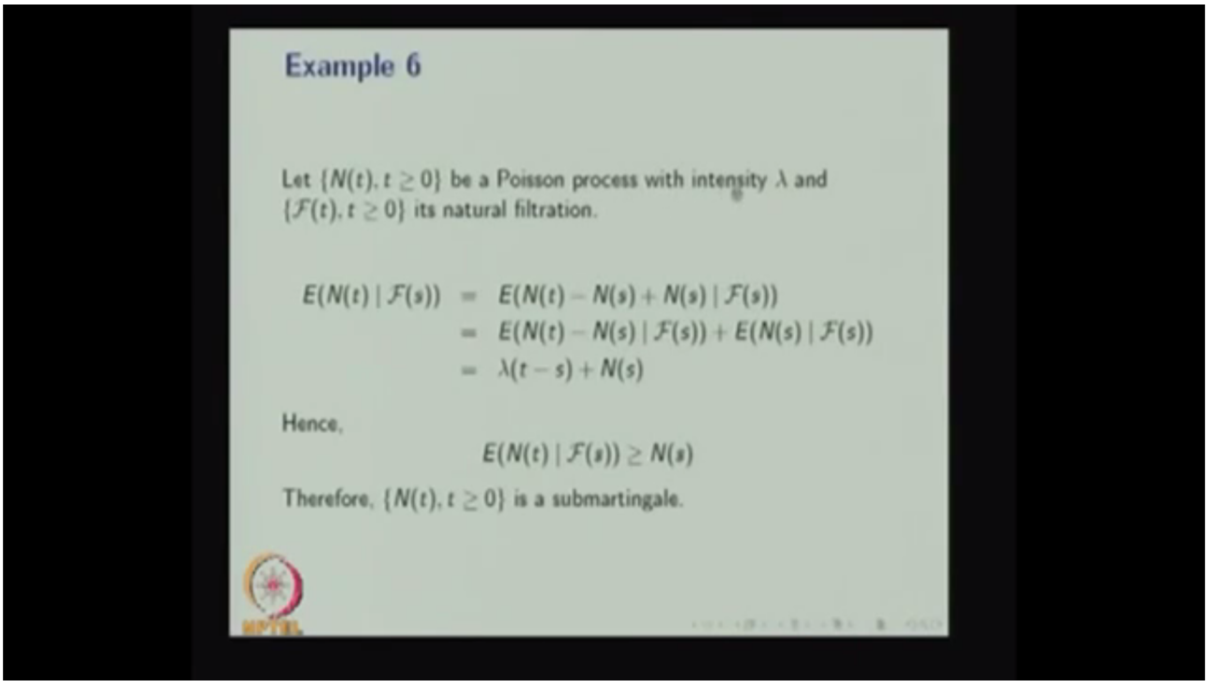

Now we present the Poisson process is a submartingale. We take the same example N(t) is a Poisson process with the intensity λ and earlier we have proved N(t) is not a martingale because conditional expectation is greater than or equal to N(s) because t - s is greater than 0, λ is strictly greater than 0. Therefore, this quantity is greater than or equal to N(s).

Hence, conditional expectation of N(t) given F(s) is greater than or equal to N(s), it's always greater than or equal to for all s less than t where t is less than or equal to infinity. So this condition is satisfied for all s and t. Hence, the given stochastic process, that is the Poisson process is a submartingale because of this greater than or equal to sign.

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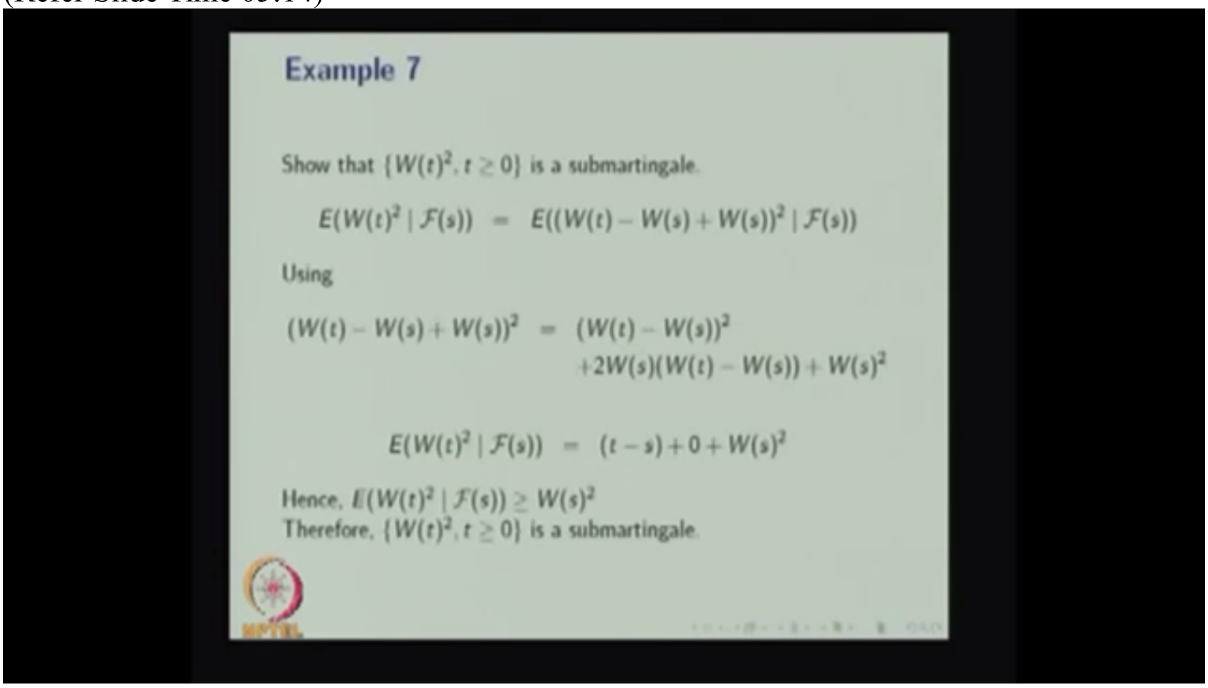

Now we present the $W(t)^2$ is a submartingale where W(t) is a Brownian motion. W(t) is a Brownian motion. So we are going to use the property of Brownian motion. That is one particular important stochastic process which has a lot of applications in financial mathematics. So here we are going to prove $W(t)^2$ is a submartingale. How?

The conditional expectation of $W(t)^2$ given F(s) is you add one term and subtract one term. That is plus W(s) minus W(s) inside the $W(t)^2$. Using the identity, $(W(t) - W(s) + W(s))^2$ will be same as $(W(t) - W(s))^2 + 2W(s)(W(t) - W(s)) + W(s)^2$.

So why I am doing this way of adjustment because I'm going to use the property of non-overlapping intervals are independent. We are directly checking the conditional expectation. The first two conditions are obviously satisfied. The first one is W(t) is integrable. The Brownian motion, for fixed W, for fixed t, W(t) is normally distributed with the mean 0 and a variance t for a standard Brownian motion. Therefore, the mean exists.

The second one, W(t) is F(t) measurable. Whenever we didn't discuss the filtration, that means that we have a natural filtration F(t). Whenever we have a natural filtration, that means that this random variable is F(t) measurable. W(t) is a F(t) measurable. Therefore, second condition is also satisfied and we are checking the third condition.

The third condition, the right-hand-side $(W(t) - W(s))^2 + 2 W(s)(W(t) - W(s))$ plus so on. So now we are applying the conditional expectation here given F(s). Therefore, the conditional expectation is a linear operator. Therefore, it is conditional expectation of this term given F(s), conditional expectation of this term given F(s) plus conditional expectation of this term given F(s). You know that F(s) is nothing but the information up to time s and (W(t) - W(s)), that is also normally distributed with the mean 0 and the variance t minus s. I am discussing about the standard normal distribution.

Later we are going to discuss the Brownian motion in detail. So here I am going to use only the distribution and the mean of Brownian motion as well as independent property. So here the W(t) - W(s) is independent of F(s). Therefore, the conditional expectation is nothing but expectation of the $(W(t) - W(s))^2$. Since W(t) - W(s) is normally distributed with the mean 0 and the variance t minus s, this conditional, sorry, this expectation is nothing but the t - s.

The second term, again, 2 times conditional expectation of W(s) multiplied by W(t) - W(s) given F(s) and the F(s) is independent of W(t) - W(s) and you have the information till s. Therefore, W(s) has to be treated as a constant. So the 2 times W(s) will be treated as a constant. So, hence, expectation of W(t) - W(s) given F(s) is nothing but expectation of W(t) - W(s) because of F(s) is independent of W(t) - W(s) and you know that the expectation of W(t) - W(s) is 0 because this is normally distributed with the mean 0, variance t - s.

The third term, the conditional expectation of $W(s)^2$ divided by F(s) is nothing but $W(s)^2$ because you know the information till time s. Therefore, $W(s)^2$ has to be treated as a constant. Expectation of constant is constant.

Hence, expectation of $W(t)^2$ given F(s) is nothing but (t - s) + 0. So, hence, (t - s) + $W(s)^2$. So this is obviously greater than or equal to $W(s)^2$. Therefore, the $W(t)^2$ is a submartingale because it satisfies the third condition with the greater than or equal to sign in the conditional expectation. Therefore, the stochastic process $W(t)^2$ is a submartingale.

Some remarks. A stochastic process let's say of random variable X_n is a submartingale with respect to the filtration F_n, n varies from 1, 2 and so on if and only if $-X_n$ is a supermartingale with respect to the same filtration F_n. If and only if is very important.

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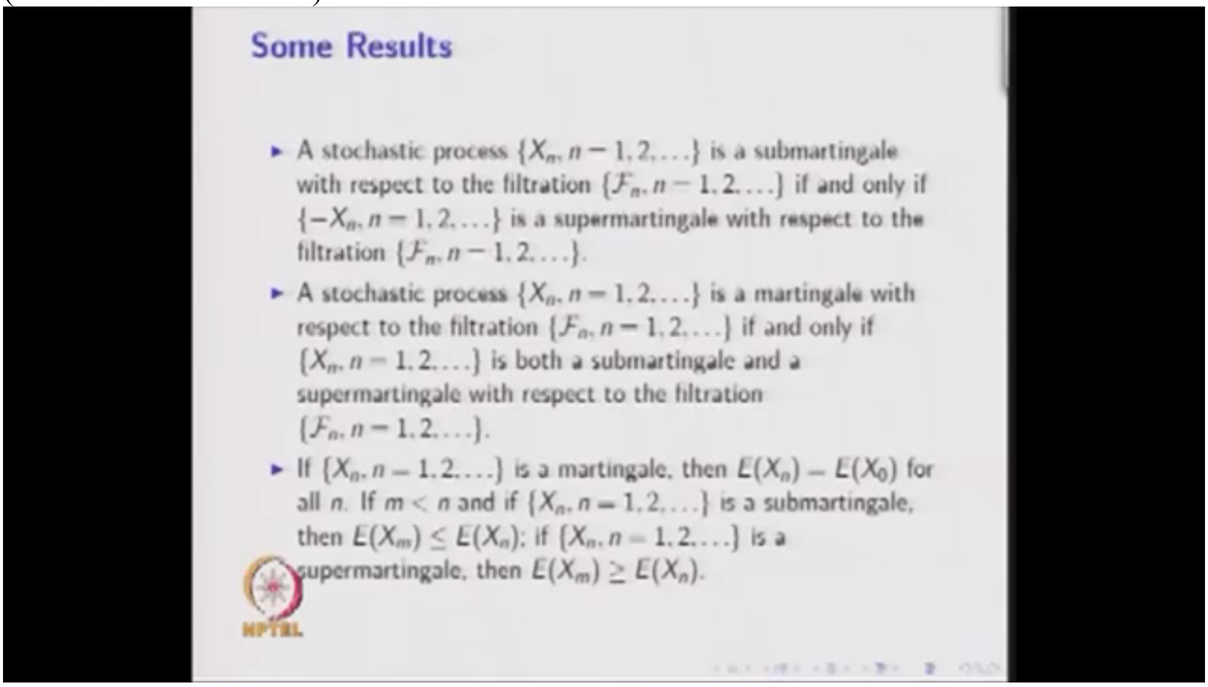

Whenever you have a submartingale, you can always convert it into the supermartingale with the minus sign or if you have a supermartingale, you have a stochastic process with the supermartingale property satisfied, then you can convert this stochastic process into the submartingale stochastic process by changing the sign.

Note that both the stochastic process we are discussing the martingale property, submartingale or supermartingale property with respect to the same filtration. That is important. You cannot convert submartingale into supermartingale with a different filtration. Not in general. If you have a same filtration, then you can transform the submartingale into supermartingale by changing the sign in both ways.

The second remark, a stochastic process X_n, n varies from 1, 2 and so on is a martingale with respect to the filtration F_n if and only if the -- if and only if X_n is both submartingale and the supermartingale with respect to the same filtration. That means if you have a given stochastic process a martingale if and only if the same random variable will be treated as a submartingale as well as supermartingale because we are changing, we are replacing the equality sign by greater than or equal to; similarly, less than or equal to. We are not replacing by strictly greater than or strictly less than. Since we have replacing the equality sign in the conditional expectation of the third condition in the definition of martingale with the less than or equal to, greater than or equal to, hence, if you have a martingale, then the same thing will be supermartingale as well as submartingale.

Similarly, if you have a stochastic process is both submartingale and supermartingale without changing the sign or without doing any change, the same random -- the same stochastic process is a submartingale as well as a supermartingale, then definitely that will be a martingale because if it is both submartingale and supermartingale, that means there is no -- the conditional expectation with greater than or equal to or less than or equal to. It must be equal to in the conditional expectation. Hence, the given stochastic process is a martingale.

These above remarks is also valid for continuous time.

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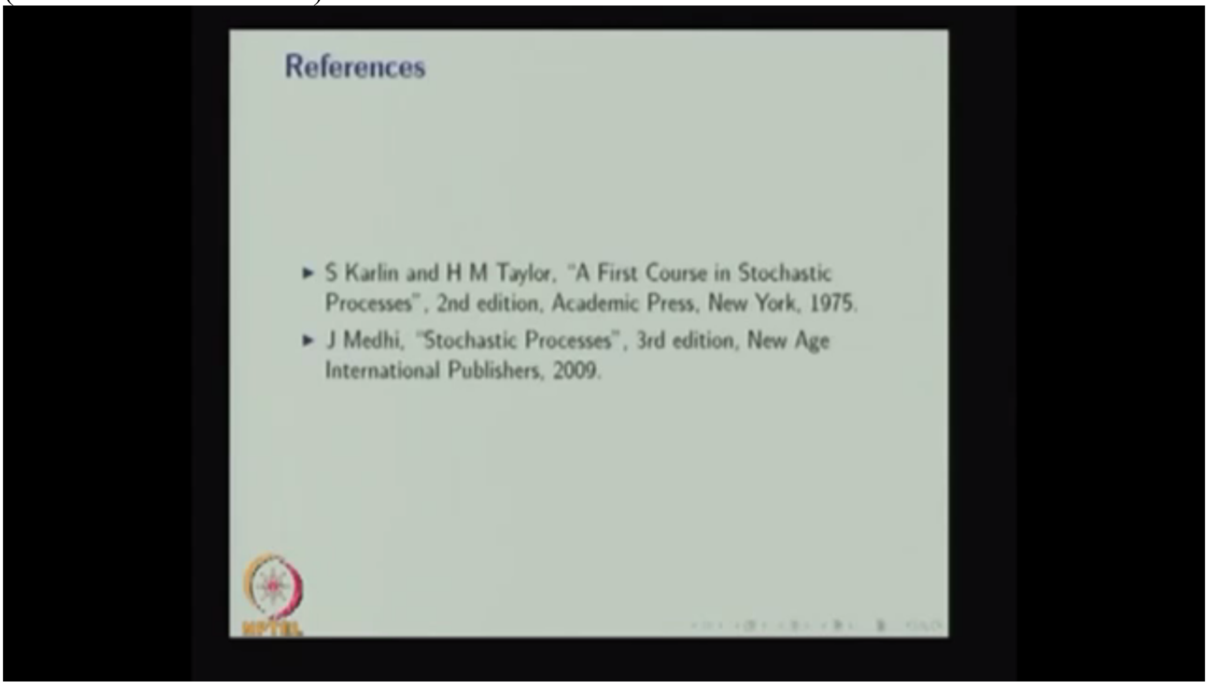

In this lecture, we have covered the definition of martingale in continuous-time, the definition of martingale in the discrete-time. Then we have discussed few examples and followed by that, we have discussed when we can say the given stochastic process is a supermartingale or submartingale. We have given few examples for the supermartingale as well as submartingale also and finally we have given some remarks over martingale, submartingale and supermartingale.

Here is the list of references. With this the Lecture 2 of Module 6 is complete.

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