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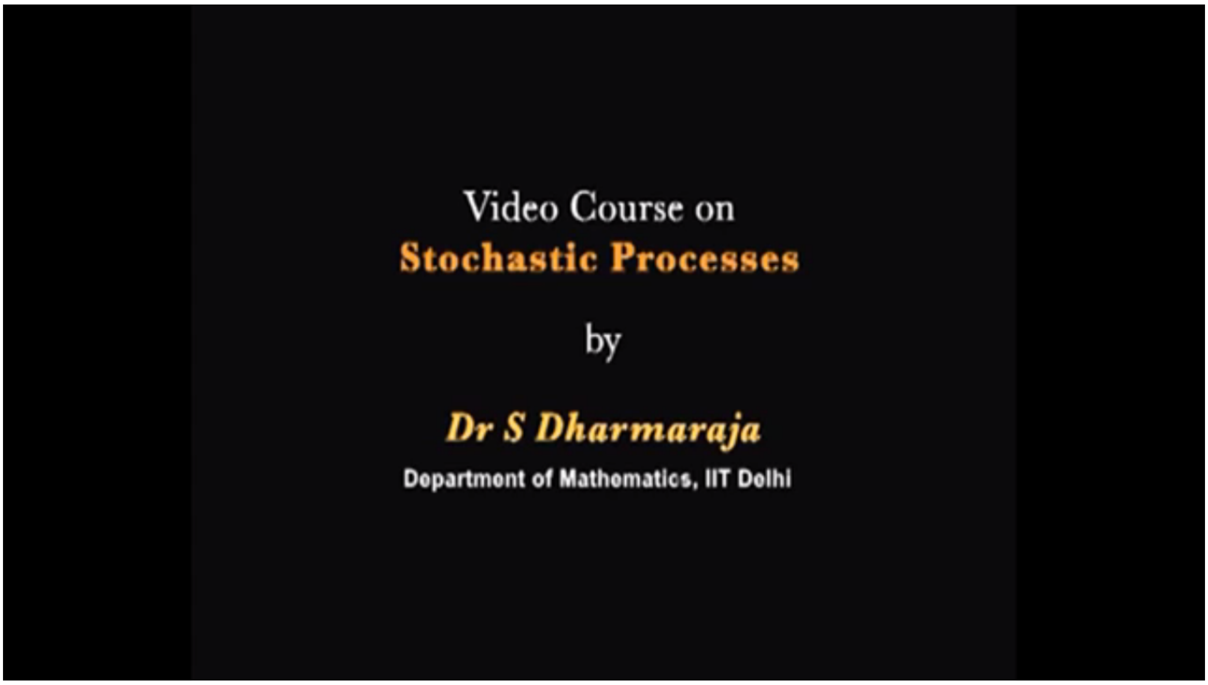

Video Course on
Stochastic Processes

by

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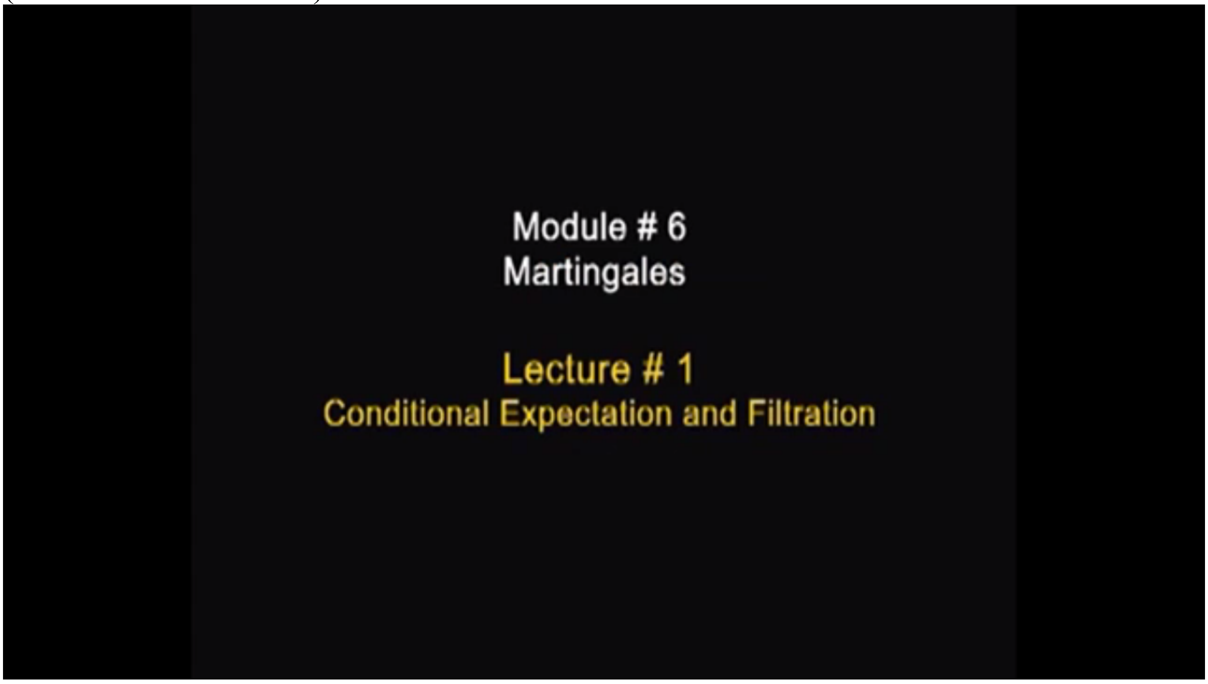

Module # 6
Martingales

Lecture # 1
Conditional Expectation and Filtration

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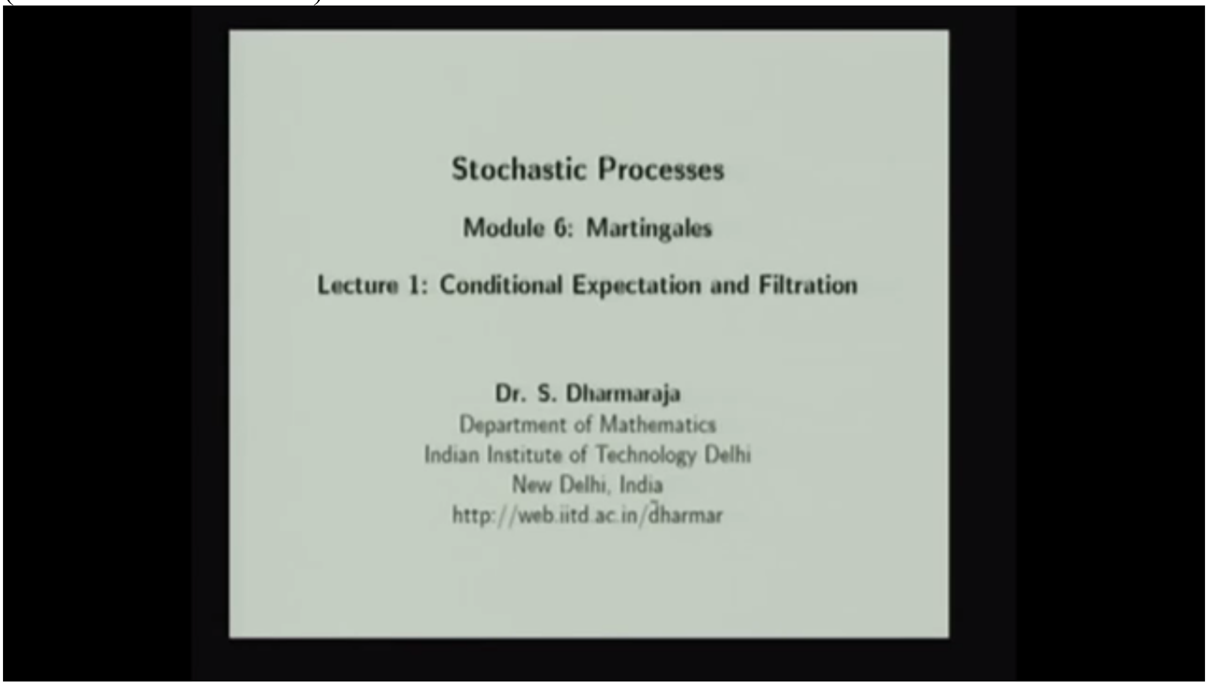

This is a Stochastic Process, Module 6: Martingales, Lecture 1: Conditional Expectation and Filtration.

In the last five models, we have discussed stochastic processes, few properties and then discrete-time Markov chains and continuous-time Markov chains.

In this module, we will discuss an important property of stochastic processes, Martingale.

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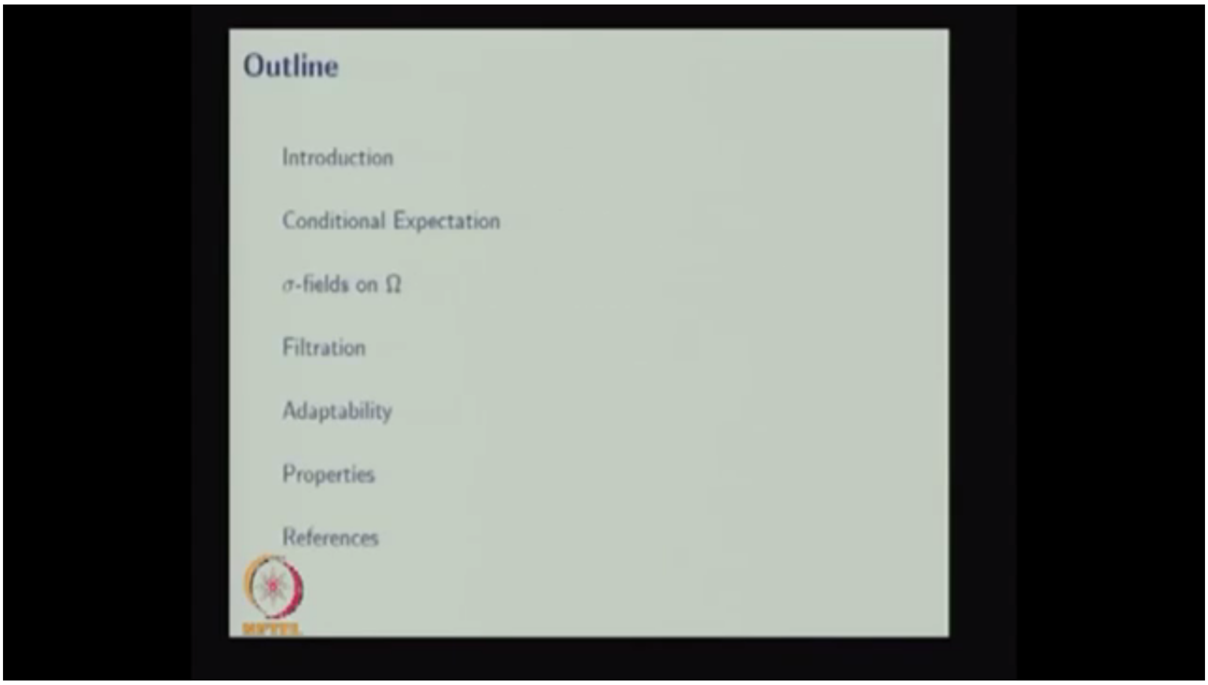

In this lecture, Conditional Expectation, few important properties, sigma-fields on omega, filtration, conditional expectation of a random variable given sigma-field, few important properties along with simple examples will be discussed.

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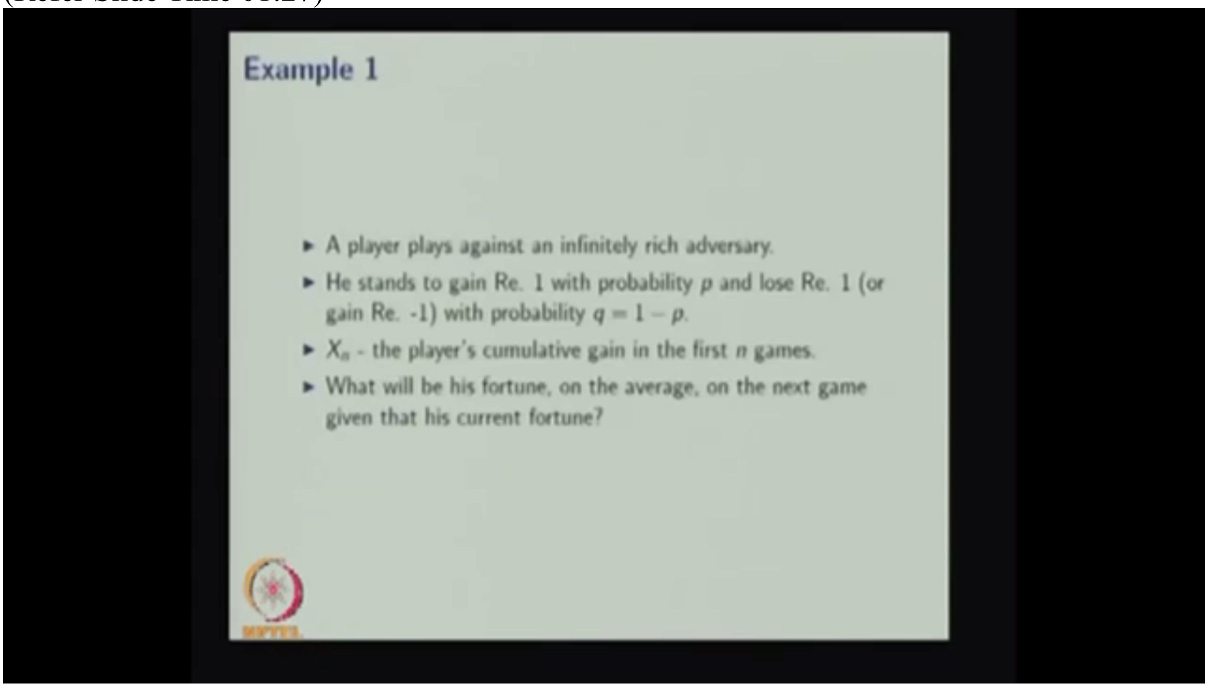

A statistic process is often characterized by the dependence relationship between the members of the family. A process with the particular type of dependence through conditional mean known as Martingale property. This property has many applications in probability.

An example, a player plays against an infinitely rich adversary. He stands to gain rupees 1 with the probability P and lose rupees 1 that's equivalent of gain rupees minus 1 with the probability q that is 1 minus p. Let X_n be the player's cumulative gain in the first n games.

The question is what will be his fortune, on the average, on the next game given that his current fortune?

We need the knowledge of Martingale to solve this problem. Martingale concept involves the conditional expectation of a random variable given sigma-field. Hence, we will introduce the conditional expectation.

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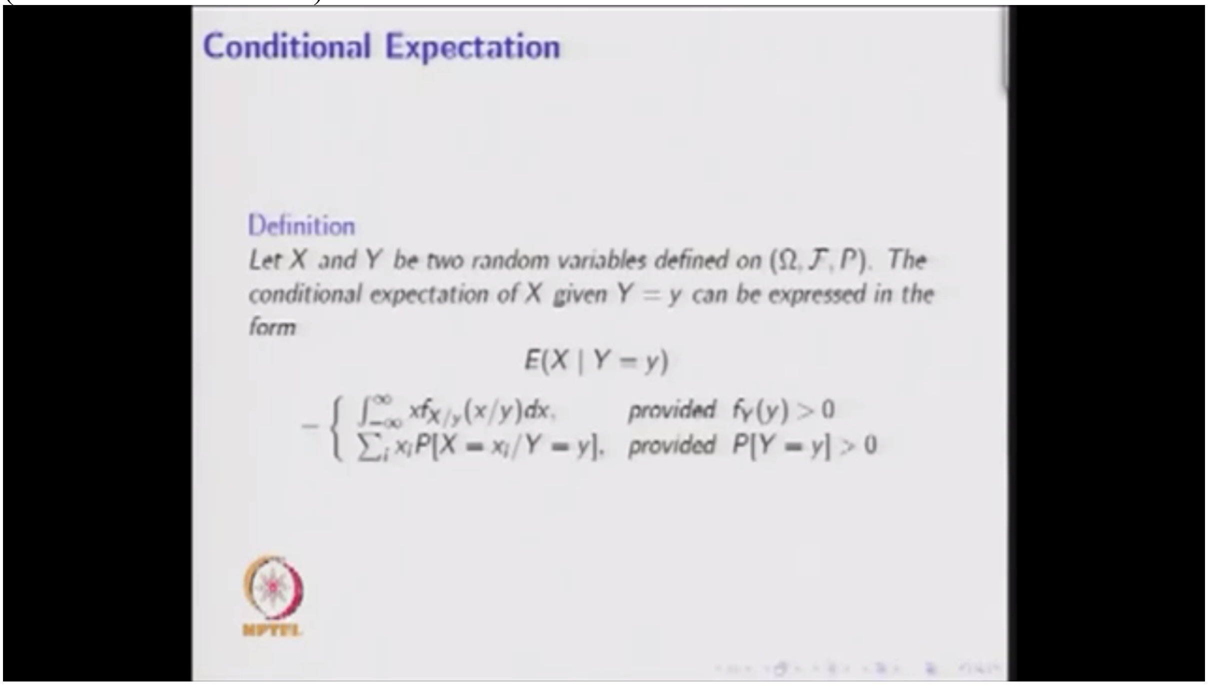

Here is the definition of conditional expectation. Let X and Y be two random variables defined on the probability space (Omega, F, P). The conditional expectation of the random variable X given the random variable Y takes the value small y can be expressed in the form expectation of X given Y is equal to small y.

Both the random variables defined on the same probability space. Omega is the collection of possible outcomes. F is the sigma-field and P is the probability measure. This triplet is a probability space. So both the random variables defined on the same probability space and we are defining the conditional expectation of the random variable given the other random variable takes the value small y.

That can be defined in the form if both the random variables are continuous, then we can make out as an integration minus infinity to infinity x times the conditional probability density function of the random variable x given that other random variable takes a value small y, integration with respect to x provided the probability density function of the random variable Y at the point y has to be greater than 0.

You know that the probability density function is always greater than or equal to 0 in the whole range of y, but here we are discussing the conditional expectation of X given the other random variable takes the value small y. So at that point small y, the probability density value, density function $f_Y(y)$ has to be strictly greater than 0. If that is the case and we assume that here both the random variables are continuous, therefore, we are having a probability density function and integration is with respect to x, x multiplied by the conditional probability density function of x given y.

Suppose both the random variables are continuous, then this conditional expectation of X given Y can be expressed in the form summation x_i's probability of X takes the value small x_i a given that Y takes the value small y. So this is the conditional probability mass function of the random variable X given the other random variable takes the value y. Here also the provided condition is provided the marginal probability, the mass function for the random variable Y at the point small y has to be strictly greater than zero. In that case, you can find out the conditional expectation of X given the other random variables small y.

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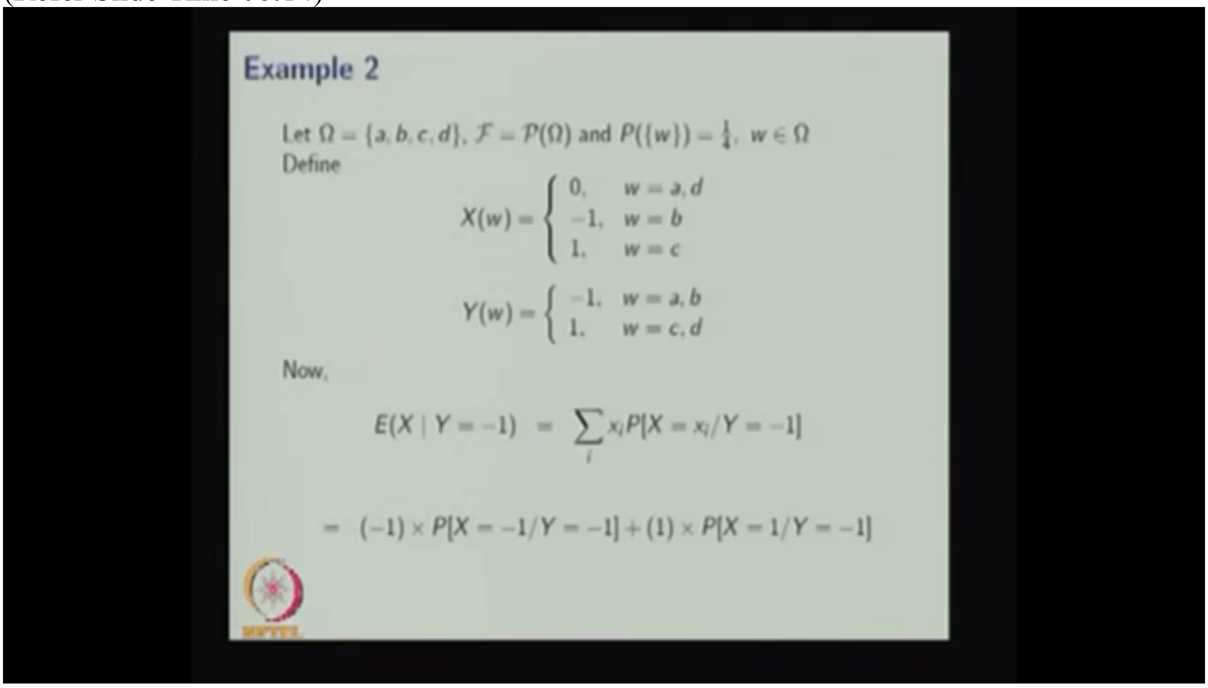

Now we present an example. So here the omega consists of four elements a, b, c, d and the sigma-field is the power set. The omega is finite, so you can create the power set. The number of elements is going to be 2 power n where n is the number of elements in the sigma, omega. So here the f is going to be the sigma-field, which is the power set. So that is the largest sigma-field also and we are defining the probability for each sample that is 1/4 where w is belonging to omega.

Now I am defining two random variables X and Y satisfying the condition of a random variable that is X inverse of minus infinity to X. That semi closed interval inverse image is belonging to F for all X belonging to omega. Then that is going to be a random variable. So here you can cross check whether this real valued function X is a random variable or not. It satisfies the random variable condition. So X is a random variable. Similarly, Y is also a random variable.

Now I am going to calculate the conditional expectation of X given Y takes a value small -- Y takes a value minus 1. So that is nothing but the summation, summation of x_i P of X equal to

x_i given Y takes a value minus 1. That means I have to find out the possible values of x_i's. Then find out the conditional probability mass function for those x_i's. Multiply it. Then summate, summate over i. That is going to be the conditional expectation of X given Y takes the value minus 1. So the X can take the value 0, -1 or 1. So I can compute the way -1 times probability of X takes the value -1, Y takes the value -1. The next X can take the value 1 into probability of X takes the value 1 given Y takes value -1, 0 into the conditional probability you don't want to write. Therefore, we have only two terms.

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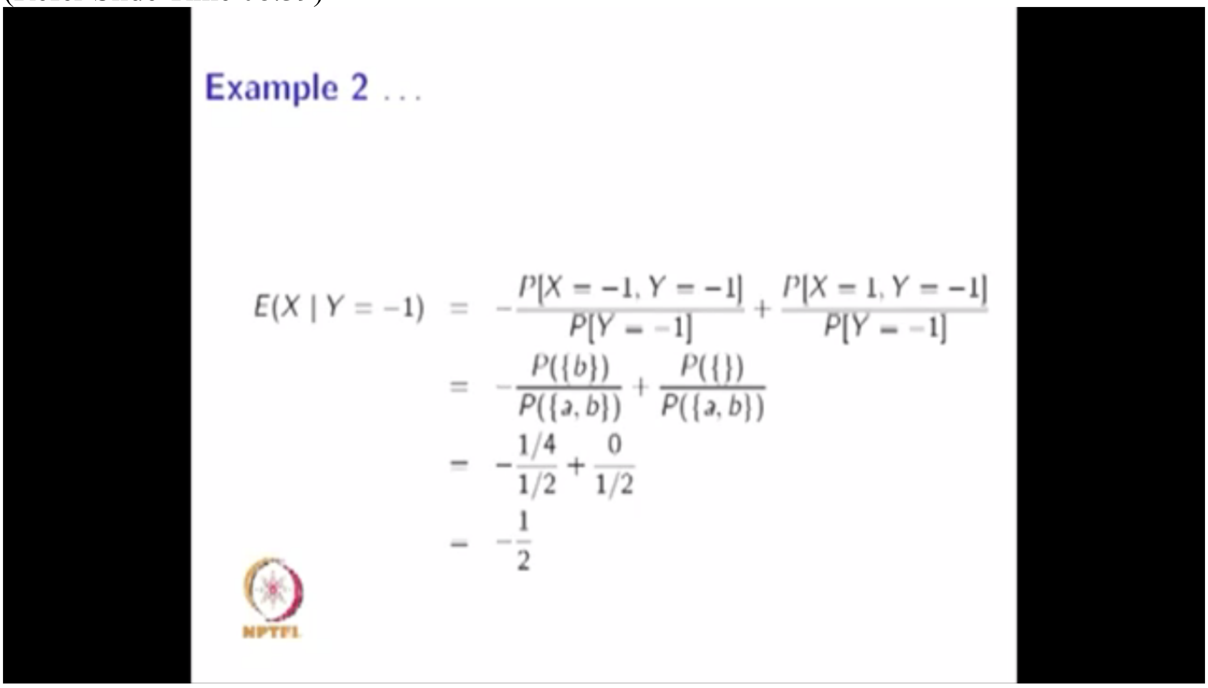

Therefore, the conditional expectation of X given Y takes a value -1, that is going to be minus times this conditional probability you know how to compute, find out the joint probability mass function of X takes the value -1, Y takes the value -1 divided by probability of Y takes the value -1 plus the second term that is a joint probability mass function of X takes the value 1 and Y takes the value -1. You know the probability of X takes the value -1 and Y takes the value -1, the only possibility is the sample is a and similarly this sample is not possible. That's an empty set.

Therefore, the probability of sample a, that is 1/4, the probability of Y takes the value -1 is possible with a and b. Therefore, it is 1/4 plus 1/4. That is 1/2. Empty set probability is 0 and the denominator probability is 1/4 plus 1/4, that is 1/2. Therefore, simplification gives the expectation is equal to -1/2.

Similarly, you can find out the conditional expectation of X given Y takes a value 1 the same way.

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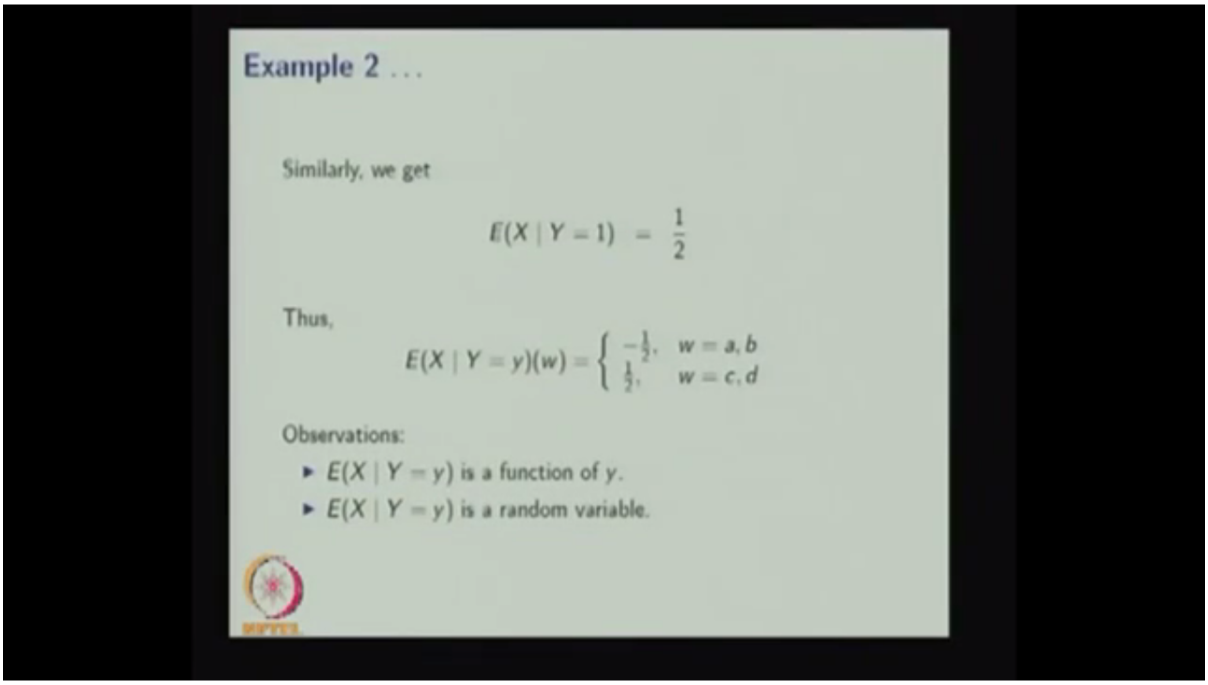

Now I can write it in the compact form. Expectation of X given Y takes a value small y for all the possibility, that is going to be -1/2 if w is equal to a and b. This is going to be 1/2 if w is going to be c or d. That means based on the w, the value changes; w is nothing but the sample; w is belonging to Omega.

So the observations are the conditional expectation is a function of Y. The conditional expectation of X given Y takes a value small y is a function of Y. Not only that, the conditional expectation is a random variable because for all possible values of w, you will get different values. Therefore, this is the random variable whereas expectation is a constant. Conditional expectation is a random variable.

Few important rules on conditional expectation will be used to verify the given stochastic process has a martingale property. So we list the rules. With a few assumptions, we have presented the proof, but these rules can also be proved without these assumptions.

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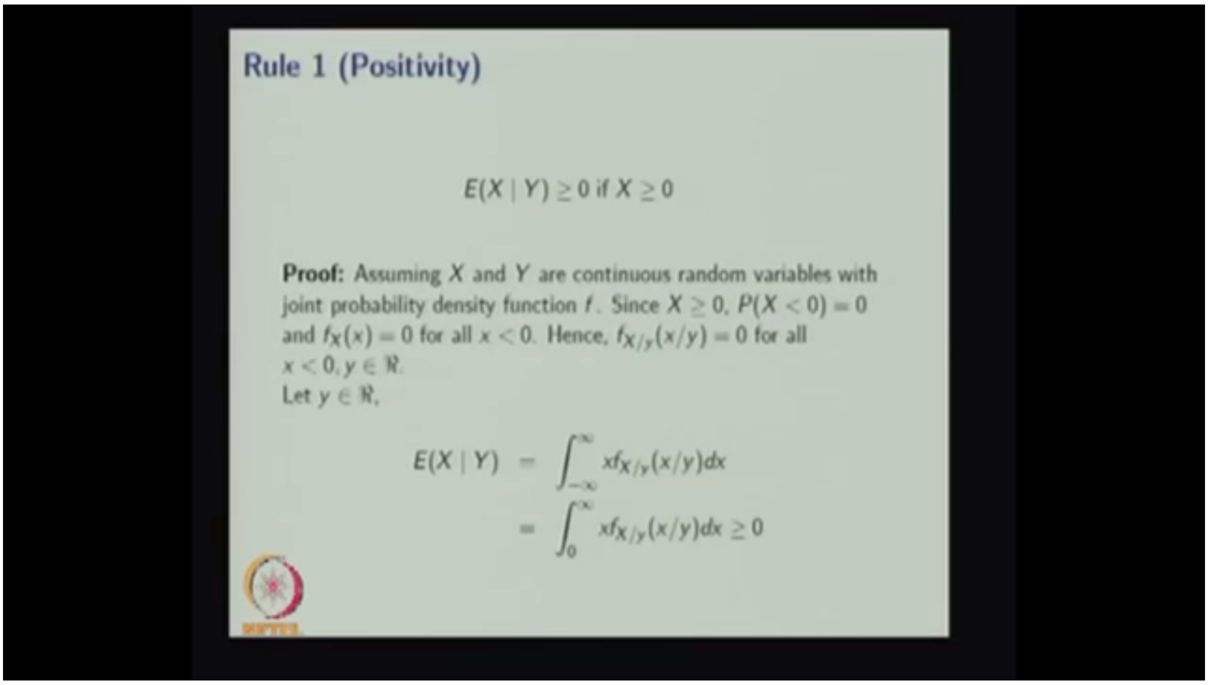

The Rule 1 says if X is greater than or equal to 0, then the conditional expectation of X given whatever be the random variable Y, that is always going to be greater than or equal to 0 if X is greater than or equal to 0 with the probability 1. Whenever X is a non-negative, then the conditional expectation is also going to be non-negative. That's a positivity property.

So the proof is given here. Assuming X and Y are the continuous random variables with the joint probability density function f, I am going to give the proof. Similarly, one can give the proof assuming both the random variables are discrete random variables with the joint probability mass function also.

Since X is greater or equal to 0, the probability of X is less than 0 is equal to 0 and the probability density function of x is going to be 0 for all x less than 0. Hence, the conditional distribution of x given y, that is also going to be 0 for all x less than 0 whatever be the y belonging to the real.

So when let y belonging to real, the conditional expectation is going to be minus infinity to infinity x times the conditional probability density function and this conditional probability density function is going to be 0 for x is less than 0. Therefore, the integration exists only from 0 to infinity because minus infinity to 0, the density function is 0.

Therefore, the conditional expectation is going to be 0 to infinity x times the conditional probability density function and you know that the probability density function is always greater than or equal to 0 in the whole range and x is here we are integrating from 0 to infinity. Therefore, this quantity is always going to be greater than or equal to 0.

So this concludes if x is greater than or equal to 0, the conditional expectation of X given Y, that is also greater than or equal to 0.