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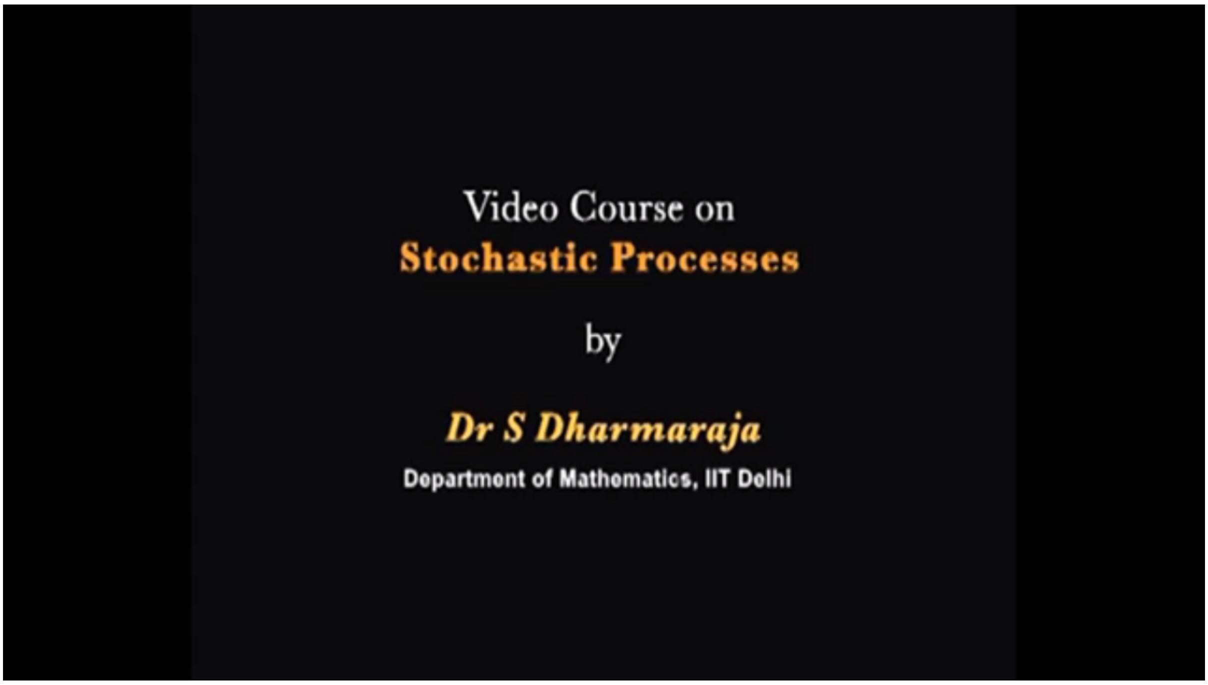

Video Course on
Stochastic Processes

by

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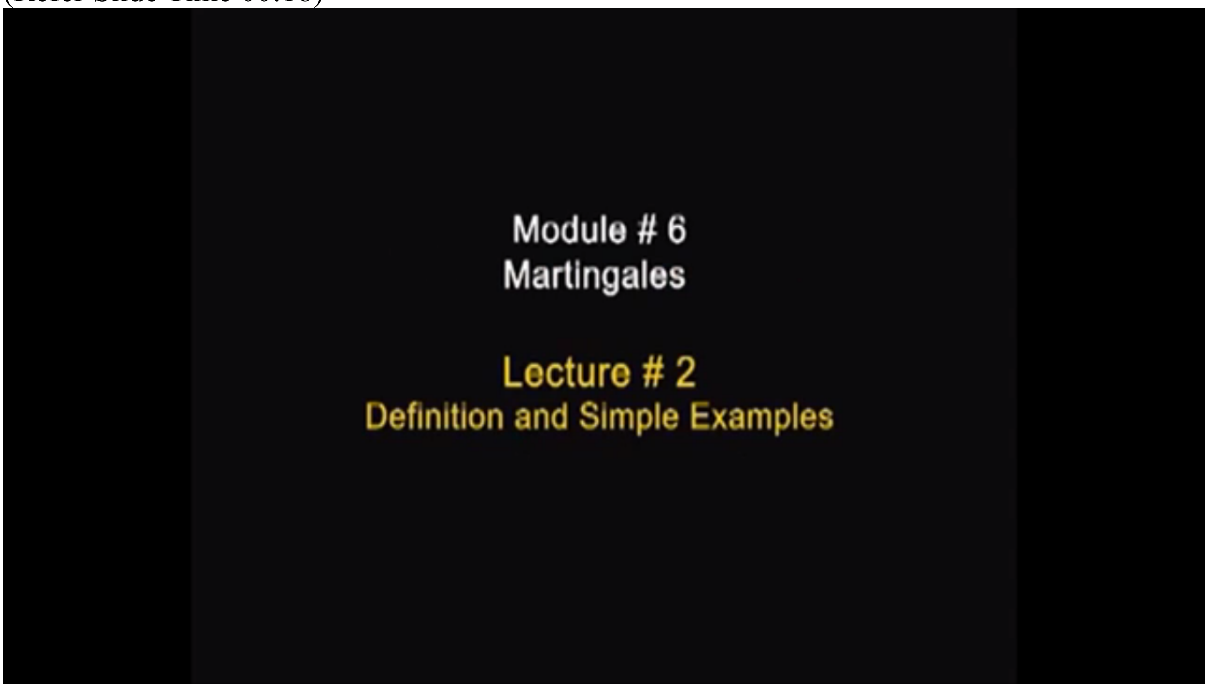

Module # 6
Martingales

Lecture # 2
Definition and Simple Examples

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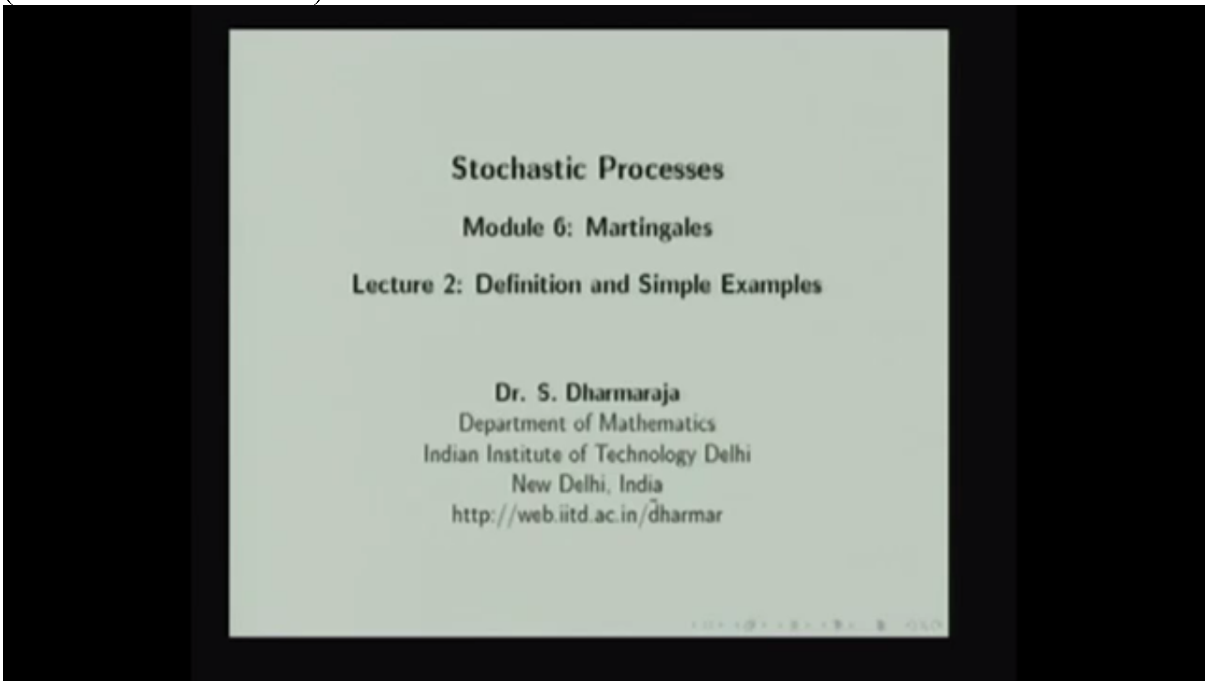

This is a Stochastic Process, Module 6: Martingales, Lecture 2: Definition and Simple Examples.

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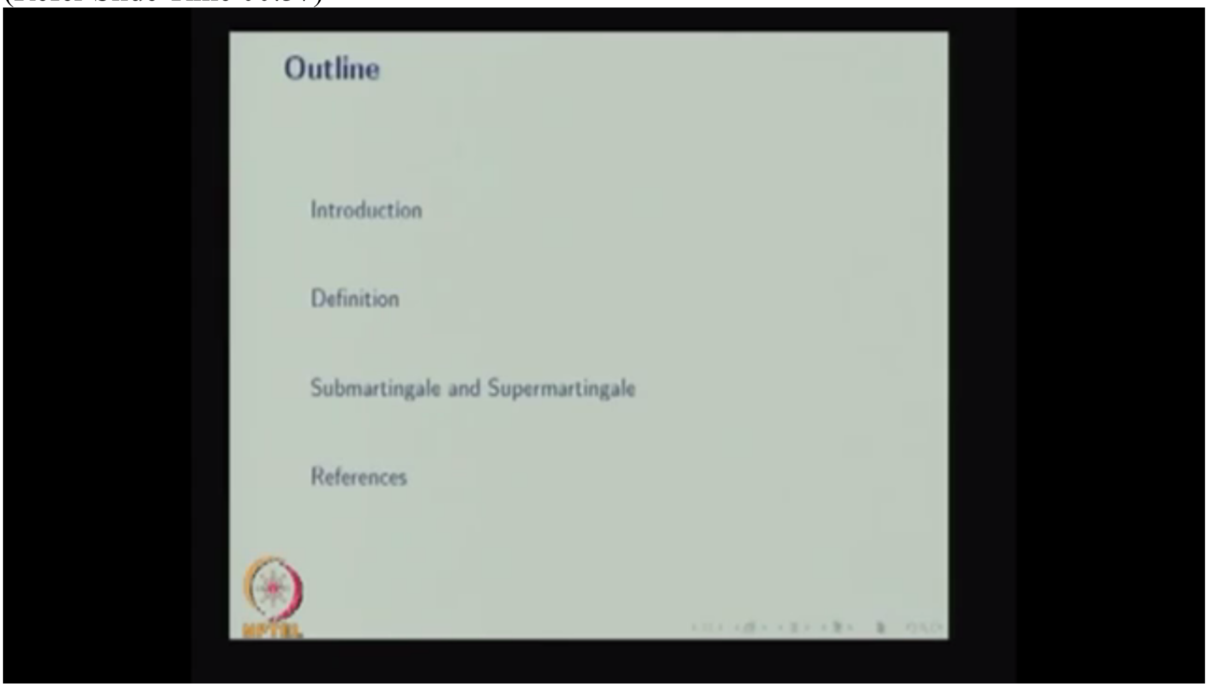

In the last lecture, we have covered conditional expectation, expectation of X given Y and its properties, filtration F of t, t over the time t over the 0 to infinity and its properties. Then conditional expectation of random variable X of t given that the filtration F of this and its properties.

In this -- in this model, we will discuss an important property of stochastic process, Martingale.

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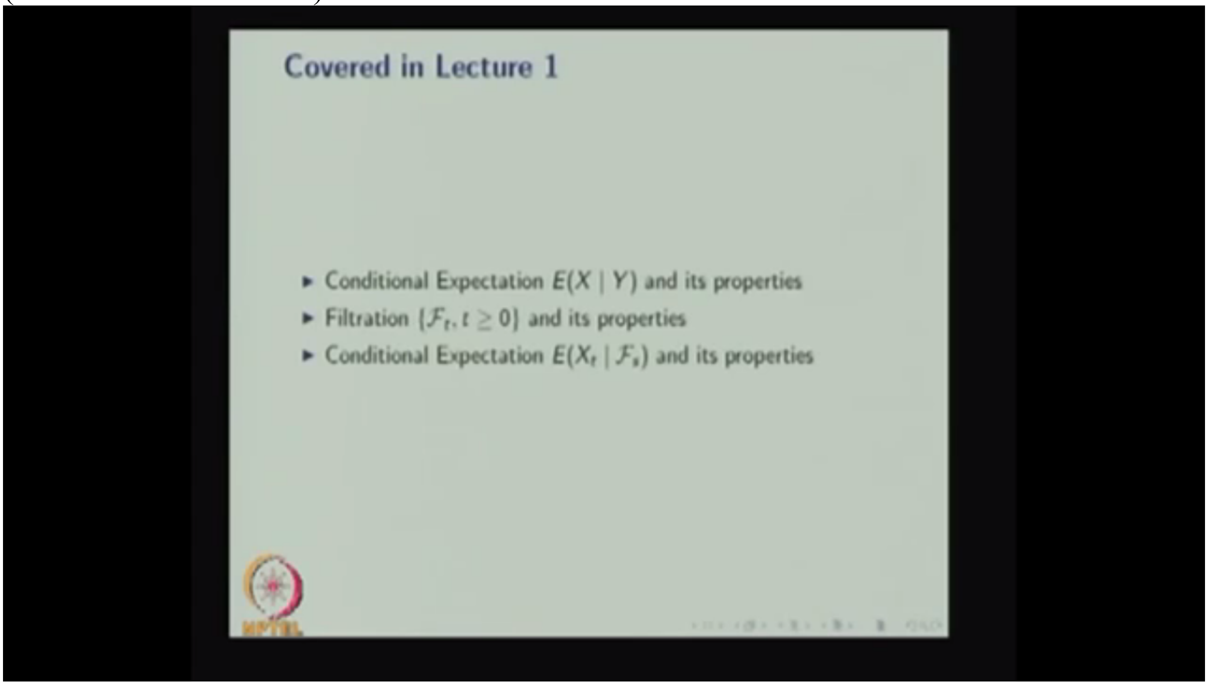

The Martingale referred to a class of betting strategies popular in 18th century. The concept of martingale in probability theory was introduced by Paul Levy and much of the original development of the theory was done by Joseph Doob. Part of the motivation for that work was to show the impossibility of successful betting strategies.

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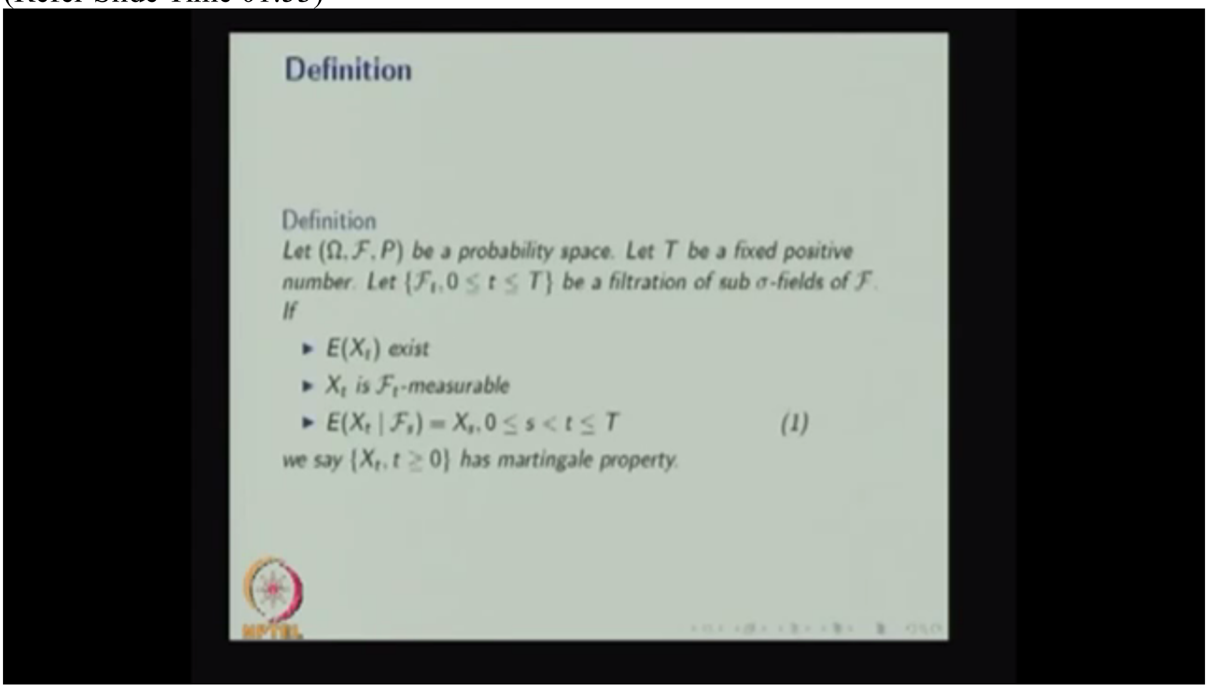

The definition of Martingale is as follows: Let (Ω, F, P) be a probability space. Let T be a fixed positive number. Let the collection of filtration F_t where t over from 0 to capital T be a sub σ-fields of F. So the F is the σ-field and the filtrations are the sub σ-fields of the F, and the probability space is defined in Ω, F and P. And T is a fixed positive number. Using that we got a filtration of sub σ-fields in the range 0 to capital T.

If the expectation of X_t exists for fixed t, X_t is a random variable, so if the expectation of X_t exists or in other words when the random variable is integrable. Also if X_t is F_t measurable, this also we've discussed in the last lecture, whenever we say the random variable is a F_t measurable that means the σ-field generated by the random variable X_t that should be contained in F_t. If this property is satisfied by the random variable for a given filtration F_t, then we say, for a given σ-field, we say X_t is F_t measurable. So the second condition is X_t is F_t measurable.

The third condition, not only for a fixed t the expectation exists and the random variable is F_t measurable, the conditional expectation that is expectation of X_t given the σ-field F_s is same as the random variable X_s. The s can take the value from zero to small t where t can take the value from s to capital T.

If these three properties are satisfied by a collection of random variables, that is stochastic process X_t, then we say the random, the stochastic process has Martingale property. So in this definition, we started with the probability space and we fix some positive integer, positive number. Using that we create a filtration and those filtrations are nothing but sub σ-field of -- sub σ-fields of F. F is a σ-field.

If you have a collection of random variables, that's a stochastic process, for fixed t, it is a random variable. So that random variable satisfies the integrable property, and F_t measurable property and the conditional expectation over the sub-σ-field F_s that is nothing but the filtration, that is same as the random variable X_s. In that property satisfied for all s and t lies between the interval 0 less than or equal to s less than t less than or equal to T, then we say the collection of random variable X_t are the stochastic process X_t has the Martingale property.

So here this stochastic process satisfies the Martingale property in the interval zero to capital T because we are checking the conditional expectation in the interval 0 to capital T. Therefore, this stochastic process has a martingale property in the interval 0 to capital T, not zero to infinity. If that is satisfied by for all t, then we can say that random variable or that stochastic process Xt, this stochastic process Xt has a Martingale property in the range 0 to infinity.

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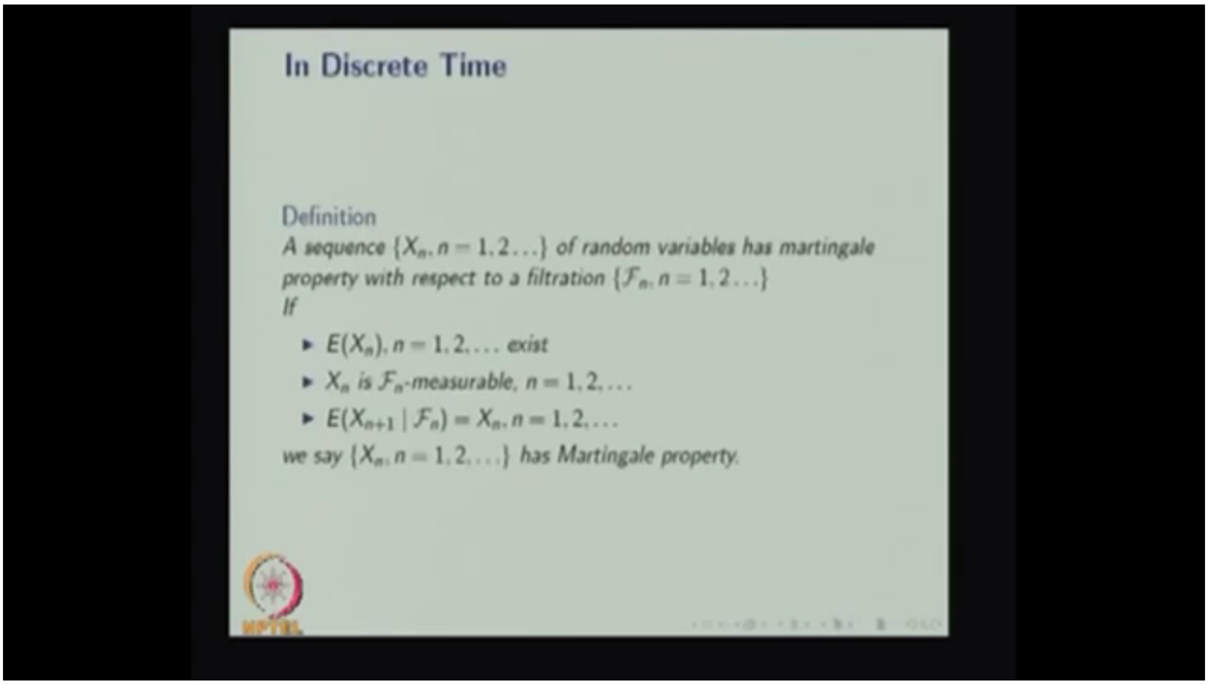

Now we present the definition of Martingale in discrete time. A sequence of random variables that is X_n, n varies from 1, 2 and so on of random variables has a martingale property with respect to the filtration F_n where n is also running from -- where n also takes the value from 1, 2 and so on if for fixed n, the random variable is integrable, there is expectation exists and also each random variable X_n is F_n measurable; the third condition, the conditional expectation of X_{n+1} given the filtration F_n is same as X_n for every n, then we say the random variable has the, sorry, then we say the stochastic process has the martingale property or the collection of random variable or the stochastic process has the martingale property.

So this is the definition corresponding to the discrete time. The previous definition is for continuous time.

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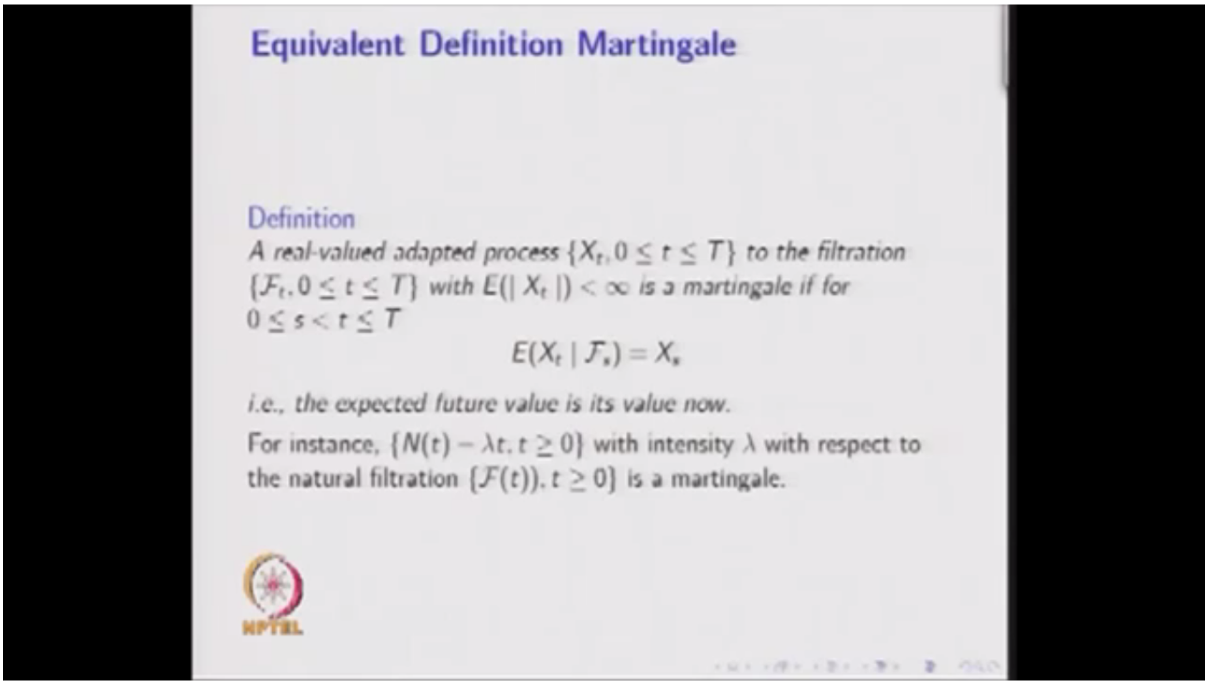

Equivalent definition of Martingale is as follows. A real-valued adapted process X_t where t is lies between 0 to capital T to the filtration F_t where t is lies between 0 to capital T with expectation is finite, expectation exists is a Martingale. For instance, N(t) minus λt for t greater than or equal to 0 with the intensity λ with respect to the natural filtration F(t), t greater than or equal to 0 is a Martingale. Here N(t) is a Poisson process.

Some remarks are as follows.

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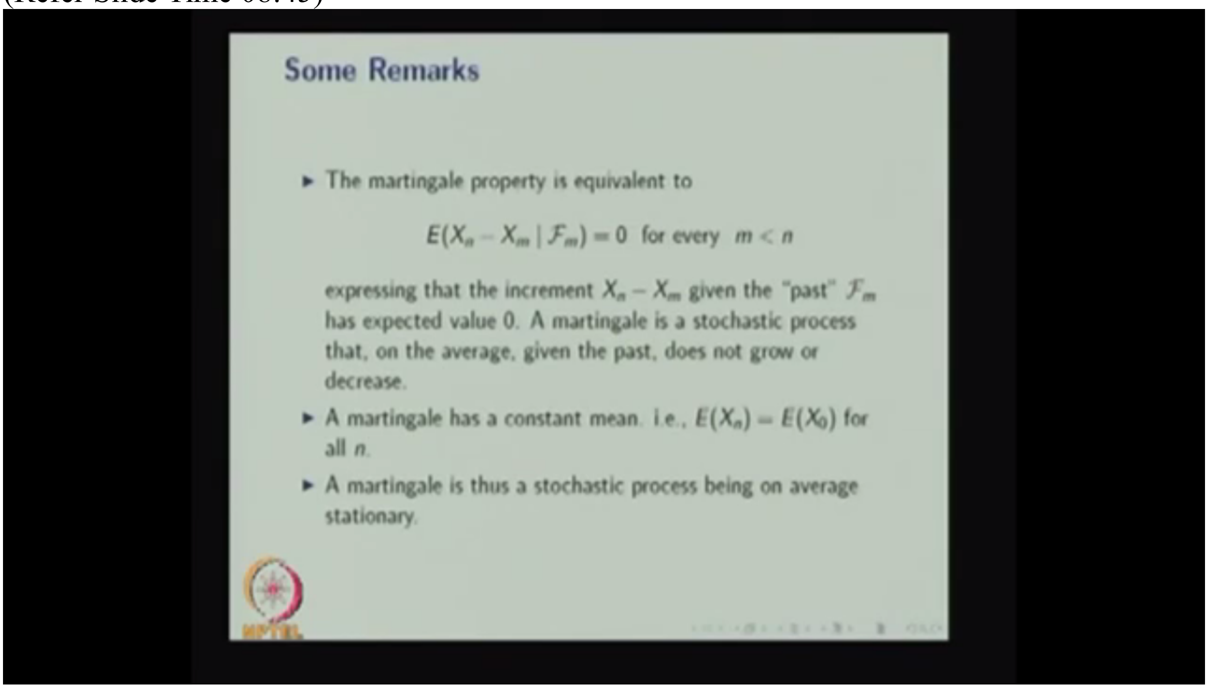

The martingale property is equivalent to the conditional expectation of X_n minus X_m given the filtration F_m is equal to zero for every m less than n. So the Martingale property is same as the conditional expectation with the difference of the random variable expressing that the increment X_n minus X_m given the past F_m has expected value 0. The martingale is a stochastic process that on the average, given the past, does not grow or decrease. That is the meaning of the above expression.

A martingale is a stochastic process that on the average, given the past, does not grow or decrease. If it increase or if it decrease, we have a different name for that particular property. Now we are discussing it does not grow or decrease. So that is called, that stochastic process is called Martingale.

A martingale has a constant mean. That means if you have a stochastic process satisfying the three conditions, hence the stochastic process has a martingale property or the stochastic process is a martingale, then the expectation is going to be constant. That is expectation of X_n is same as expectation of X_0 for all n.

Note that Markov property can also be given in terms of expectations. In other words, the expectation of X_n is same as expectation of X_0 for all n. That is same as a martingale is thus a stochastic process being on average stationary. Average stationary means it has the time invariant property in average or in mean. Usually the stationarity property or the time-invariant property is discussed in the distribution, but here in the Martingale, it has the -- it has an expectation of X_n is equal to expectation of X_0 for all n. Therefore, a Martingale is thus a stochastic process being on average stationary. These above remarks is also valid for continuous time.

While Martingale concepts involves expectation, the Markov process concepts involves distribution. Whenever you discuss a stochastic process with the martingale property, it involves the conditional expectation whereas whenever you discuss the stochastic process with the Markov property, it involves the distribution, conditional distribution.

A Markov process need not necessarily be a Martingale because the stochastic process having a Markov property, therefore, it is going to be a Markov process. A stochastic process having a martingale property, martingale concepts involves the conditional expectation whereas the Markov property concept involves the distribution. Hence, a Markov process need not necessarily be a Martingale.

The martingale has a lot of applications in branching processes and finance.

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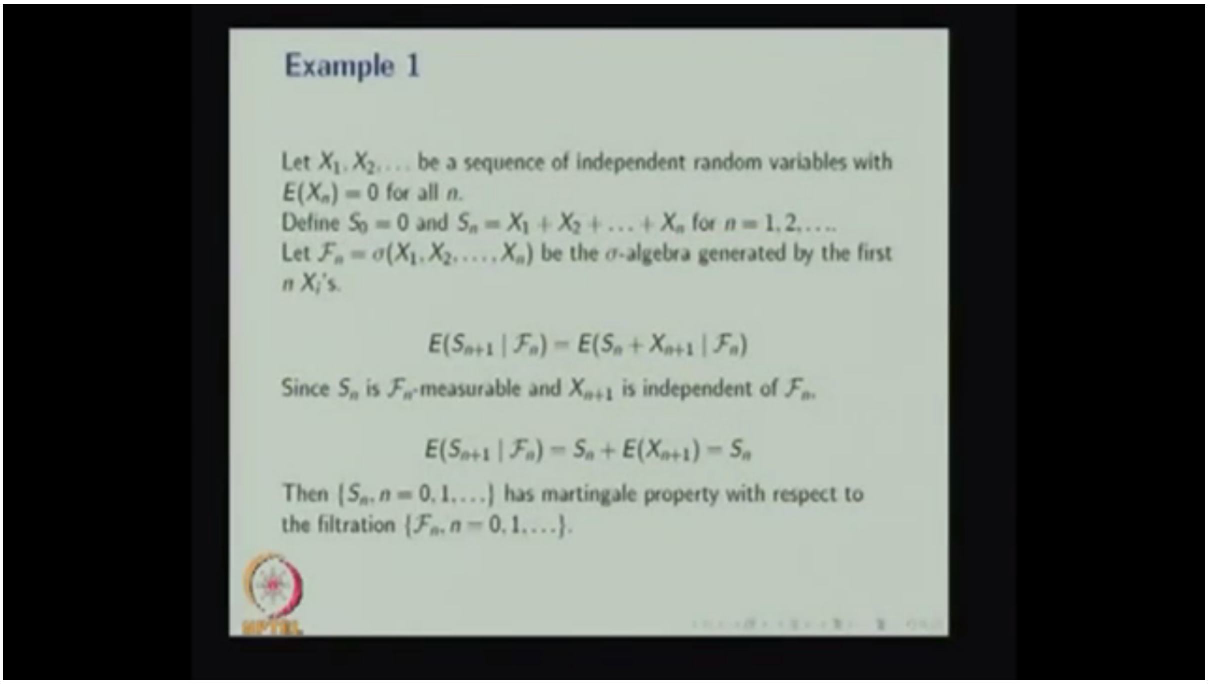

Now we are going to consider the few examples. Example 1. Let X_1, X_2 be a sequence of independent random variables with the expectation of X_n is equal to 0 for all n. We are not giving the distribution of the random variables. We have said it is a sequence of random variable and all the random variables are mutually independent and you know the expectation of the random variable is equal to 0 for all n.

Now we are defining a new random variable S_0 is equal to 0 and the S_n is equal to sum of first n random variables X_i's for all n is equal to 1 to n. The filtration F_n is defined on the σ-field generated by the n random variables X_1 to X_n be the Sigma algebra generated by the first n X_i's.

Now we are trying to compute what is the conditional expectation of S_{n+1} given the filtration F_n. That is same as conditional expectation of you can replace S_{n+1} by S_n+X_{n+1} because that is the way we defined S_n. S_{n+1} is going to be first n plus 1 random variables. The first n random variables will be S_n. The last term will be X_{n+1}.

The way we created the σ-field, σ, sorry, the way we created the filtration F_n is nothing but the σ-field is generated by the first n random variables. Therefore, the S_n random variable is the F_n measurable because each X_i's are F_n measurable, each Xn is F_n measurable. The S_n is nothing but the first n random variable -- X_i's random variable summation. Therefore, S_n is also F_n measurable.

X_{n+1} is independent of F_n because F_n is the information till first n X_i random variables. Therefore, X_{n+1} is independent of F_n. Therefore, a conditional expectation of S_{n+1} given the filtration F_n, the information up to n, that is nothing but since S_n is F_n measurable, therefore, S_n is known. Because we know the information till n, that means S_n is also known. Therefore, S_n has to be treated as a constant. So the conditional expectation of S_n given F_n is going to be S_n.

The second term, the conditional expectation of X_{n+1} given F_n, since F_n is independent of X_{n+1}, it is not -- the information up to n is not going to affect the value of X_{n+1} for the random variable X_{n+1}. Therefore, it is just instead of conditional expectation, it is an expectation of X_{n+1}, but already we have made expectation of n is equal to 0.

Therefore, the expectation of n+1, it is all -- for all n expectation of n is equal to 0. Therefore, expectation of n plus -- X_{n+1}, this is also 0. Therefore, the conditional expectation is going to be Sn. Since you know the information up to n, the S_n is a value. So the conditional expectation of S_{n+1} given F_n is equal to S_n. This is nothing but the martingale property, the last condition of martingale property in a discrete time.

Whenever you want to conclude the given stochastic process is a martingale, has a martingale property, three conditions has to be checked. The first one is expectation exists. You can find out the expectation of S_n. Since expectation of X_n is 0, expectation of S_n is also 0. The second condition, the S_n has to be a F_n measurable. That is also verified. The third condition, the conditional expectation has to be S_n. This is also verified. So since three conditions of the definition which we discussed for the discrete-time satisfied, we conclude the given stochastic process S_n, the collection of or the sequence of random variables S_n has a martingale property.

This martingale property is with respect to the filtration $F_{n.}$ Because this martingale property is with respect to this filtration, there is a -- there may be a possibility this stochastic process may not be a -- may not have the Martingale property with respect to some other filtration. So the given stochastic process S_n has a martingale property with respect to this filtration F_n.