Video Course on
Stochastic Processes

Filtration in Discrete time

by

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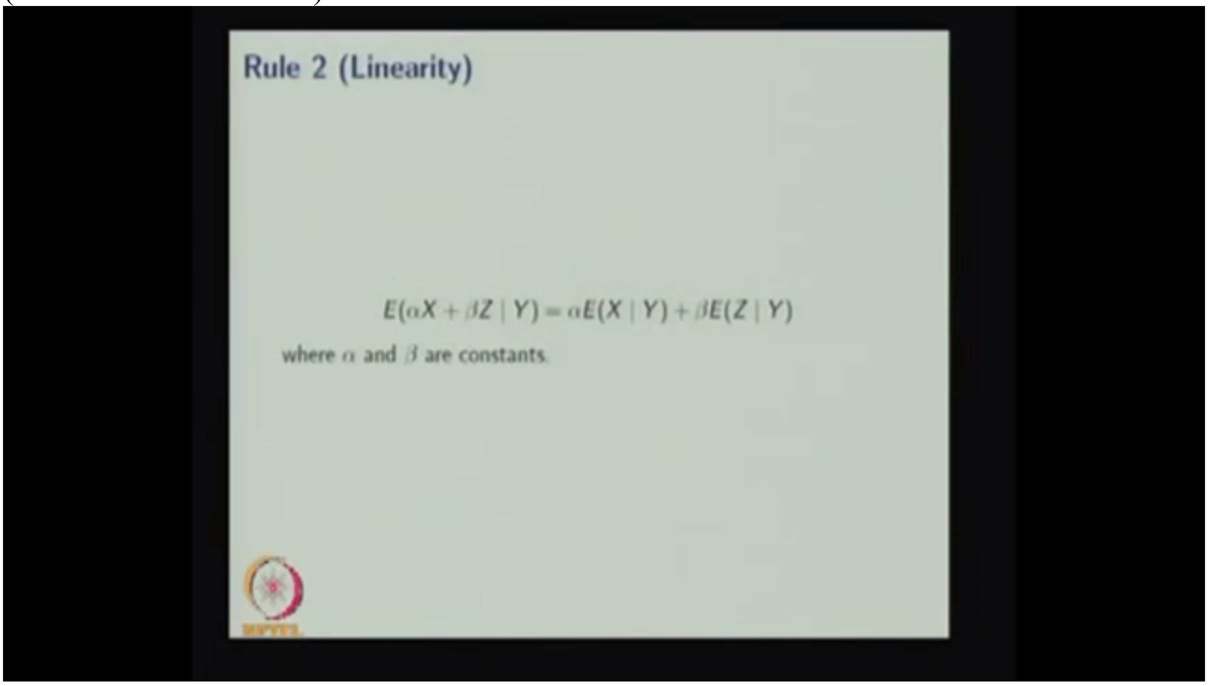

The Rule 2, it is a Linearity property. The linear property says the expectation of alpha times X plus beta times Z given that Y takes a value some small y, that is same as alpha times the conditional expectation of X plus beta times the conditional expectation of Z where alpha and beta are constants.

It's similar to the linear property of expectation. The same thing holds good for the conditional expectation also. Therefore, no need to give the proof.

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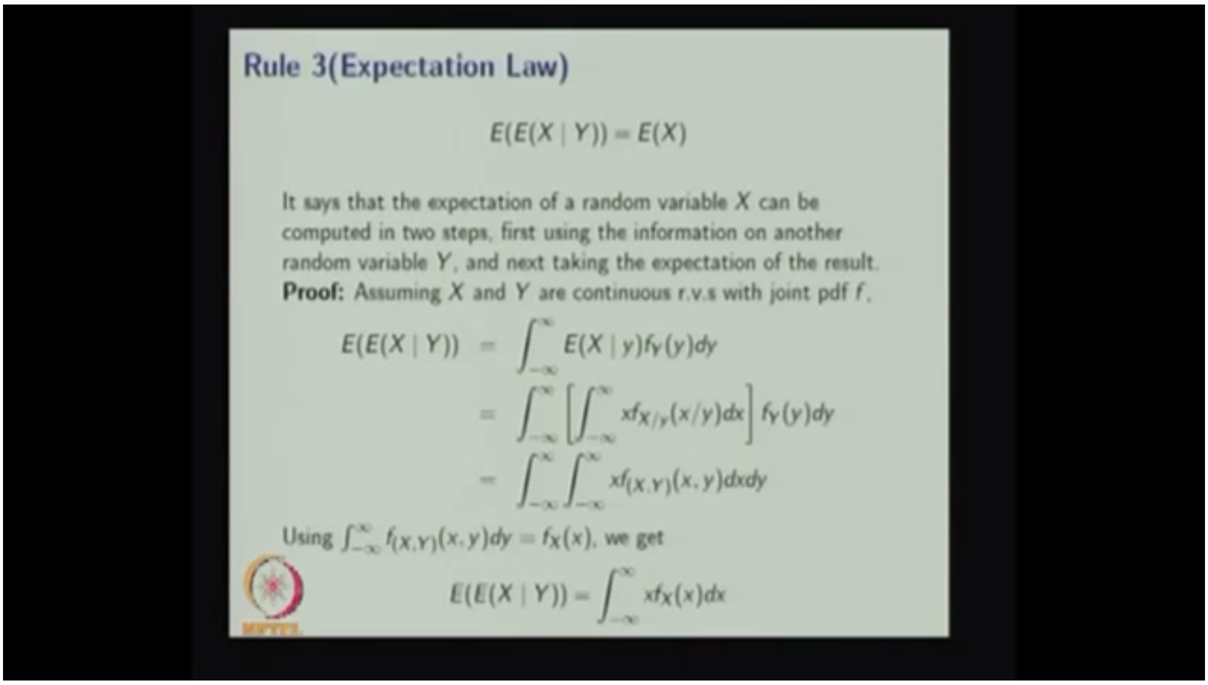

The third one, that is the Expectation Law. Expectation of conditional expectation is same as expectation. It says the expectation of a random variable X can be computed in two steps: first using the information on another random variable Y and next taking the expectation of the result.

You can visualize in the other way around. Expectation of X can be computed as a expectation of conditional expectation of X given Y. That means you can take any random variable Y as long as the conditional expectation possible, find out that conditional expectation, then find -- since the conditional expectation is a random variable, so find out the expectation of that random variable. That is same as the expectation of X.

The proof is given with the assumption both the random variables are continuous with the joint probability density function f. So I'm starting with the left hand side expectation of expectation X given Y that is same as you know how to compute the expectation. Here the provided condition is expectation exists. Similarly, in the definition of conditional expectation also you have to make the assumption the expectation is exist. Then only we are finding the conditional expectation.

So this expectation exists. Therefore, minus infinity to infinity conditional expectation, this is a random variable and you are finding the expectation of that. Therefore, you multiply the probability density function for the random variable Y because this expectation of X given Y is a function of y. Therefore, you should multiply with the probability density function of y, integrate with respect to y between the limits minus infinity to infinity and by definition the conditional expectation is nothing but x times the conditional probability density function integration with respect to x between the limits minus infinity to infinity. So you substitute that.

Now you can come to the conclusion, the integration of minus infinity to infinity, the joint probability density function of x and y is nothing but the marginal distribution. So here the integration is with respect to y. Therefore, you get the marginal distribution or marginal probability density function of x. So here it is a conditional probability density function multiplied by the probability density function.

Therefore, this product will give the joint probability density function of x and y. Any joint -- any two random variables joint probability density function can be written as the product of marginal distribution into the conditional distribution. So using that I am getting the joint probability density function.

Now this integration is f(x). Therefore, the one integration and this much will give a marginal. Therefore, it is a minus infinity to infinity x times that is going to be the marginal probability density function of x. Therefore, you will get expectation of X.

So the right hand side, right hand side is going to be the expectation of X. So you can find out the expectation of X by computing the conditional expectation with some other random variable, then find the expectation. So this rule has a lot of importance.

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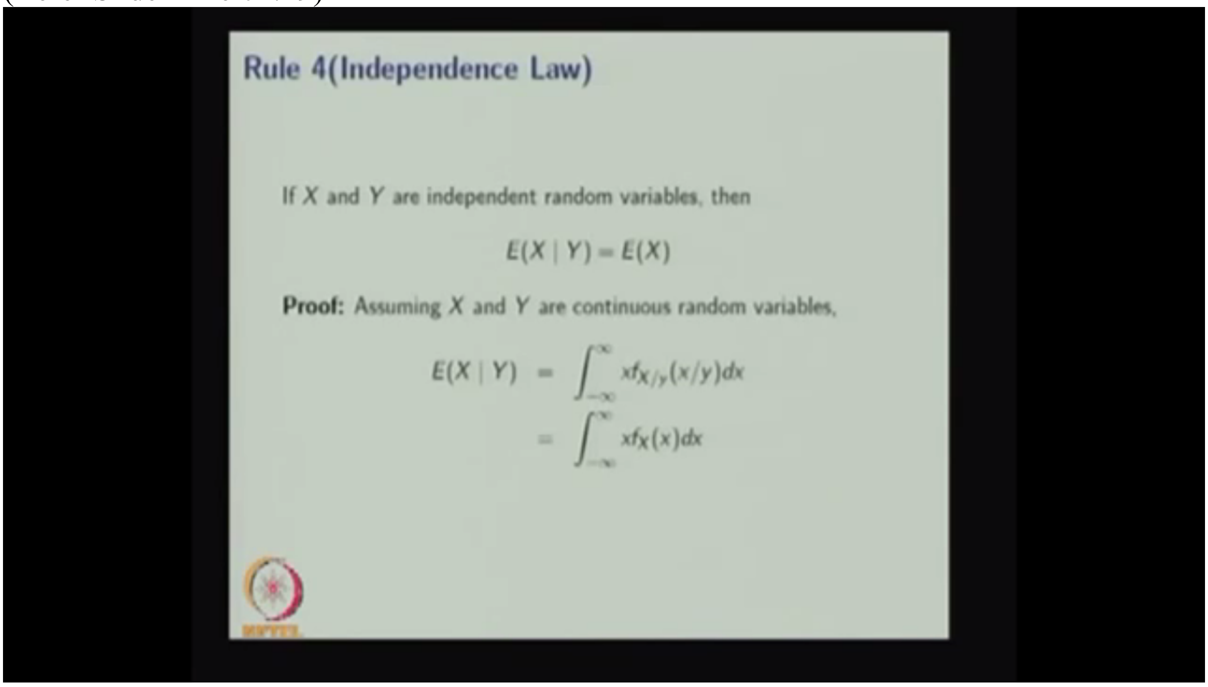

The next one is Independence Law. If two random variables are independent, then the conditional expectation and the original expectation are both are same.

The proof is assuming both the random variables are continuous, so the conditional expectation, this is by definition and you know that both the random variables are independent. Then the conditional distribution is same as the marginal. Therefore, you can replace this way the marginal distribution. So x times the f of x, that is nothing but the expectation of X.

That means if two random variables are independent, then the conditional expectation is not a random variable. It is a constant because the right hand side expectation of X is a constant. Therefore, this is also a constant.

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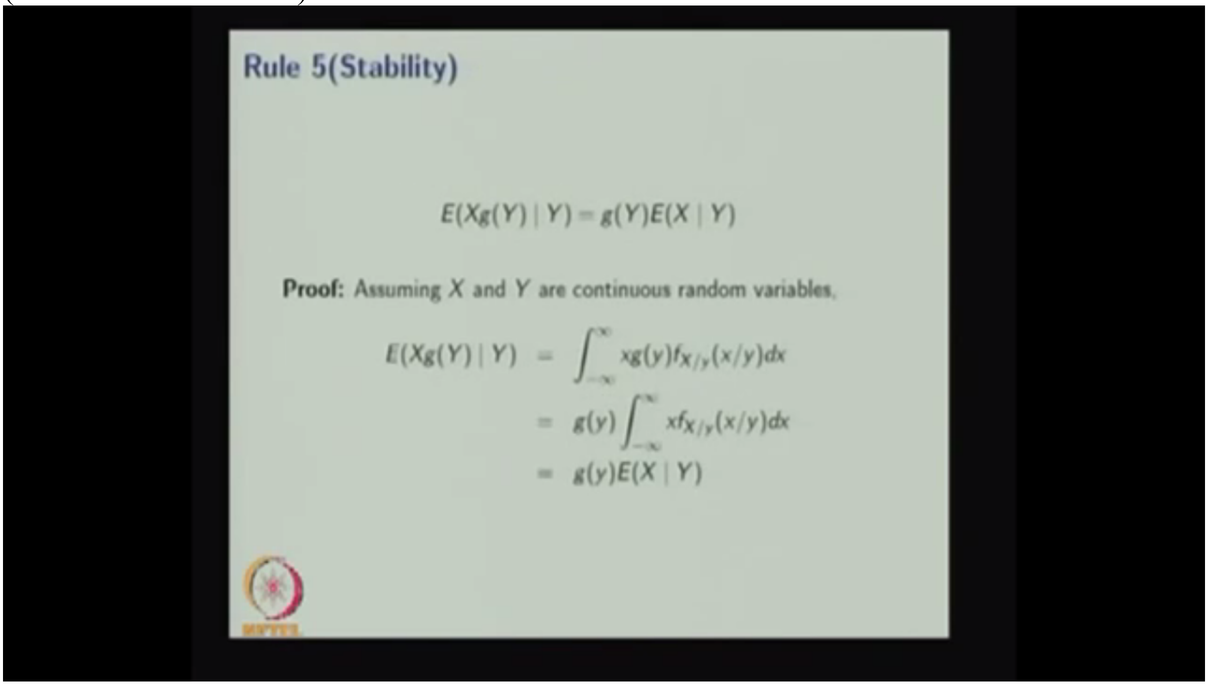

The next rule is Stability. Suppose you have the function g and g(Y) is also going to be a random variable, that means g is a Borel measurable function, so the expectation of X times g(Y) given Y that is same as the g(Y) will be out, g(Y) times the conditional expectation of X given Y.

That means the later we are going to use the property called known is out. That means that the expectation of X times g(Y) given Y takes a value something, some small y, that means this is going to be treated as a constant. So g(Y) has to be treated as a constant. Therefore, the constant will be come out. Therefore, g(Y) times the expectation of X given Y. The same thing I have written in the proof with both the random variables are continuous.

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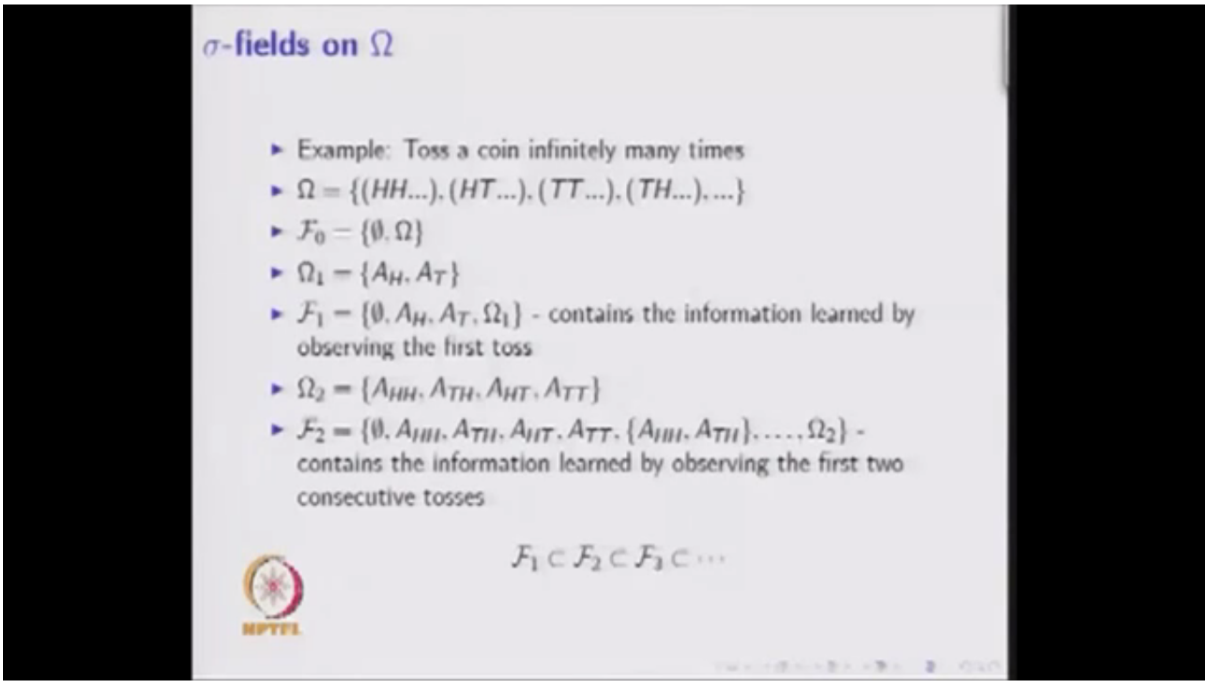

Now we introduce sigma-fields on omega through an example because this is very important concept for the Martingale. Example start with tossing a coin infinitely many times. Tossing a unbiased coin infinitely many times. Let Omega be the collection of possible outcomes HH and so on, HT and so on, TT and so on, TH and so on. Let F_0 be the trivial sigma-field consists of two elements empty set and the whole set. F_1 is the smallest sigma-field containing in Ω_1. F_2 is the smallest sigma-field containing in Ω_2 containing the information learned by observing the first two consecutive tosses.

If you observe, you will find the Ω_1 contained in, sorry, F_1 contained in F_2, F_2 is contained in F_3 and so on. Since we are tossing a unbiased coin infinitely many times, if you find out the limit of F_n, that is going to exist and that is going to be F infinity in notation and that in notation we can make out it's F.

So this consists of the information learned by infinitely many tosses observation. That is the sigma-field F. So this is the way one can create the sigma-fields on Omega. So Omega is consisting of all the possible outcomes in infinitely many tosses observation over the infinitely many tosses whereas the Ω_1 consists of only two elements. Therefore, we are creating a first sigma-field on Ω_1 and after you create four elements, the possible outcomes, after framing the possible outcomes into the four elements, you get the Ω_2. So using Ω_2 we are creating a larger sigma-field F_2.

So like that you can create $\Omega_3 F_3$, $\Omega_4 F_4$ and so on and all those F_i's satisfies this property and the limit exists as n tends to infinity. This sigma-field is going to be denoted by the letter F. So this is the way one can create the sigma-fields on Omega.

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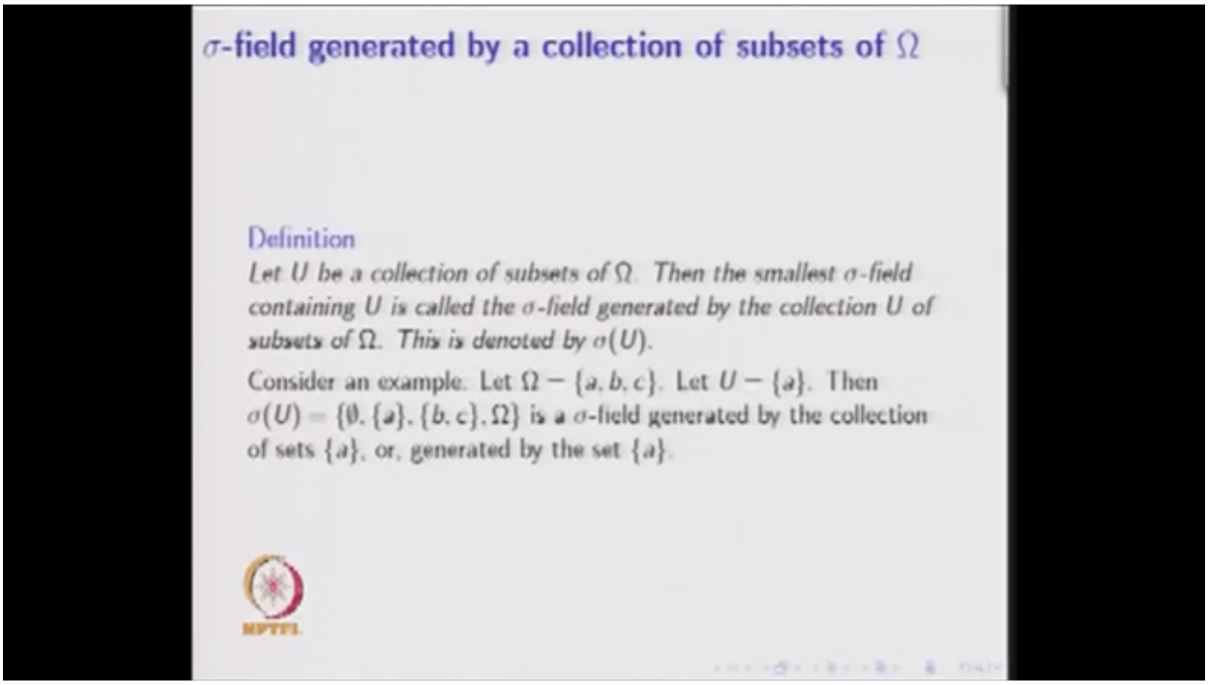

Now we present the sigma-field generated by the -- by a collection of subsets of Omega. Let U be a collection of subsets of Omega. Then the smallest sigma-field containing U is called the sigma-field generated by the collection U of subsets of Omega. This is denoted by σ(U).

Consider an example where Omega is equal to {a, b, c}. Let U is {a}. Then σ(U) is a empty set, element a, element {b, c} and the whole set is the sigma-field generated by the collection of sets {a}.

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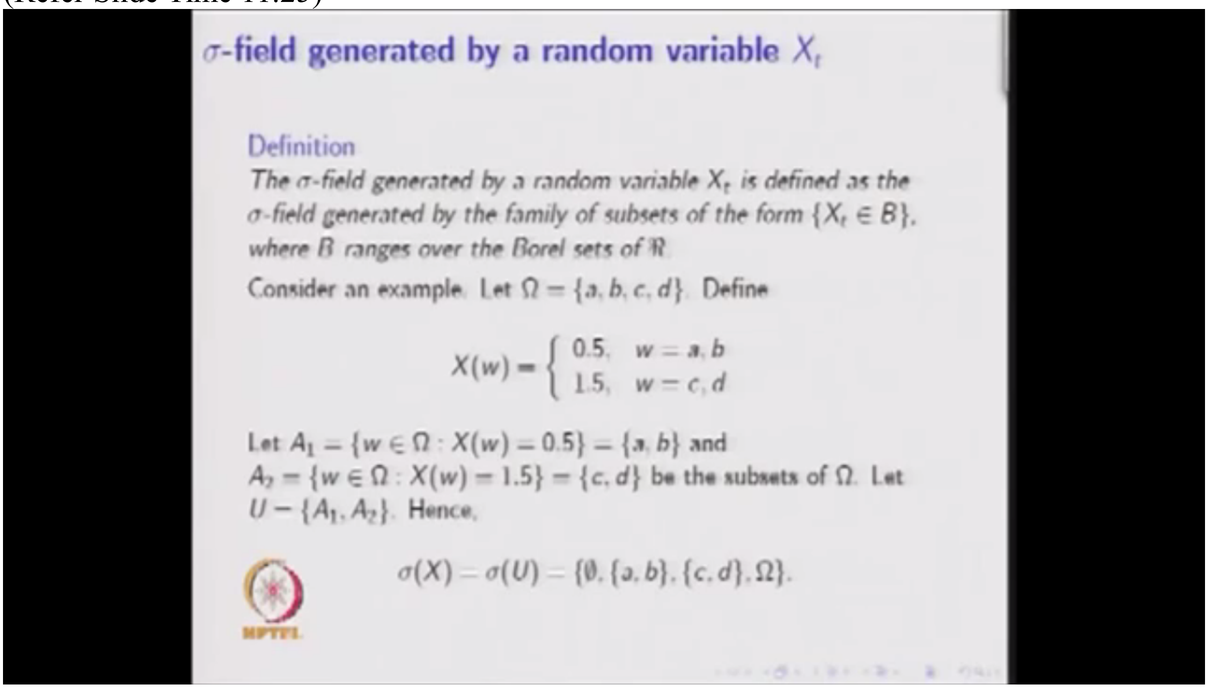

Consider an example where Omega is equal to {a, b, c, d}. Define the random variable X, which takes a value 0.5 for w is equal to a, b. It takes a value 1.5 when w takes a value c, d.

Let A_1 be the set which takes the value -- which is the collection of possible outcomes in which the X(w) takes a value 0.5. Therefore it is {a, b}.

Similarly, let A_2 to be the set for the collection of w belonging to Omega in which X(w) is equal to 1.5. Hence it is {c, d} be the subsets of Omega.

Let U is equal to $\{A_1, A_2\}$. Hence, the σ-field generated by the random variable X that is same as σ(U) that is empty set, {a, b} is one element and {c, d} and the fourth element is the whole set.

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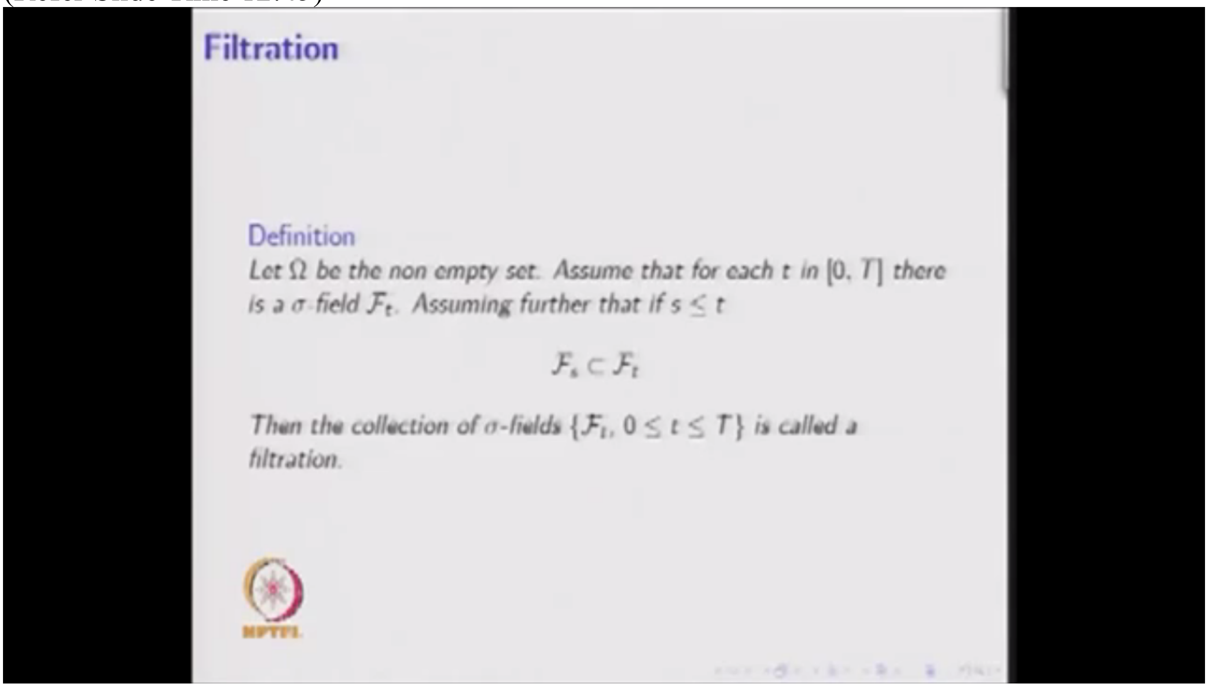

Assume that for each t in [0, T] where T is a positive real number, then for each t between the interval 0 to T, you are creating a sigma-field F of t and those sigma-fields F of t for possible values of t between the interval 0 to T, it satisfies the condition F_s is contained in F_t. If this condition is satisfied over the interval 0 to T by s and t where s is less than or equal to t, then this collection of -- this collection of random variables, sorry, this collection of sigma-fields is called the filtration.

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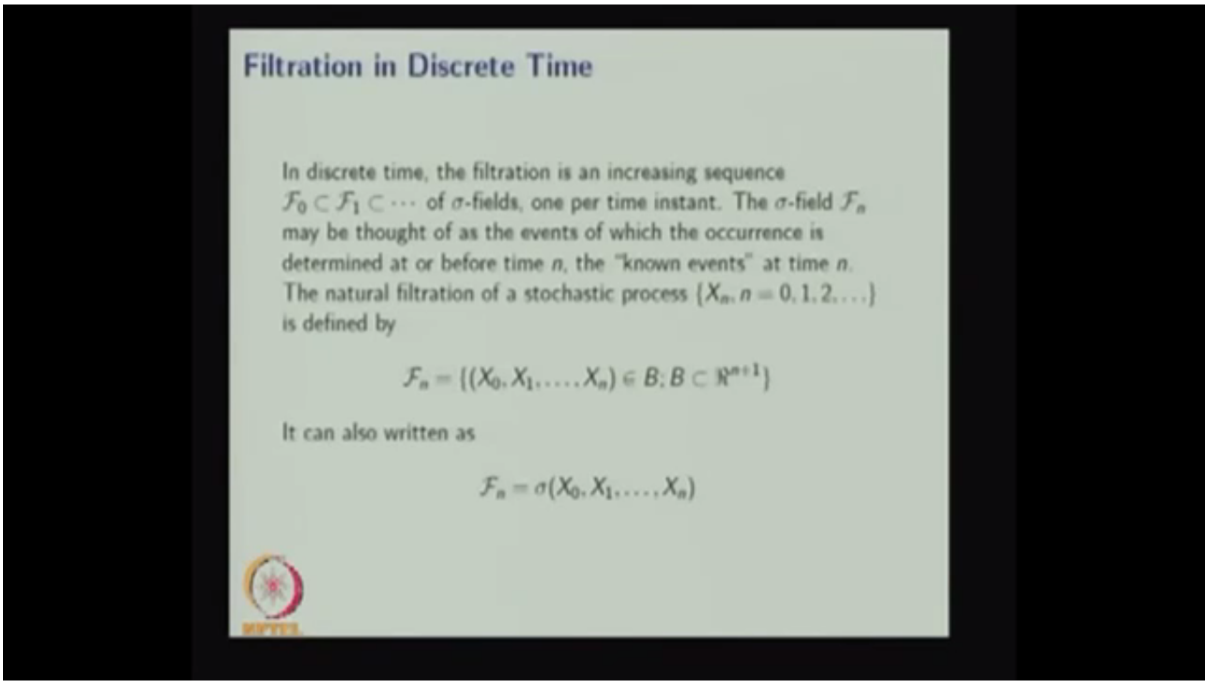

The definition of filtration in real time is as follows. In discrete time, the filtration is an increasing sequence F_0 contained in F_1 contained in F_2 and so on of sigma-fields, one per time instant. The sigma-field F_n may be thought of as events of which the occurrence is determined at or before time n, the "known events" at time n.

The natural filtration of a stochastic process X_n is defined by F_n is the collection of n+1 dimensional random variables belonging to B where B is contained in R_{n+1}. This is also written as F_n is a sigma-field generated by the n+1 random variables or you can think of a random vector with n+1 dimension. So the F_n is a sigma-field created by the random vector X_0 to X_n or the random variables X_0 and X_1 and so on till X_n. So this is the filtration in discrete time.

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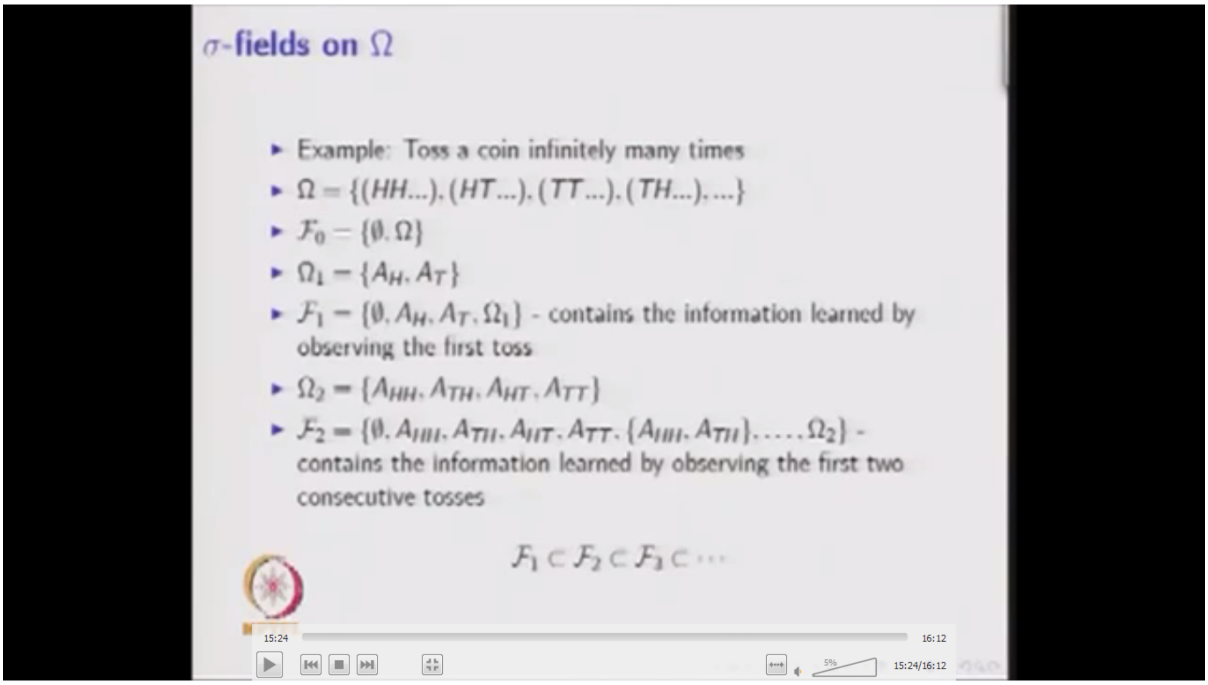

Let us generate a filtration for this example for discrete situation. So take the same example, tossing a unbiased coin infinitely many times and the Ω_1 is having two elements. This is the sigma-field on Ω_1 and the F_2 is the sigma-field on Ω_2 and also satisfied sigma -- F_1 is contained in F_2 which is contained in F_3. Therefore, this collection of -- the collection of random -- the collection of sigma-fields is called the filtration. So this is an example of creating sigma-field in real time.