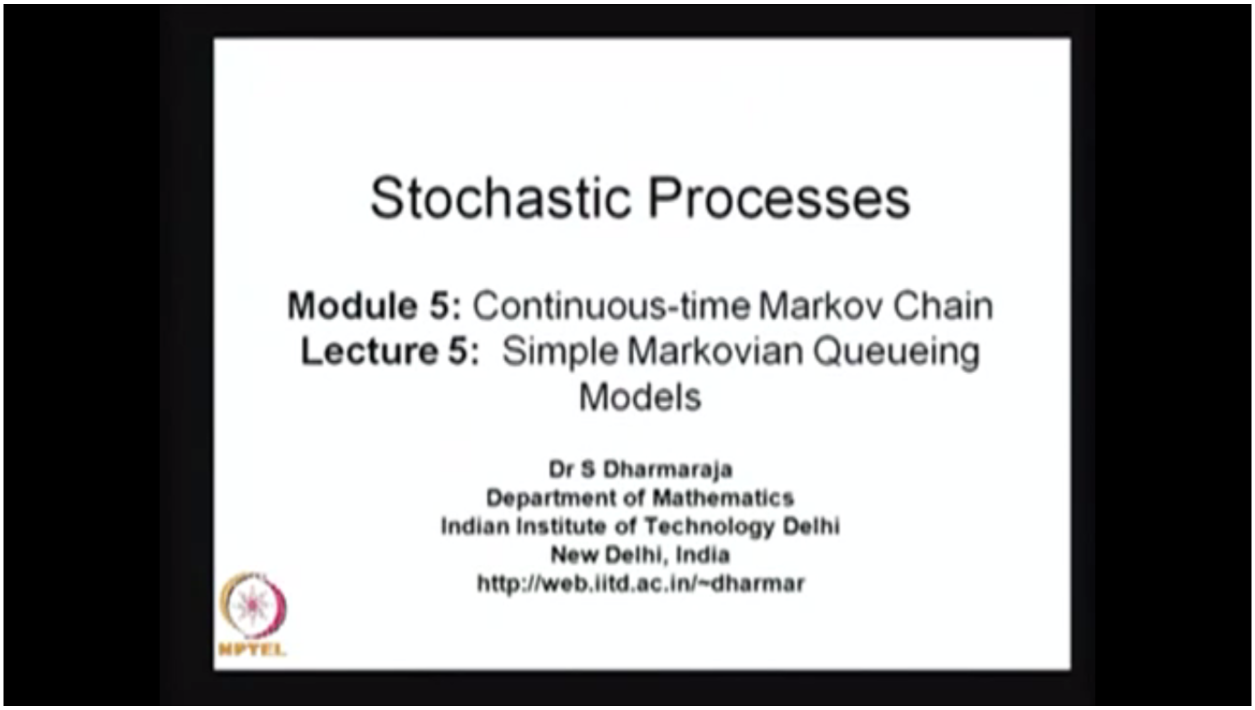

Video Course on

Stochastic Processes -1

By

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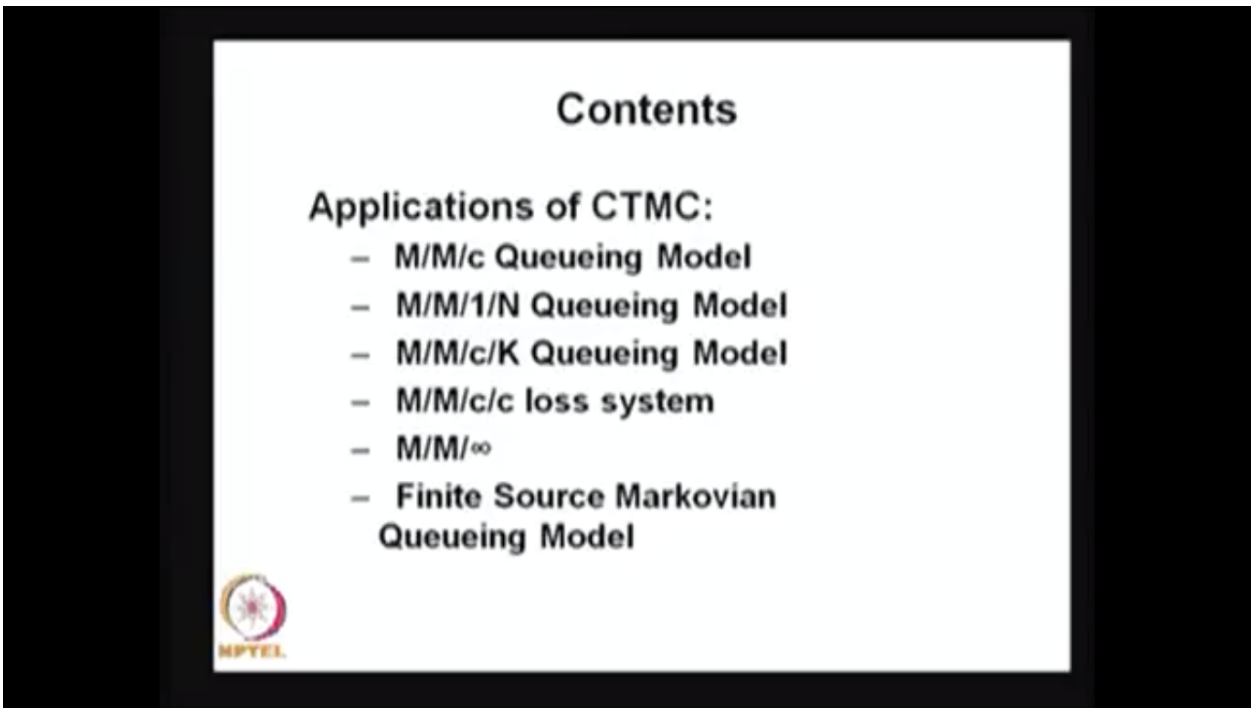
Module 5 : Continuous-time Markov Chain

Lecture#5

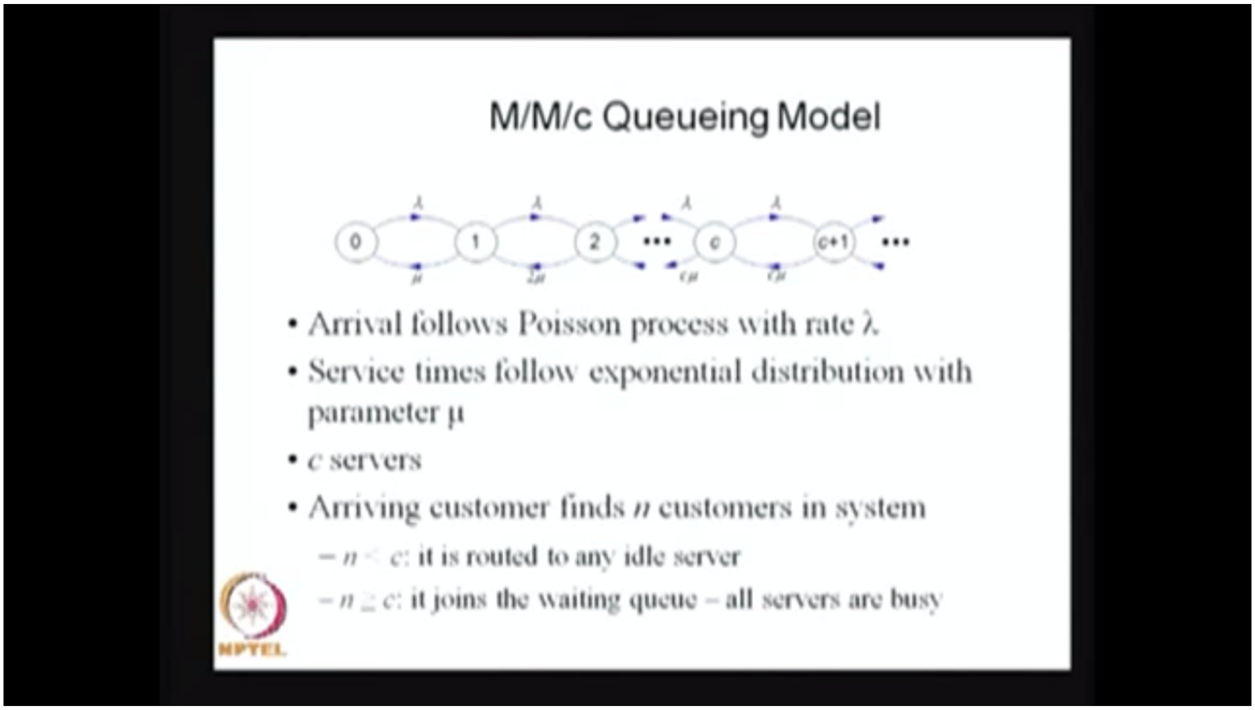
Simple Markovian Queuing Models

This is a lecture 5; application of a continuous time Markov chain in simple Markovian queuing models. The first lecture we have discussed the definition of a stochastic process in particular continuous-time Markov chain. Then we have considered the Kolmogorov differential equation, Chapman Kolmogorov equations, the transient solutions for the CTMC.

The second lecture we have discussed the special case of continuous time Markov chain that is birth-death process we have discussed in lecture 2. lecture 3 the special case of birth-death process which is very important stochastic process that is a Poisson process is discussed in the lecture 3. In the lecture 4 we have discussed the MM1 queuing model that is a very special and important queuing model and the underlying stochastic process for the MM1 queuing model that is a birth-death process with the birth rates are lambda and death rates are mu. In the fourth lecture we have discussed only the MM1 queuing model.



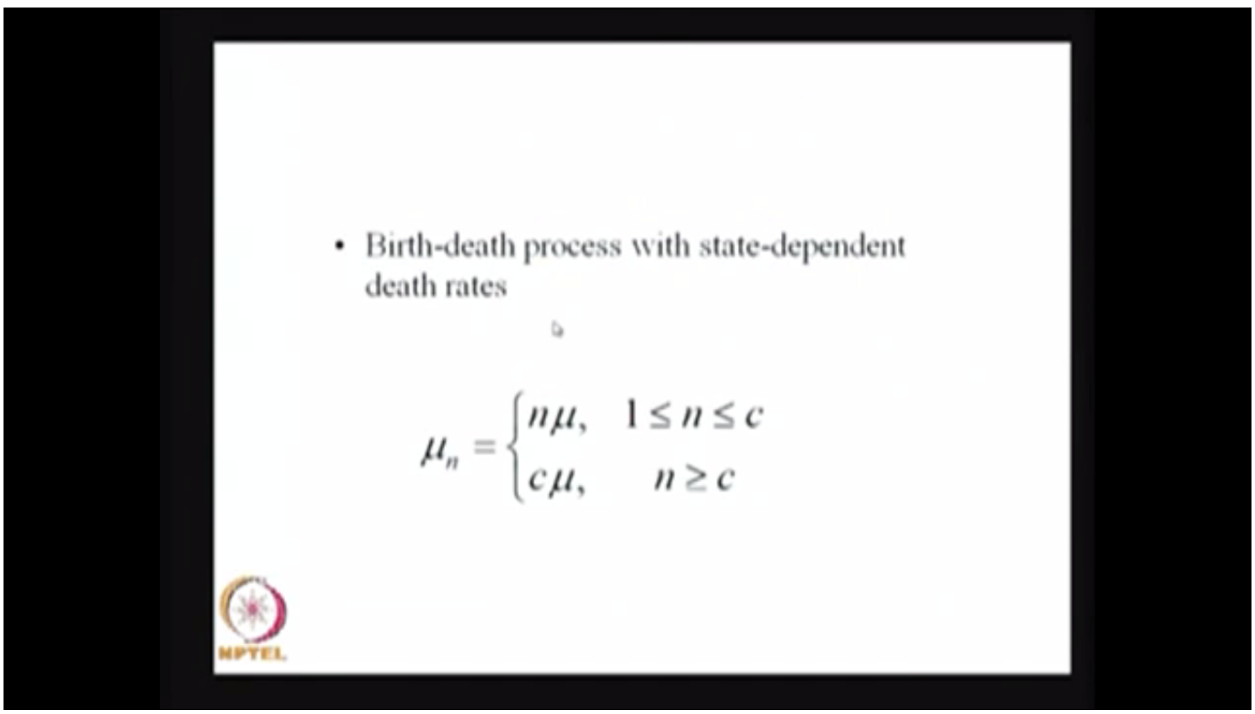
In this lecture we are going to consider the other simple Markovian queuing models as an application of a continuous-time Markov chain. So in this lecture I am going to discuss other than the MM1 queuing model I am going to discuss simple Markovian queuing models starting with the MMC infinity queuing model. Then the finite capacity model Markovian setup. MM1N queuing model. Then I am going to discuss the multi-server finite capacity model that is MMcK queuing model. After that I am going to discuss the last system that is here MMcc model for a infinite server model that is M infinity also I'm going to discuss. At the end I'm going to discuss the finite source Markovian queuing models whereas the other five models the population is infinite source so the last one is a finite source Markovian queuing model also I'm going to discuss as the application of continuous time Markov chain.



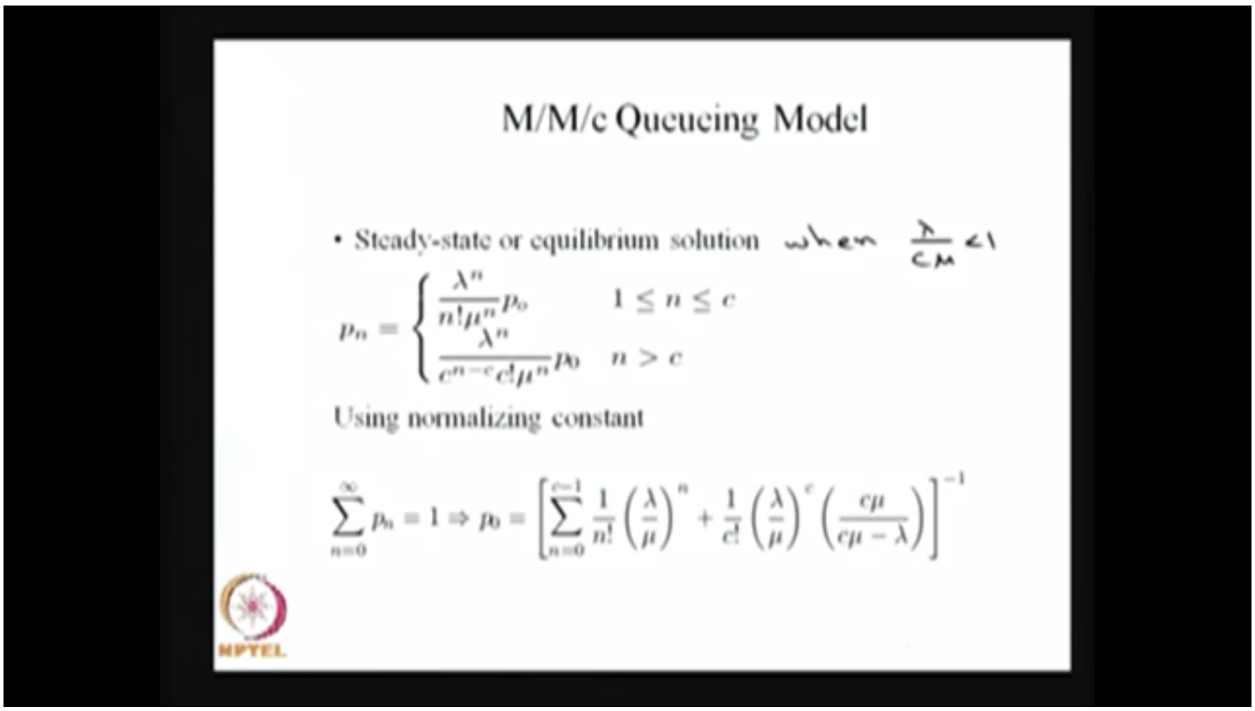
The first model is a multi server infinite capacity Markovian queuing model. The letter M denotes the inter-arrival time is exponentially distributed with parameter lambda. The service time by the each server that is exponentially distributed with the parameter mu and all we have more than one servers suppose you consider as a c where c is a positive integer and all the servers are identical and each server is doing the service which is exponentially distributed with the parameter mu which is independent of the all other servers and the service time is independent with the inter-arrival time also. With these assumptions if you make a random variable X of t is the number of customers in the system at any time t that is a stochastic process. Since the possible values of number of customers in the system at any time t that is going to be 0, 1, 2 and so on therefore it is a discrete state and you are observing the queuing system at any time t therefore it is a continuous time. So discrete state continuous times stochastic process and if you observe the system keep moving into the different states because of either arrival or the service completion from the any one of the c servers.

So suppose there are no customer in the system and the system moves from the state 0 to 1 by one arrival. So the inter-arrival time is exponentially distributed therefore the rate in which the system is moving from the state 0 to 1 is lambda. Like that you can visualize the rates for the system moving from 1 to 2, 2 to 3 and so on whereas whenever the system size is 1, 2 and so on till c since we have a c number of the servers in the system whoever are entering into the system they will get they will start getting the service immediately. Suppose the system goes from state 1 to 0 that means that the customer enter into the system and he get the service immediately and the service time is exponentially distributed with the parameter mu therefore whenever the service is completed the system goes from the state 1 to 0. Therefore the rate is mu; whereas a from 2 to 1 there are 2 customers in the system and both are under service. At any time if any one of the servers complete the service then the system moves from 2 to 1. So the service completion will be minimum of the service time of the both the servers. Since each server is doing the service exponentially distributed with the parameter mu therefore the minimum of two exponential both are independent also therefore that is also going to be exponentially distributed with the sum of parameters. So it is going to be parameter will be mu plus mu that is 2mu. So the system moves from the state of 2 to 1 will the rate will be to 2mu. Like that it will be keep going till the state from c to c minus 1. That means we have c servers therefore whenever the system size is also less than or equal to c that means all the customers are under service.

Now we'll discuss the rate in which the system is moving from the state c plus 1 to c. This is some state is a c plus 1 that means and the number of customers in the system that is c plus 1. We have c servers therefore one customer will be waiting for the service, waiting the queue. Therefore, the system is moving from c plus 1 to c that is nothing but one of the servers completed the service out of c service therefore the rate will be the service time completion, service time will be exponential distribution with the parameter c mu not c plus 1 mu it is we have only c servers therefore the minimum of exponentially distributed with the parameters mu and so on with the c exponentially distributed random variables. Therefore, that is going to be exponential distribution with the parameter mu plus mu plus there are c mus therefore it is going to be cmu.



Like that the rate will be the death rate will be cmu after c plus 1 onwards whereas from 0 to c it will be mu, 2mu, 3mu and so on till cmu after that it will be cmu from the state from c plus 1 to c plus 2 to c plus 1 and so on. And if you see the state transition diagram you can observe that this is a birth-death process. So before that let me explain what is a MMC infinity means whenever c customers or c servers or any one of the c servers are available then the customers we get the service immediately. If all the c servers are busy then the customer has to wait till any one of the c servers are going to be completing their service. So that is the way the system works therefore you will have the system size, the system size the underlying stochastic process is going to be a birth-death process. It's a special case of a continuous time Markov chain because the transitions are only the neighbors transition with the forward rates that is lambda and backward rates or the death rates are going to be mu, 2mu and so on therefore this is a special case of a continuous time Markov chain the underlying stochastic process for the MMC infinity model that is a birth-death process. The birth rates are lambda whereas the death rates depends on the n. The mun is a function of n therefore it is called a state dependent and death rates. It may not be the function n times mu it can be a function of n and then we can use the word state dependent. So here it is a linear function so state dependent death rates and the death rates n times mu whenever n is lies between 1 to c and the mun is going to be c times mu for n is greater than or equal to c that you can observe it from the state transition diagram also the death rates are going to be cmu, here also cmu and so on. Therefore this is a birth-death process with the state dependent death rates.



Now our interest is to find out the steady state or equilibrium solution. Since it is a infinite capacity model if you observe the birth-death process with the infinite state space then you need a condition so that the steady state probabilities exist. So whenever lambda by cmu is less than 1 whenever lambda by cmu is less than 1 you can find out the limiting probabilities. So sometimes I use the letter P and sometimes I use the word Pn both are one and the same. So you find out the steady state probability by solving a Pq is equal to 0 and the summation of Pn is equal to 1 and if you recall the birth-death process the steady state probabilities the Pi not has the 1 divided by the series whenever the denominator series converges then you will get the Pn's. So either I use a Pn's or Pin's both are one and the same so here summation of Pn is equal to 1 and P if you make a vector P, P times Q is equal to 0 if you solve that equation and the denominator of P naught that expression that is going to be converges only if lambda divided by cmu is less than 1. So therefore whenever this condition is there the queuing system is stable also. If you put c is equal to 1 you will get the MM1 queue.

So using the normalizing condition you are getting the P naught and the P naught is 1 divided by this. So this is a series. So this series is going to be converges only if this condition is satisfied. So by solving that equation so you are getting P and c in terms of P naught and using normalizing and constant you are getting a P naught. Therefore this is a steady state also known as the equilibrium solution for the MMC infinity model. So here we are using the birth-death process with the birth rates are lambda and the death rates are given in this form and use the same logic of the stationary distribution for the birth-death process. Using that we are getting the steady state or equilibrium solution for the MMC model.