Slide 1: Now I'm explaining the transient solution of a finite birth death process. So using this one can find out the transient solution of the birth death process, which I have discussed in today's class M/M/1/N, M/M/c/K, and M/M/c/c also. So the logic is same. That means you have a birth death process with the finite state space, therefore the Q matrix is going to be a degree, whatever be the number of states in the state space, and it is going to be a dry diagonal matrix, and you know the lambda N and Ms birth rates as well as the death rates. And the birth rates and their death rates are going to be different for the… these three models. There are many literature over the transient solution of a finite birth death process started with Murphy and O'Donohue. He uses the polynomial method and in the 1978 Rosenlund also found the transient solution for the finite BDP using again different polynomial methods. And Chiang in 1980, he made a matrix method to get this transient solution. Then later Van Doom gave the solution using a spectral representation method. And Nikiforov et al 1991, he also gave the transient solution using orthogonal polynomial. And later Kijima also gave the solution using Eigenvalue methods. So these are all the literatures for getting the transient solution of a finite birth death process.

Slide 2: And here I am going to explain how to get the transient behavior of M/M/1/N queue and this is by O.P. Sharma and U.C. Gupta. It appears in stochastic processes and their applications Volume 13, 1982. So what this method work, you start with the forward Kolmogorov equation, that is pi-The that is started with pi-T=pi of T in the Q matrix, where PI is the matrix and pi- is the derivatives and the Q is the infinite decimal matrix take forward Kolmogorov equation. Then use the Laplace transform for each by N of T you take the… sorry here the PI - of T is the vector, is a distribution of X of T therefore this is a vector and this is a vector and Q is the matrix, not the matrix, which I said wrongly. So this is a vector and this is a vector and Q is the matrix. So take a Laplace transform for each probability, where the PI And of T, that is nothing but… so the PI of T is a vector that started with the PI not of 0 The, PI 1 of T and so on PI N of The, where PI N of T is nothing but what is the probability that the same notation I started when I discuss the birth-death… sorry continuous time Markov chain, what is the probability that N customers in the system at time T. It's an unconditional probability distribution. So PI N of T is the probability that N customers in the system at time T and using PI B of The, you get the vector and you make a forward Kolmogorov equation, PI- of T = PI of T times Q. And take a Laplace transform for each N of T, that exist, because this is a probability and the conditions for the Laplace transform of this function satisfies, you can cross check that for you are taking a Laplace transform, this is going to be a function of theta. Before taking a Laplace transform, you need initial condition also. So at time 0, you assume that no customer in the system, at time 0 no customer in the system that means the X of 0 = 0, therefore that probability is going to be one and all other probabilities are going to be 0, that is the initial probability vector. So use this initial probability vector and a plate over the forward Kolmogorov equation taking a Laplace transform, you will get the system of algebraic equation. Since you are using the PI not of 0 is equal to 1 you will get the first equation with the term 1 and all other terms are going to be 0. And you know the Laplace transform of derivative of the function, so you substitute… you take a Laplace transform over the forward Kolmogorov equation with this initial condition as well as PI Ns of 0 is equal to 0 for N not equal to 0. So you will have a algebraic equation that is N plus 1 algebraic equations. It's a function of theta. You have to solve this algebraic equation, system of algebraic equation in terms of theta. Once you are able to solve these and take inverse Laplace transform, and that is going to be the system size at any time T. You can start saying that this is going to be of the solution A times alpha N and B times beta power N, where alpha and beta are given in this form, where alpha is equal to this plus something and the beta is equal to minus of this… minus square root of this expression. So you will get… you have alpha as well as beta, now what do you want to find out, if you find out the constant A and B, you can get the Laplace transform of PI N of T. Then you take a inverse Laplace transform and you get the PI N of T.

Slide 3: So for that you need the determinant of matrix of this form. And here this is nothing but all these values are death rates and these are all the birth rates and this is corresponding to the M/M/1/N model and the same logic goes for the transient solution of M/M/c/K as well as M/M/c/c. So instead of this lambdas and mu, you will have corresponding birth rates and death rates, but ultimately you will have N plus 1 matrix determinant as a function of theta. And since these three models are going to be irreducible positive recurrent, the stationary probability and limiting probabilities exist, therefore this determent is going to be always of the form theta times some other function as a degree, as a polynomial of degree N and the function of theta. So this theta is corresponding to the stationary probabilities or the limiting probabilities. Therefore always you can get the N plus 1, the degree matrix month order matrix determined, that is theta times the polynomial of degree N as a function of theta. For the M/M/1/And model, the birth rates are lambda and the death rates are mu, and you can get this polynomial also in the form of product. The product of theta plus lambda plus mu times alpha of N, K square root of lambda mu, where alpha of N, K is nothing, but the K roots of Nth degree Chebyshev's polynomial of second kind. There is a relation between the birth death process with the orthogonal polynomial, for instant the M/M/1/N model, the yen model the finite capacity M/M/1/N model, the corresponding orthogonal polynomial for this birth-death process is the Chebyshev's polynomial of the second kind. Similarly you can say the orthogonal polynomial corresponding to the M/M/c/c model that is a Charlier polynomial. Like that we can discuss the orthogonal polynomial… corresponding orthogonal polynomial for the finite capacity birth-death processes. So here for the M/M/1/N model this is related to the Chebyshev's polynomial of second kind, that is U N of X. So once you are able to get the

Chebyshev's polynomial roots and that root is going to play a role in the product form and that is going to be the polynomial. Note that this polynomial has a distinct real factor. Therefore you can use the partial fraction then you take a inverse Laplace transform, finally you can get the pi N of T, skipping all the simplification part and main logic is this N plus 1th order matrix determinant and that determinant has the factors and those factors are related to the Chebyshev's polynomial roots.

Slide 4: So once you use all those logics and use the partial fraction, then finally you take a inverse Laplace transform. For Lambda is not equal to mu, you will get a steady state or stationary probabilities, plus this expression and this is a function of T. E power minus lambda plus mu times T plus 2 times square root of lambda mu times T cos of R by by N plus 1 and denominate this expression multiplied by this and here this result is related to the initial condition 0, that means at time 0 the system is empty. If the system is not empty then you have one more expression here sign of this minus another term, so that's why you will have a little bigger expression for system size is not empty. And these theta times these that will give the corresponding partial fraction and so on, inverse Laplace, it will give the terms which is independent of T and that is related to the steady-state probabilities, because if you put T tends to infinity, and these quantities are greater than 0, so as T tends to infinity the whole terms will tends to 0. Therefore as T tends to infinity you will have PI N of T is equal to this expression and this is valid for Rho is less than 1, with that condition Rho is less than 1, with the condition Rho is less than 1 those terms will tends to 0 and you will have only this term and that is going to be the steady state or limiting probabilities for M/M/1/N model. If you make also N tends to infinity along with the T tends to infinity, you will have PI Ns, that is the steady-state probability for the M/M/1 infinity model.

Slide 5: So even though I have explained M/M/1/N transient solution in a brief way, but the same logical goes for the M/M/c/c model also, the only difference is this determent has the lambdas and instead of mus, you will have mu, 2 mu 3 mu and so on and instead of the Chebyshev's polynomial you land up with the Charlier polynomial. But there is a difference between the M/M/1/N model and the M/M/c/c model transient solution since the Chebyshev's polynomial has a closed form roots you can find out the factors, so here these are all the factors and you know the factors as well as you can get the closed form expression further M/M/1/N transient solution, whereas the Charlier polynomial does not have a closed form roots, therefore you will land up with the numerical result for the transient solution for M/M/1… M/M/c/c model.

Slide 6: In the case of a continuous time Markov chain that is a finite source Markovian queuing models. This model is also known as a machine repairman model and you can think of these PCs are nothing but the machines and this is nothing but the repairman. And here the scenario is, we have K PCs and each PC can give a print job and inter arrival of print jobs, that is exponentially distributed by the each PC, therefore the print job said… that is follow a arrival process, that is the Poisson process with the parameter lambda from each PC. And once the print jobs come into the printer, it will wait for the print. And the time taken for the each print, that is also exponentially distributed with the parameter muscle. And here there is another assumption, before the first print is over by the same PC it cannot give another print command. Therefore after the print is over by any one particular print job of any PC then those things will go back to the same thing then with the enter-arrival of print jobs generated, that is exponentially distributed, then the print job can come into the printer. So with these assumptions, you can think of the stochastic process, that means the number of print jobs at any time T in the printer, that is going to form a stochastic process and with the assumption of inter arrival of print jobs, that is exponential and actual printing job that is exponentially distributed and so on. Therefore this is going to be a birth-death process with the birth rates or K times lambda and K minus 1 times lambda and so on whereas the death rates that is mu, because we have only one repair. So this is nothing but system size number of jobs in the print job printer, so therefore that varies from 0 to capital K because we are making the assumption more than one print job cannot be given by the same PC before the print is over. And from 0 to 1, the arrival rate will be any one of the K PCs, therefore the arrival rate is K times lambda and already one print job is there in the system, printer. Therefore out of K minus 1 PCs one print job can come, therefore the inter arrival time that is exponentially distributed with the parameter K minus 1 times and lambda and so on so this is the way you can visualize the birth rates whereas the death rates are mu. Once you know the birth rates and their death rates you can apply the birth-death process concept to get the steady-state probabilities. So here we are getting the PI Is in terms of PI not, and the using summation of PI is equal to 1 you are getting the PI not also. And once you know the steady-state probability, you can get the all other measures. So the difference is in this model it is a finite source therefore the birth rates are the function of… it's a state dependent birth rates, whereas the death rates are mus only.

Slide 7: Simulation of the queuing model, I will do it in the next lecture. The summary of today's lecture, I have discussed the simple Markovian queuing models, other than M/M/1 infinity, that I have discussed in the lecture… previous lecture and the stationary distribution and all other performance measures using the birth-death process we have discussed for these queuing models. And finally I discussed the finite source Markovian queuing model also. These are all the reference books. Thanks.