Indian Institute of Technology Delhi

NPTEL

National Programme on Technology Enhanced Learning

Video Course on
Stochastic Processes

Examples of Martingale (contd.)

by

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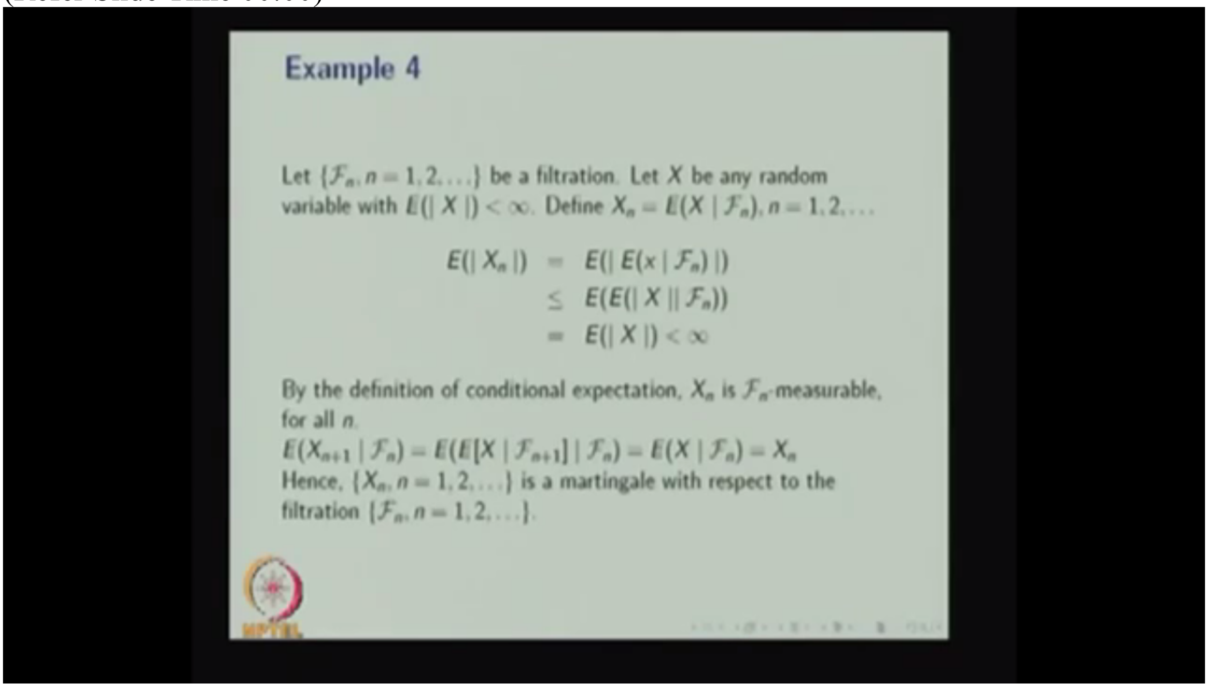

Now we are moving into the fourth example. The fourth example we started with the filtration. F_n is a sequence of σ-fields and this is the filtration. Let X be any random variable with the random variable is a integrable. Define X_n is a conditional expectation of X given F_n. So you are defining a sequence of random variables with the help of the conditional expectation of the random variable with a given the information up to n or the filtration n.

So suppose you want to prove this sequence of random variable or the stochastic process is a martingale, then it has to satisfy the three conditions, which we discussed in the discrete-time of a martingale property.

So the first we are checking whether the random variable X_n is integral. So if you want to prove the X_n is a integrable, then you have to prove the expectation in absolute random variable has to be a finite. That is finite. Then the random variable is integrable.

So this is same as expectation of in absolute you can replace X_n by expectation of X given F_n. You can take the absolute inside the expectation and this is nothing but it is an absolute of X because that is a way we define since X_n be any random variable with the expectation is a finite and you know the definition of expectation of expectation X given Y is going to be expectation of X. So we are using that property.

Hence, expectation of expectation of X given the filtration F_n is same as expectation of that random variable. So here the random variable is absolute X and this is already proved that it is a finite. Therefore, this is also going to be finite value. Hence, the expectation of absolute X_n is finite. The random variable X_n is integrable. So the first condition is verified.

By the definition of conditional expectation the, X_n is F_n measurable. The way we have written X_n is expectation of X given F_n, so this is the definition of conditional expectation whenever you write conditional expectation of X given F_n and that exists with the $X_{n,}$ that means the random variable X_n's are F_n measurable for all n. So by the definition of conditional expectation, X_n is F_n measurable for all n. So the second condition also satisfied.

Now we are going to verify the third condition. The expectation of X_{n+1} given F_n is same as you can replace X_{n+1} by the definition, that is expectation of X_n given F_{n+1}. You are replacing X_{n+1} with the conditional expectation given F_n.

You know the property of the filtration. The filtration property is F_1 contained in F_2, F_2 is contained in F_3 and so on. Therefore, F_n is contained in F_{n+1} if you have two σ-fields and one is the sub-sigma field of other one. F_n is a sub σ-field. F_n is the sub-sigma fields of F_{n+1}. Then the conditional expectation of conditional expectation X given F_{n+1} given F_n is same as conditional expectation of X given $F_{n.}$ We are using the conditional expectation given σ-fields with two σ-fields F_n contained in F_{n+1}. We are using the property. Hence, expectation of X given F_n, by the definition, expectation of X given F_n is nothing but X_n. Therefore, this is equal to X_n.

So left hand side we started with the expectation of X_{n+1} given F_n. The right hand side we land up X_n. This is the third property of -- this is the third condition we have defined it in the Martingale property in discrete time. So, hence, all the three conditions are satisfied by the sequence of random variable X_n's. Therefore, the stochastic process X_n is a Martingale with respect to the filtration F_n because we have used this filtration to conclude X_n is a F_n measurable and find out the conditional expectation is same as X_n and the random variable X_n is a integrable. Hence, the stochastic process is a Martingale with respect to the filtration F_n.

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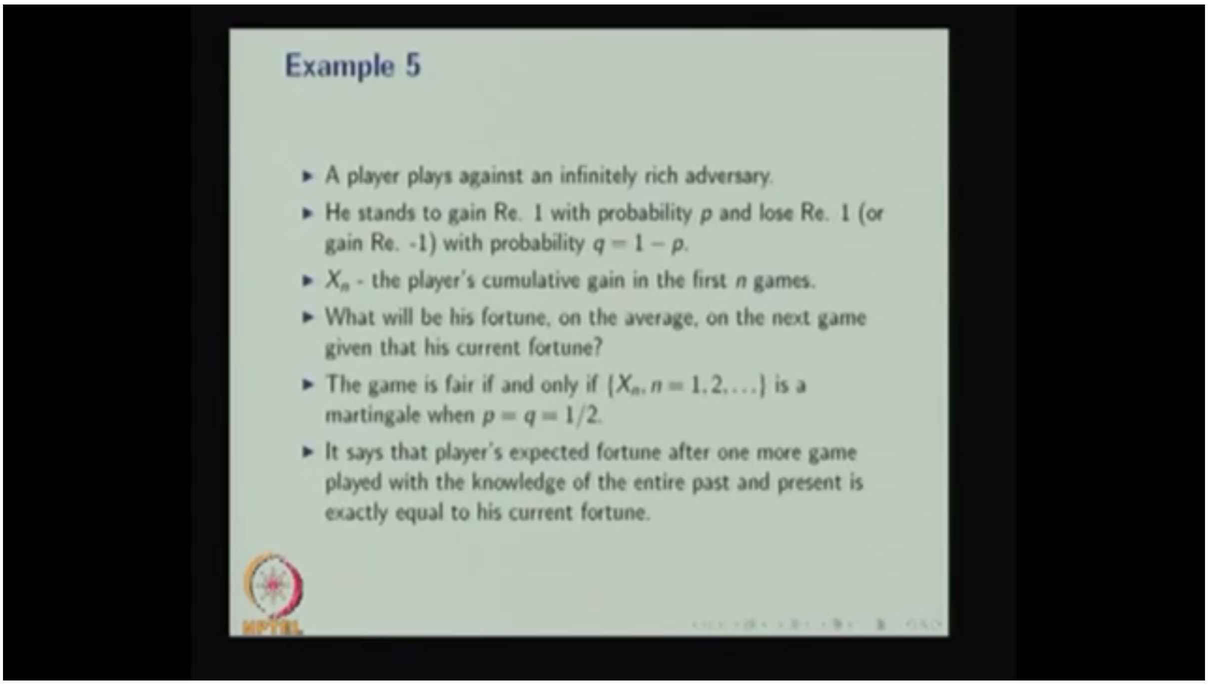

Now we present the example which was introduced in the beginning of lecture in this module. So this is the example we have given as a motivation for the model Martingale. A player plays against an infinitely rich adversary. He stands to gain rupees 1 with the probability p and lose rupees 1 with the probability q.

We are defining the random variable X_n, the player's cumulative gain in the first n games. What will be his fortune, on the average, on the next game given that his current fortune? We are asking the measure in a conditional expectation. The game is fair if and only if this sequence of random variable X_n is a Martingale and that is -- this sequence of random variables or this stochastic process will be a martingale when p and q is equal to 1/2.

That means when the game is fair, when the game is fair, that means whether he will gain one rupee or he lose one rupee with the equal probability 1/2 and 1/2, then when the game is fair, whenever the game is fair, the given stochastic process is a martingale and also whenever the given stochastic process is a martingale, in that case, the game is -- the game will be a fair game, that means the p and q will be half, 1/2.

The conclusion is it says that the player's expected fortune after one more game played with the knowledge of entire past and the present is exactly equal to his current fortune. The conditional expectation of his one more game expected fortune given the knowledge of entire past and present, that means the filtration till time t or till time n in the discrete case, that is same as exactly equal to his current fortune. That means it is same as expected -- that is same as X suffix n for a discrete case or it is X of t for a continuous case.

So whenever the game is fair, that is a p and q is equal to 1/2, then the given stochastic process X_n is a Martingale and the conclusion of the -- conclusion of this problem is the player's expected fortune after one more game played with the knowledge of entire past and present is exactly, exactly is important because later we are going to say more than or less than. For that we are going to name the different. So here it is exactly equal to his current fortune.