Remarks of Conditional Expectation and Adaptability

by

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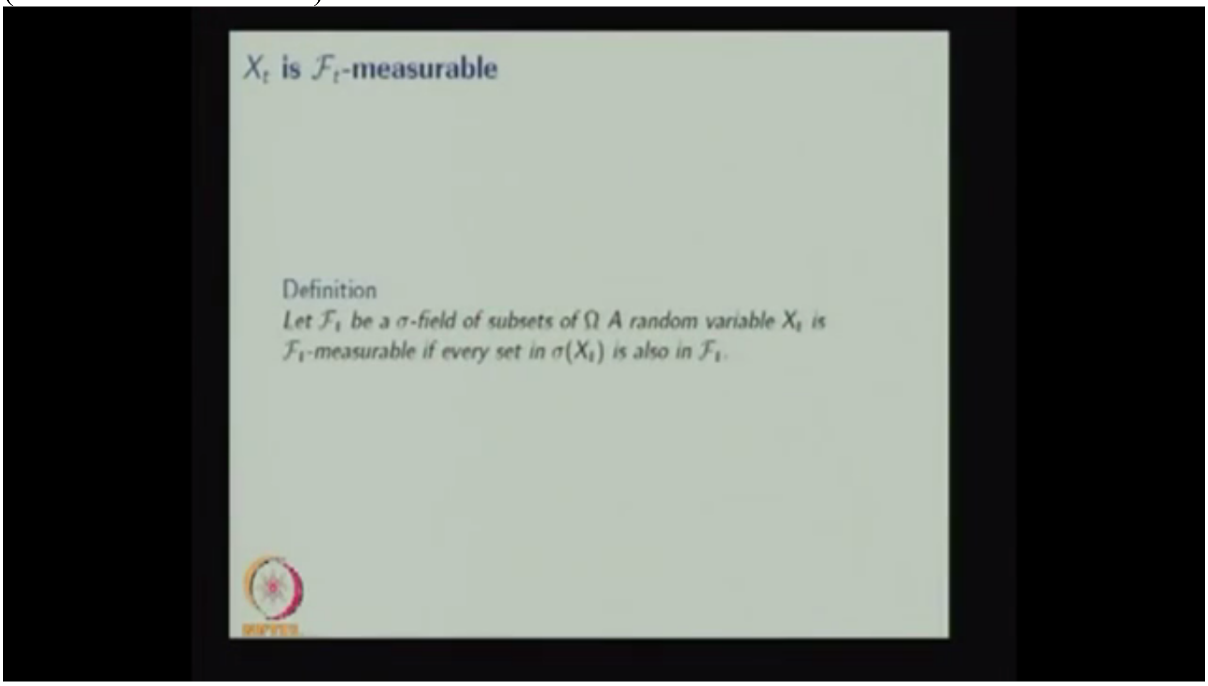

To verify the given stochastic process as the Markov, sorry, as the Martingale property, each random variable X_t must be a F_t measurable. So we will see the definition of when the random variable X_t is going to be the F_t measurable. For that the definition is as follows.

Let F_t be a σ-field of subsets of Ω. A random variable X_t is F_t measurable if every set in σ-field generated by the random variable X of t is also in F_t. We have a non-empty set, that is a collection of possible outcomes and we have created the σ-field on Ω and the random variable is said to be a F_t measurable, the random variable X_t is said to be a F_t measurable if the σ-field, if every set in σ-field generated by the random variable X_t is also in F of t. If this condition is satisfied, then this random variable is going to be call it as a F_t measurable.

Obviously, F of t is contained in $\sigma(X_t)$. So here every element, every set in σ-field generated by the random variable X_t is also in F_t. That means it's other way around. So if that property is also satisfied, then this random variable is a F_t measurable.

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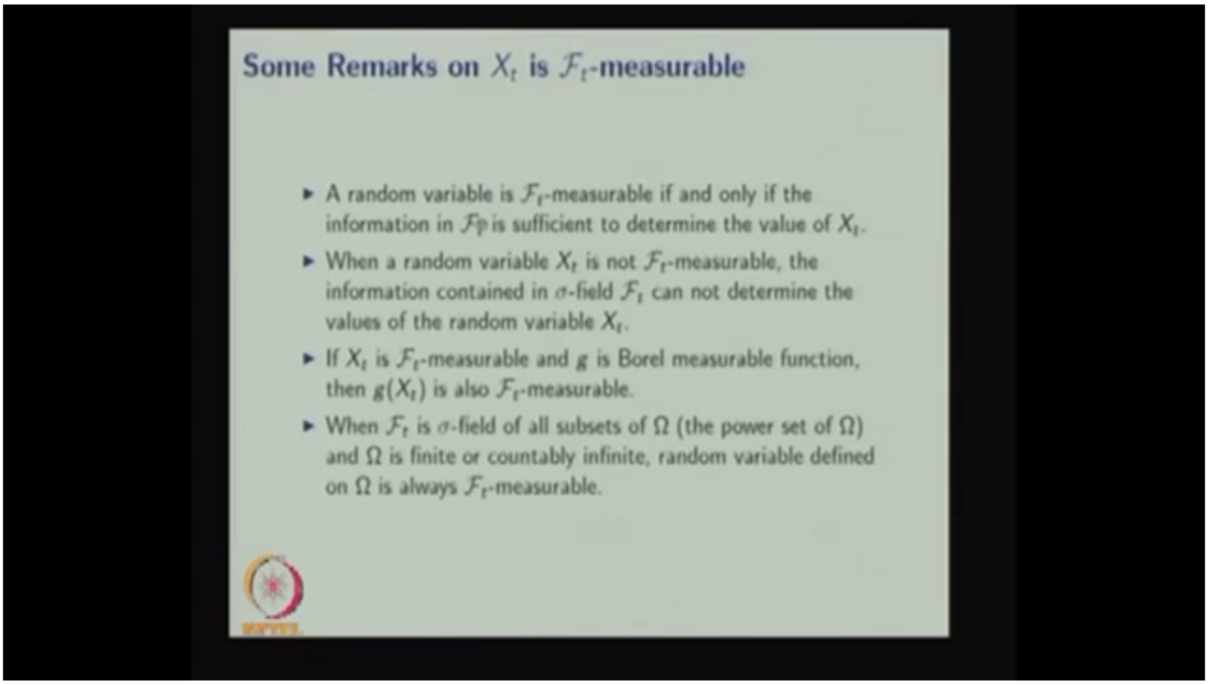

Some remarks on X_t is F_t measurable. The first remark, a random variable is -- a random variable X_t is F_t measurable if and only if the information in F of t is sufficiently to determine the value of X of t. The F of t is nothing but the collection of the information up to the time. So whenever we say the random variable is F_t measurable if and only if the information in F_T is sufficient to determine the value of X of t.

The second remark, when a random variable X_t is not a F_t measurable, then the information contained in σ-field F_t cannot determine the values of the random variable X_t. Whenever the random variable is not a F_t measurable, the conclusion is the information contained in the σ-field F of t cannot determine the values of the random variable whereas X_t is a random -- is a F_t measurable if and only if it has the sufficient information to determine the value of X of t.

Third remark, if X_t is F_t measurable and g is a Borel measurable function, then $g(X_t)$ is also F_t measurable. We know that X is a random variable. Then, and g is a Borel measurable function. Then g(X) is a random variable. So here we are saying if X_t is F_t measurable and g is a Borel measurable function, then $g(X_t)$ is also a F_t measurable. Obviously, $g(X_t)$ is also a random variable. Since XT is X_t is F_t measurable, then $g(X_t)$ is also F_t measurable.

The fourth remark, we are not giving the proof for this remark. The fourth one, when F_t is a σ-field of all subsets of Ω, that is the power set of Ω and the Ω is finite or countably infinite, then the random variable defined on Ω is always F_t measurable. Whenever the σ-field F_t is the power set or the largest σ-field, in that case the random variable defined on Ω is always a F_t measurable. We know that whenever F_t is the largest σ-field, then any real valued function is going to be a random variable. Here, whenever F_t is a σ-field, which is a largest σ-field or the power set of Ω and additional condition, and Ω is a finite or countably infinite, then the random variable is always F_t measurable.

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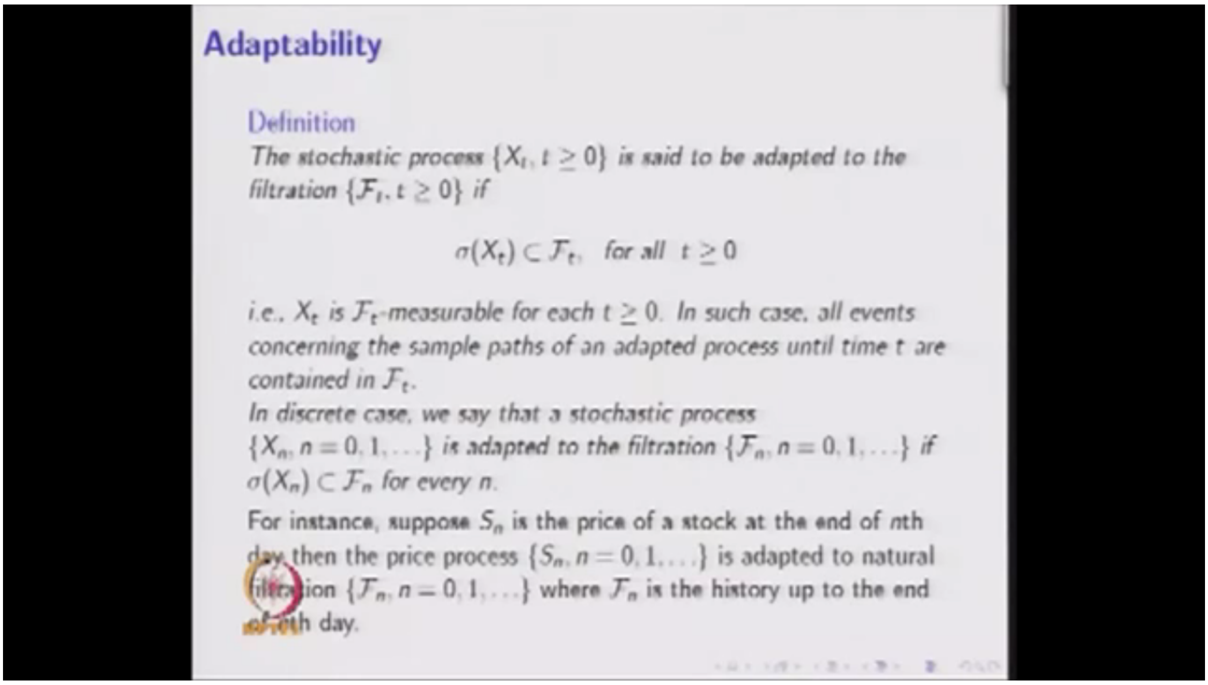

Based on X_t is F_t measurable, we discuss adaptability of the stochastic process. The stochastic process X_t over the t greater than or equal to zero is said to be adapted to the filtration F of t if the σ-field generated by the random variable X_t, which is contained in F of t for all t greater than or equal to zero. So this condition is nothing but X_t is F_t measurable. Whenever X_t is F_t measurable, then the collection of random variable X of t is adapted to the filtration F of t. If this condition is not satisfied, in that case X_t is not a F_t measurable. If X_t is not a F_t measurable, then the collection of random variable, the collection of random variables X of t is not going to be adapted to the filtration.

Suppose it is X_t is F_t measurable, in such cases, all events concerning the sample paths of adapted process until time t are contained in F of t. It says F_t has the collection -- has the information up to the time t. Whenever X_t is F_t measurable, then all the events concerning the sample paths of the adapted process until time t are contained in F_t. It's the same meaning here. The σ-field generated by the X of t which is contained in F of t whenever the stochastic process is adapted.

In a discrete case, we say that stochastic process X suffix n, n can takes a value 0, 1, 2 and so on is adapted to the filtration F suffix n. So instead of F of t, I am using F suffix n for discrete type and similarly the random variable is also discrete type. Instead of t I'm using small n. If the σ-field generated by the random variable X_n which is contained in the σ-field F_n for every n, it has to satisfy for every n, then only this collection of a stochastic, sorry, this collection of random variable or this stochastic process is adapted to this filtration F_n.

For instance, suppose S_n is the price of a stock at the n^{th} day, then the price process S_n, n is equal to 0, 1 and so on is adapted to the natural filtration F_n, n 0, 1 and so on where F_n is the history up to the end of n^{th} day.

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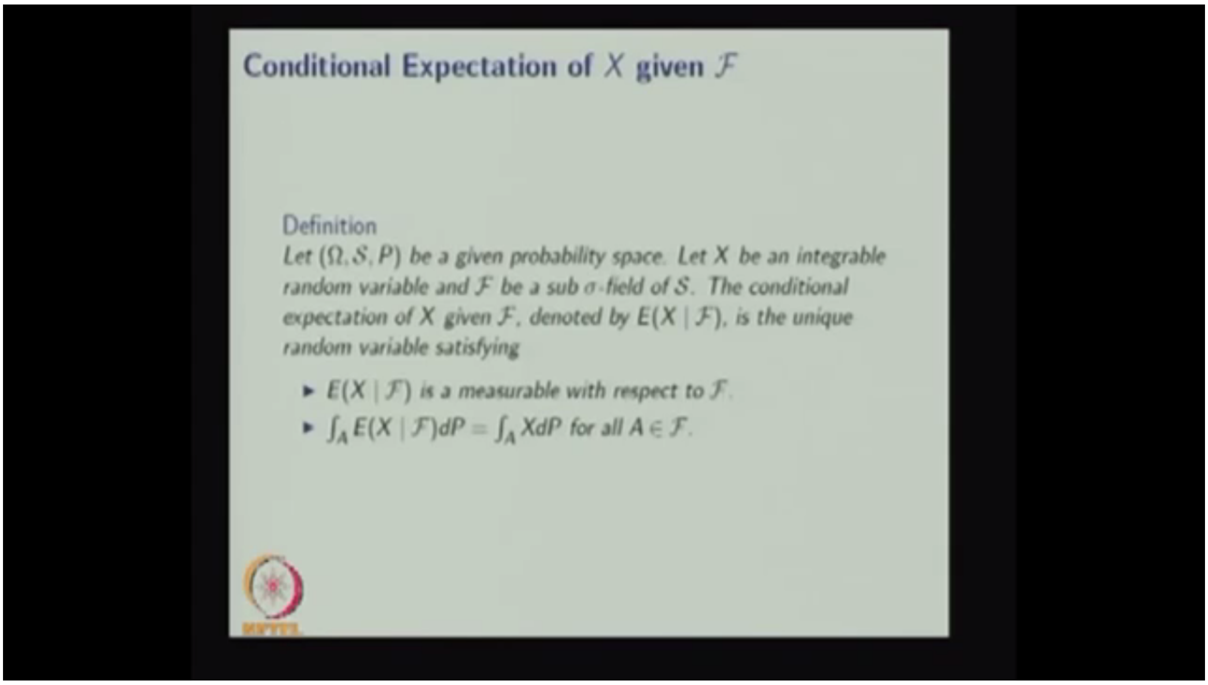

Now we see the definition of a Conditional Expectation of random variable X given σ-field F. Earlier we have defined the conditional expectation of the random variable X given other random variable Y takes the value small y. The definition as follows.

Let (Ω, S, P) be the given probability space. Let X be an integrable random variable and F be a sub σ-field of S. So here I am defining a probability space with the Ω and this is the σ-field S and P is the probability measure and the random variable which is integrable and F be the sub σ-field of S.

The conditional expectation of X given F, so to distinguish the σ-fields F and S, I am giving the S is a σ-field and F is the sub σ-field, and we are defining the conditional expectation for the random variable X given the sub σ-field of S, that is F, that is denoted by X, expectation of X given F. It's a unique random variable. The way I discuss the conditional expectation is a random variable. So here the conditional expectation of X given the σ-field, that is also a random variable. It's a unique random variable satisfying the conditional expectation of X given σ-field is a measurable with respect to the σ-field F. This is a measurable.

Also the integration over any set A where A is belonging to F, the expectation of X given F integration with respect to the probability measure P, that is same as integration with respect to the probability measure P of the integrand is simply X integration over A. Both are one and the same.

Note that the conditional expectation given the σ-field F is a random variable satisfying these two properties and we are defining the conditional expectation given the sub σ-field of S where this is the probability space.

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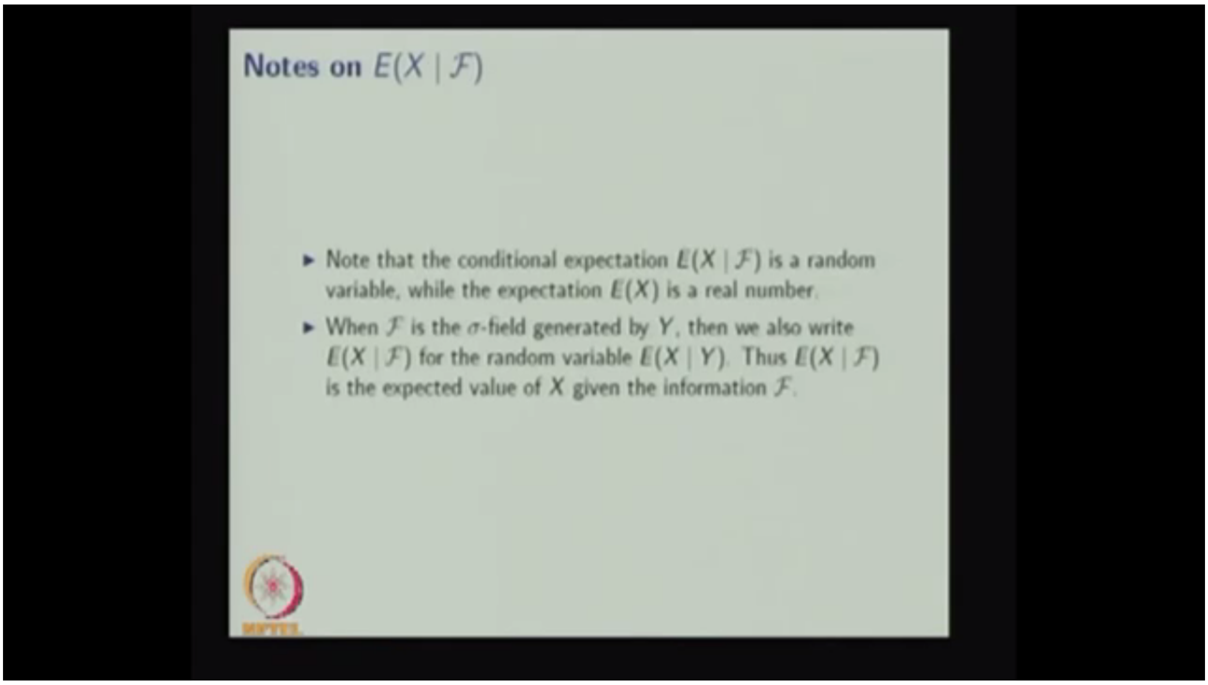

Notes on conditional expectation of X given the σ-field F. Note that the conditional expectation, expectation of X given F is a random variable while expectation X is a real number. Similarly, conditional expectation of X given other random variables also a random variable, not a constant.

When F is the σ-field generated by Y, then we also write conditional expectation of X given F for the random variable X, expectation of X given Y. Whenever F is the σ-field generated by Y, you can replace F by Y. Thus expectation of X given F is the expected value of X given the information F. That means you can replace the random variable Y by F whenever the F is the σ-field generated by the random variable Y. That is same as the expectation -- expected value of X given the information F. Whenever we say the σ-field, that is nothing but the information.

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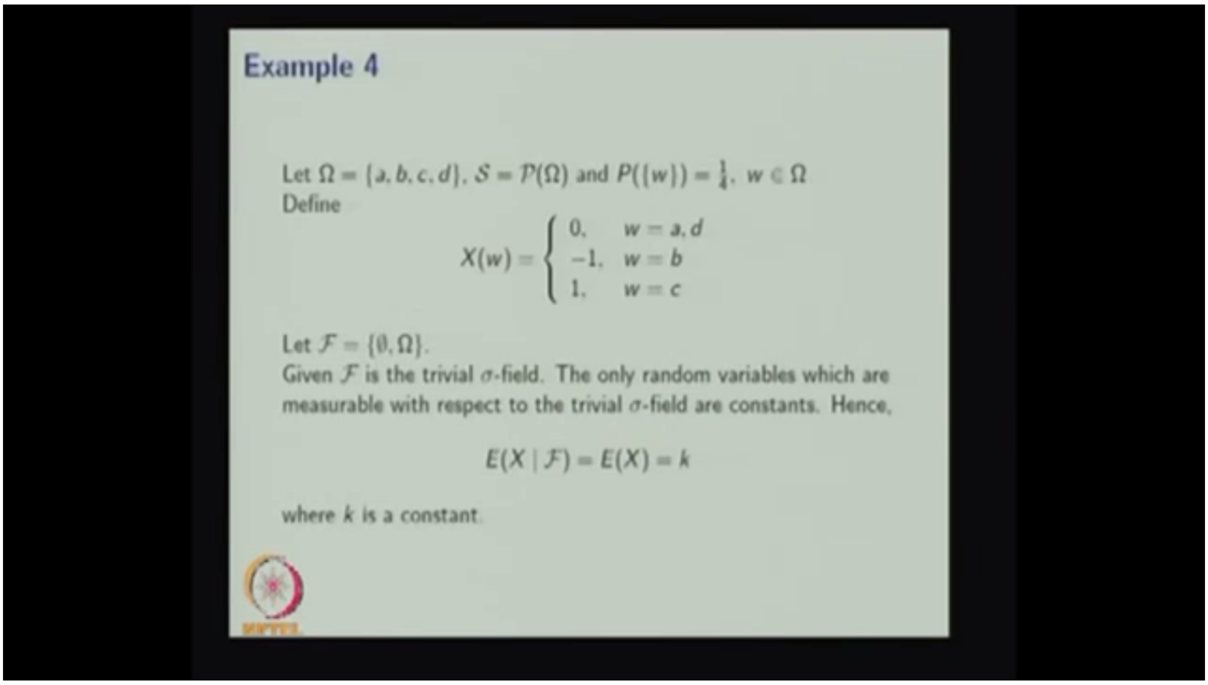

We discussed the conditional expectation of the random variable given σ-field through two examples. The first example is as follows. Omega consists of four elements. S, that is a largest σ-field, power set on Ω and the P of {w} is equal to 1/4 where w belonging to Ω. So, therefore, this is the set function probability measure. We are defining a real valued function. You can cross-check this is a random variable.

Let F that's a σ-field, it's a trivial one. It consists of two elements, empty set and the whole set. Given F is a trivial σ-field, the only random variables which are measurable with respect to the trivial σ-fields or constant. The only random variables which are measurable with respect to the trivial σ-fields are constant. Hence, the expectation of X given the σ-field F where F is a trivial one, that is same as the expectation of X that is equal to constant. That constant you can find out by using the probability and the possible values of 0, -1 and 1, you can find out what is the constant.

So here the conclusion is whenever the σ-field is a trivial one, then the conditional expectation over the trivial σ-field, that is a constant and that constant is same as the expectation of X because it is no more random variable.

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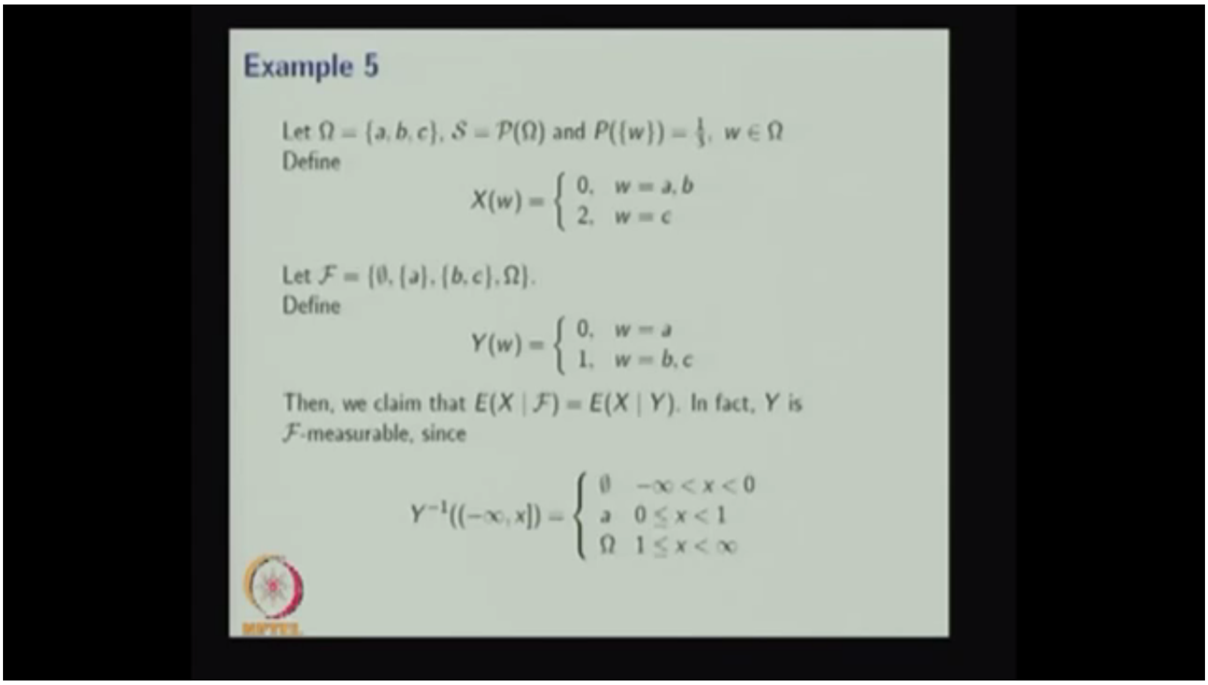

The second example is as follows. Here the Ω consists of three elements. S is the largest σ-field. Probability measure is defined on Ω in each sample itself with the probability 1/3. X is the random variable. F is not the trivial one here. F is the σ-field which is not a trivial one, and I am defining another random variable Y and that takes a value 0 or 1, and here I am claiming that expectation of X given F is same as expectation of X given Y because the Y is the σ-field generated -- because the F is the σ-field generated by the random variable Y.

If you -- if you create the σ-field generated by the Y, you may land up empty set, element a, element b and c and the whole set and that is same as F. Therefore, you can replace expectation of X given F by expectation of X given Y. Here Y is the F measurable. You can check Y is a random variable also by finding the inverse images.

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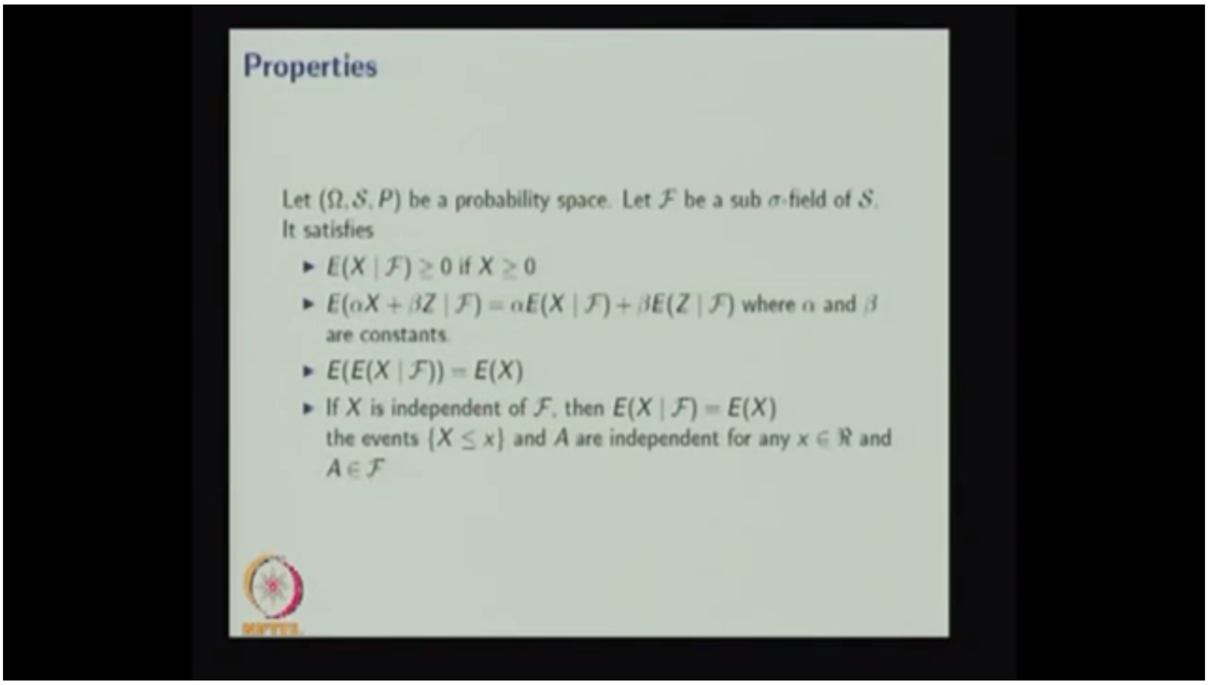

The other properties or the first one, the conditional expectation is always greater than or equal to zero if X is greater than or equal to zero. Then the linear property similar to the conditional expectation as I discussed earlier. Then this also I have discussed. Instead of a σ-field, I have discussed with the random variable. If they are independent, then both are same.

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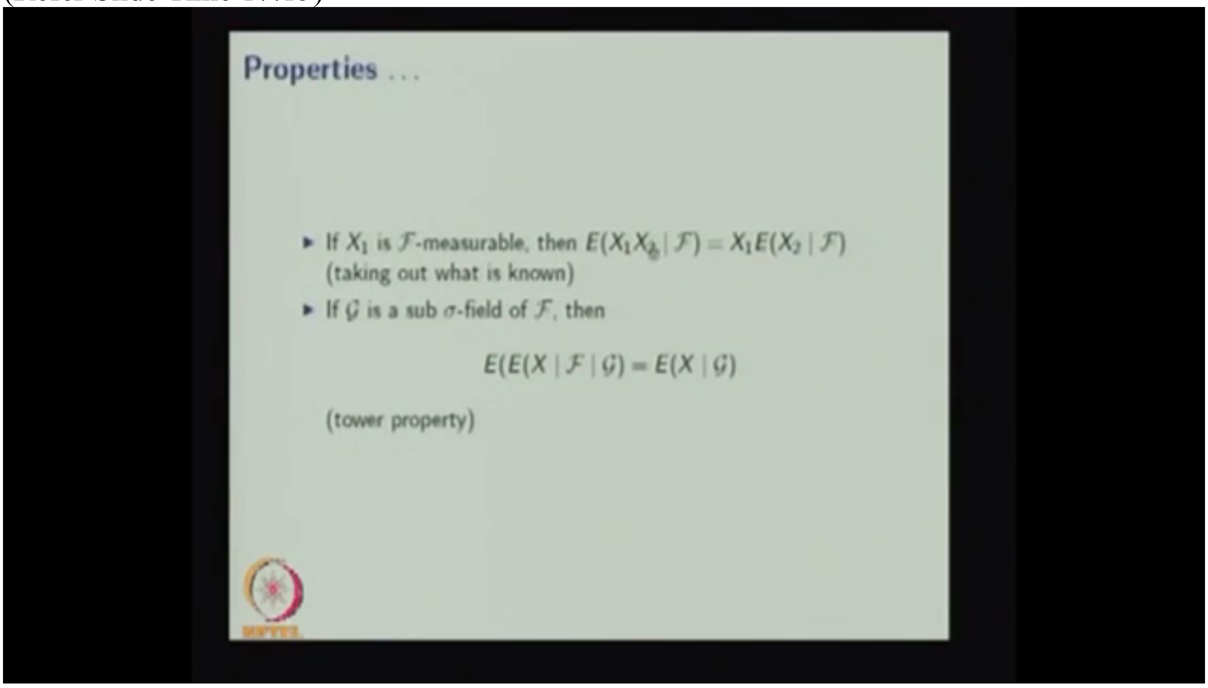

The last two properties, if X_1 is F_t measurable, then the multiplication, the X_1 taken out which is what is known. So since X_1 is known because X_1 is a F measurable, so X_1 will be coming outside, X_1 times the conditional expectation.

If G is a sub σ-field, then this expectation, conditional expectation is same as the conditional expectation and this is called the tower property. I am not going to give the proof of this.

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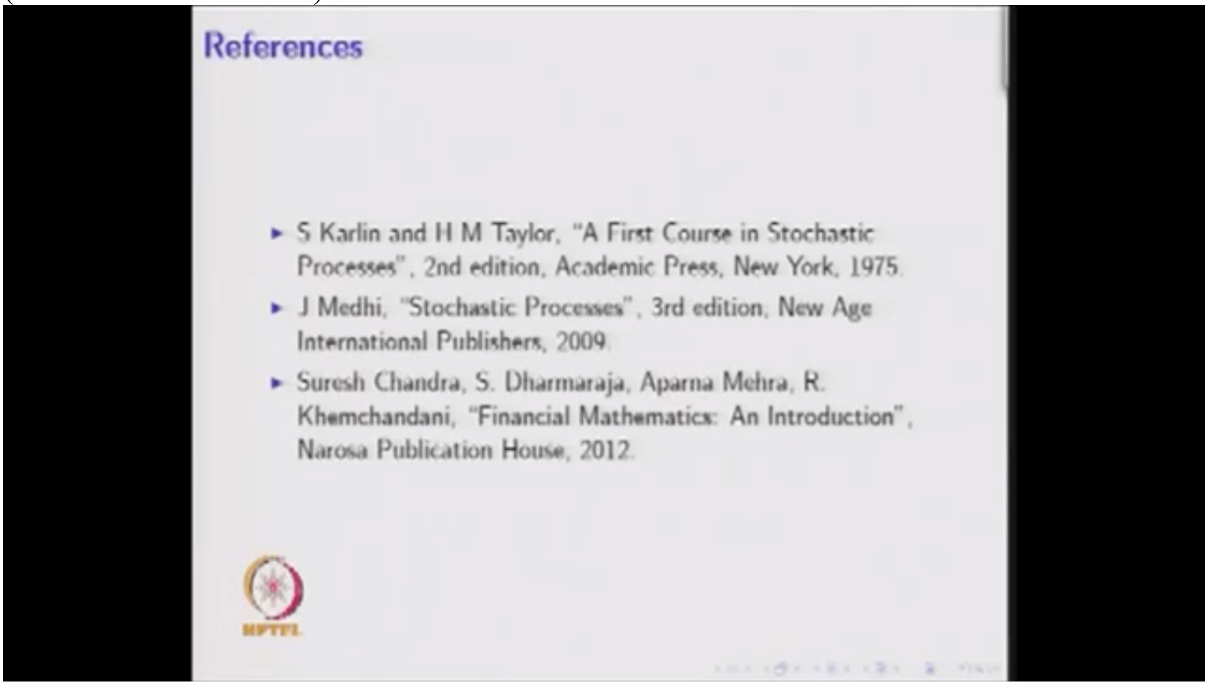

Here is the reference. Thanks.

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