Indian Institute of Technology Delhi

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National Programme on Technology Enhanced Learning

Video Course on
Stochastic Processes

Examples of Martingale contd.

by

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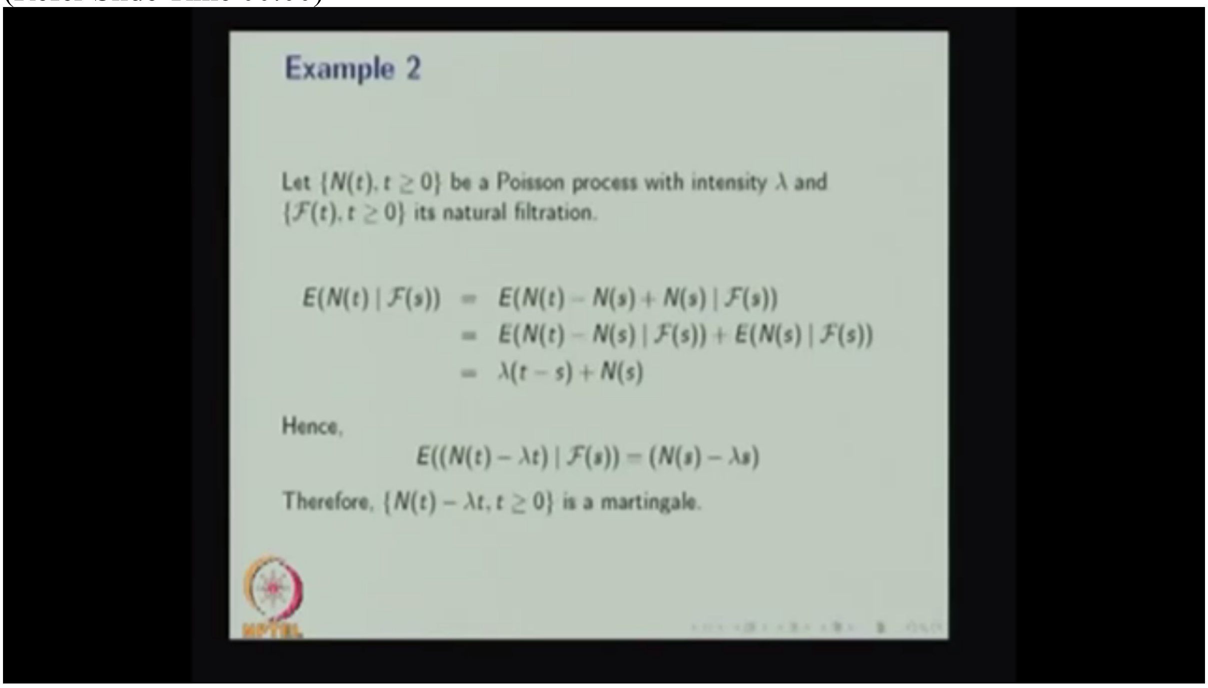

The second example, this stochastic process is a continuous-time discrete state stochastic process. So here let N(t) where t varies from 0 to infinity be a Poisson process with the intensity λ or parameter λ and the F(t) is its natural filtration. The natural filtration means it has the information till time t and each random variable N(t) for fixed t is F(t) measurable. That is the meaning of a natural filtration.

We know that for fixed t, N(t) is a Poisson distributed random variable with the parameter λ. Therefore, the mean of N(t) that is same as the parameter λ. So the first condition is satisfied.

The second condition, for fixed t, N(t) has to be F(t) measurable. Since it is a natural filtration, the N(t) is a F(t) measurable.

The third condition we are going to verify. The conditional expectation of N(t) given F(s) where F(s) is the filtration at time s or the information till s. Obviously, here s is less than t.

So to find out this conditional expectation, what we do? We add and subtract N(s) with the N(t). So instead of a conditional expectation of N(t) given F(s), we subtract N(s) and add N(s).

Expectation is a linear operator. Therefore, you can split this, these three terms into two different expectations, conditional expectations. Therefore, the first one you can keep N(t) - N(s). The second one you can keep it separately N(s). Hence, you have conditional expectation of N(t) - N(s) given the filtration F(s) plus conditional expectation of N(s) given F(s).

In the first term, the first term, this conditional expectation is nothing but you know the information till time s and we are asking the conditional expectation of N(t) - (s) given F(s). That means this t minus s and s, this is a non-overlapping intervals. S is the point. T minus s is the non-overlapping intervals. So this is the random variable corresponding to the non overlapping interval with respect to s.

Therefore, non -- you know the property of Poisson process. For s less than t, N(t) - N(s) is nothing but the increments. The increments are stationary and independent. Therefore, the N(t) - N(s) is independent of F(s). If it is independent, the conditional expectation is nothing but expectation of N(t) - N(s). You know the Poisson process, the properties. Since N(t) is a Poisson process, N(t) - N(s) for a fixed t and s, this is a Poisson distributed random variable with the mean λ times (t-s).

Therefore, this conditional expectation will be λ times (t-s) based on the N(t) - N(s) is independent of F(s) and for fixed s and t, N(t) - N(s) is a Poisson distributed random variable with the mean λ times (t-s) whereas the second term, conditional expectation of N(s) given F(s) that means for information till time s, what is the expectation of N(s) at the same time s? So since you know the information till or up to the time s, N(s) is constant. N(s) is a constant.

Therefore, expectation of constant is a constant. Therefore, it is N(s). It's not a λ times s because you know the information till time s. Once you know the information till time s, that means you know the value of N(s) also. Once you know the value of N(s), therefore, N(s) is no more a random variable. So it's a constant. So expectation of a constant is a constant. So it is N(s).

Hence, conditional expectation of N(t) given F(s) is same as you can take λt in this side, so conditional expectation of N(t) - λt given F(s) is same as N(s) minus λ times s. You see this is the expectation of N(t) - λs given F(s). That is same expression. Therefore, as such expectation of N(t) given F(s) is not N(s). It has the some positive value. (t - s) is always greater than 0. Therefore, λ times (t - s) will be greater than 0. Therefore, this conditional expectation is always greater than or equal to N(s).

Hence, N(t) is not a Martingale whereas if you treat N(t) - λt as a stochastic process over the t ranges from 0 to infinity, then this stochastic process is a Martingale. The N(t) is not satisfying is same as N(s). The conditional expectation is not equal to N(s), but it is greater than or equal to N(s). Therefore, N(t) is not a Martingale whereas if you make another stochastic process, that is nothing but N(t) - λ times t for t greater than or equal to 0, then this stochastic process satisfies the third condition and also satisfies the other two conditions. Therefore, the N(t) - λt is a Martingale.

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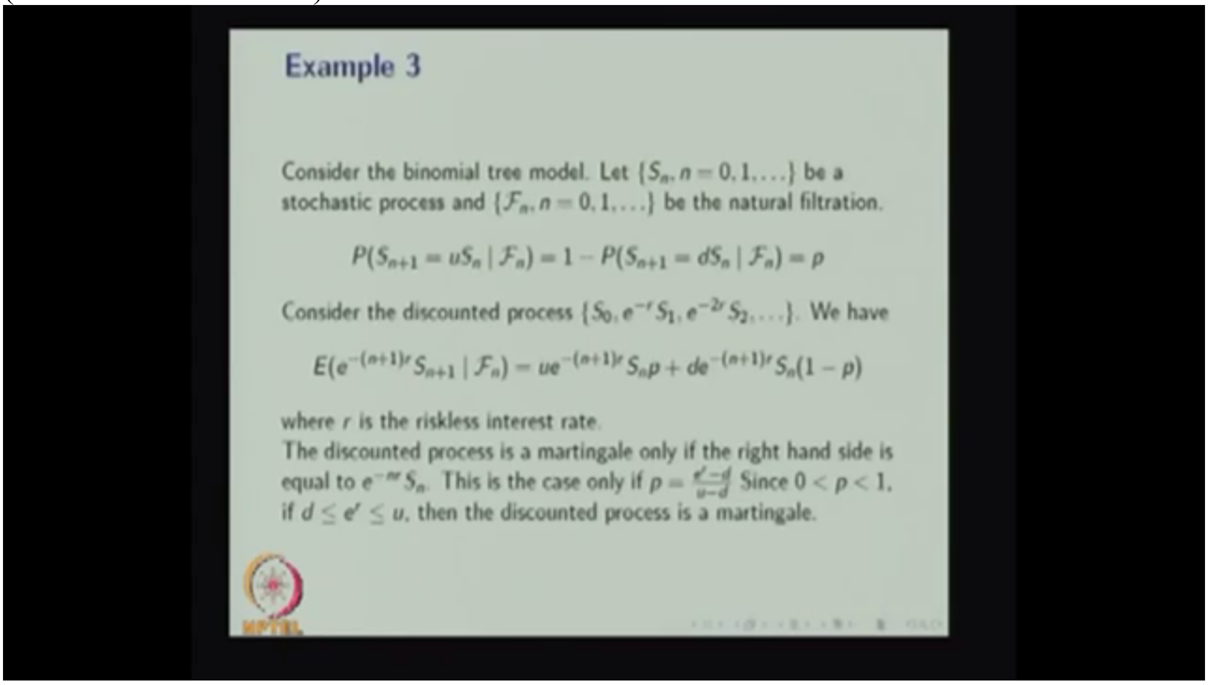

Third example. This is related to the application of finance. Consider the binomial tree model. Let S_n be a stochastic process and the F_n be the -- its natural filtration. Define the probability of S_{n+1} given u times S_n given F_n that is same as -- that is P and the probability of S_{n+1} is equal to d times S_n given F_n is equal to 1 - p where u and d are the next value of S_n with the probability p and q respectively.

Therefore, suppose the previous, the n^{th} value was S_n, then the next value will be, it is decremented with d, therefore, d times S_n or it would have been incremented with u. Therefore, u times S_n will be the $S_{n+1}{}^{th}$ value. Therefore, this -- this stochastic process is called the binomial tree model.

Now I am considering the discounted stochastic process that is nothing but e power minus r times, basically e power rt S_t. Since it is a discrete-time stochastic process, the first S_1 is multiplied by e power minus r whereas the second random variable S_2 is multiplied by e power minus 2 times r and so on. Therefore, the n^{th} random variable S_n will be e power minus n times r where r is the riskless interest rate. So whenever you multiply the e power minus r times t, the corresponding stochastic process is called the discounted stochastic process.

The discounted stochastic process is a martingale if -- only if the right hand side is equal to e power -n times r times s because here it is a conditional expectation of $e^{-(n+1)r}$ S_{n+1} given Fn. If this quantity is same as e^{-nr} multiplied by S_n, then this discounted stochastic process will be a martingale or it has the martingale property.

So this is the case only if -- if the p value takes e^r times -d and divided by u minus d if the p is the probability of incremented by u, if the p is equal to e^r -d divided by u-d, then the discounted stochastic process is a Martingale. So that is possible because since p lies between 0 to 1, since the -- whenever the r is the riskless interest rate, e^r is also lies between d to u. Therefore, the u - d and e^r -d, these values is going to be lies between 0 to 1. So, therefore, with the proper value of r, d and u, the p can be -- if the p is of this form, then the discounted stochastic process is a martingale.