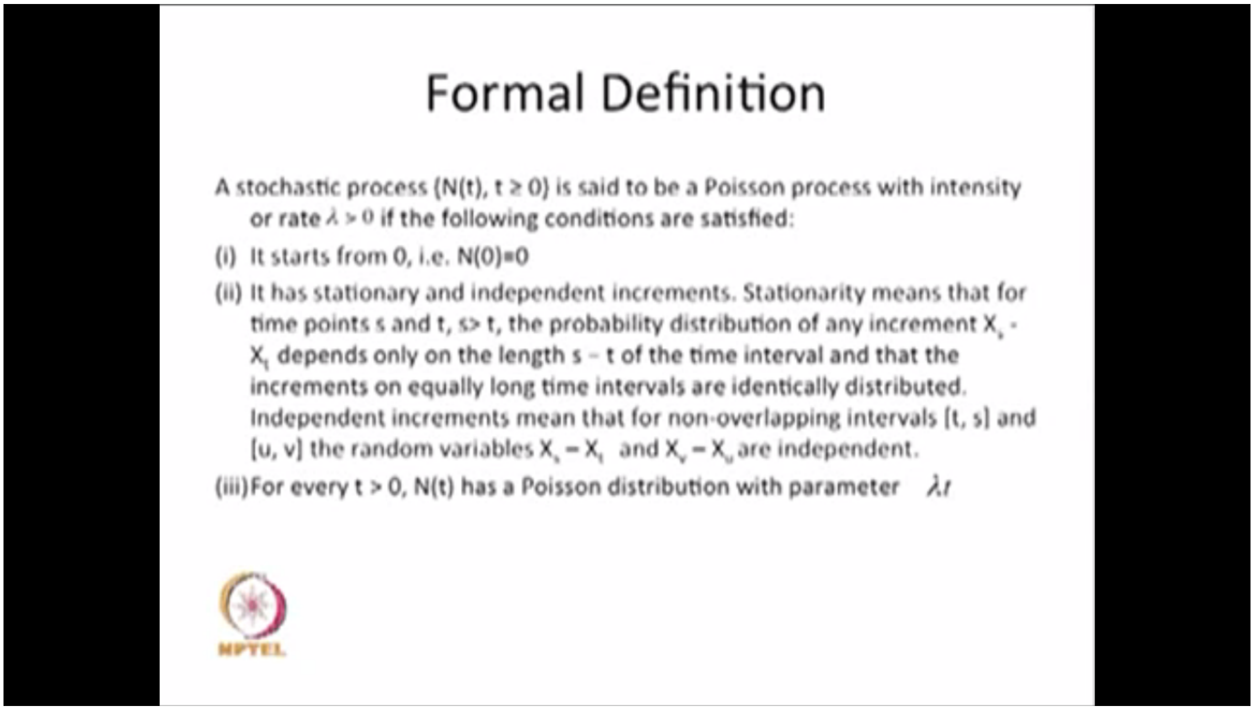


Formally we define Poisson process as follows a stochastic process N of T T greater or equal to 0 is said to be a Poisson process with the intensity or rate lambda greater than 0 if the following conditions are satisfied. First condition it starts from 0; that is N of 0 is equal to 0.

Second condition the increments are stationary and independent. Stationarity means that for time points S, N T is greater than the probability distribution of any increment N of S minus N of T depends only on the length S minus T of the time interval and that the increments equally long time intervals are identically distributed.

Independent increments means that for any non-overlapping intervals T,S and U, V the random variables n of s minus n of t and n of v minus n of you are independent. For t greater than 0 n of t has a Poisson distributed random variable with the parameter lambda t.

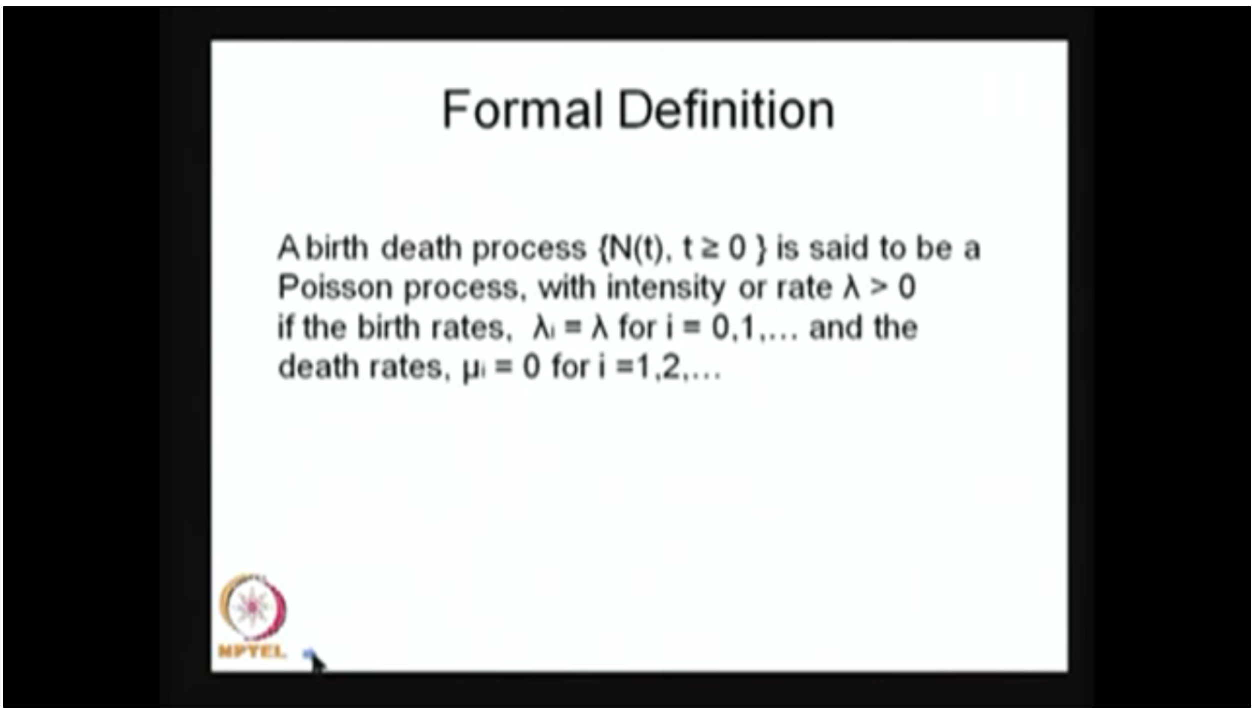
And the difference of the random variables defined over non-overlapping intervals are independent. Lambda t is the cumulative rate t time t. the Xi's are independent and identically distributed random variables with some distribution function G independent of the Poisson process n of t, t greater or equal to 0.

It is Markov in nature because the two queues act independently and are themselves M/M/1 queue system which satisfies the Markov property assuming that each queue behaves as the M/M/1 queue. The details of the proof can be found in the reference books. because QAJs are obtained by differentiating the PAJs.

For every t greater than 0 n of t has a Poisson distribution with the parameter lambda t like that you can go for many more increments also. For illustration I have made it with the two increments that means in the occurrence of arrival during this non-overlapping intervals are independent and the stationary means it is a time invariant only the length matters not the actual time.

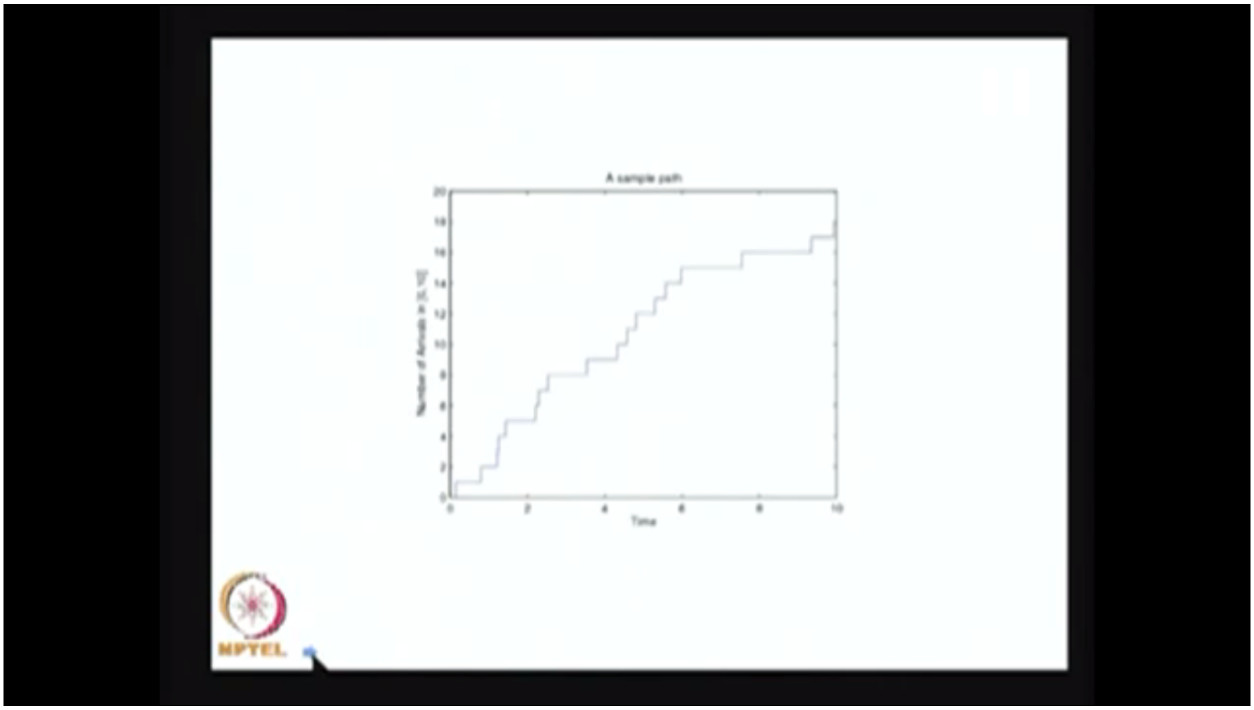
Third one, for every t n of t has a Poisson distribution in the parameter lambda t. So the Poisson logic is coming into the fourth condition only. The first condition start at 0 increments are stationery and increments are independent. The third condition for [Indiscernible] [00:03:47] n of t is Poisson distribution random variable with the parameter lambda T. Therefore this stochastic process is called a Poisson process.

Now we can relate the way we have done the derivation we have taken care these three assumptions starting at time 0,0 increments are stationary that we have taken and increments are independent that is non-overlapping intervals are independent then when we have derived we are getting the distribution of the random variable n of t is a Poisson distribution random variable. Therefore, this is a Poisson process.



The another way of defining the Poisson process we can start with the birth-death process. You know that birth the death process is a special case of a continuous time Markov chain. Also it's a special case of sorry, it's a special case of Markov process also. So you can think of a stochastic process then this special case is the Markov process. Then the special case is a continuous time Markov chain. Then you have a special case that is a birth-death process.

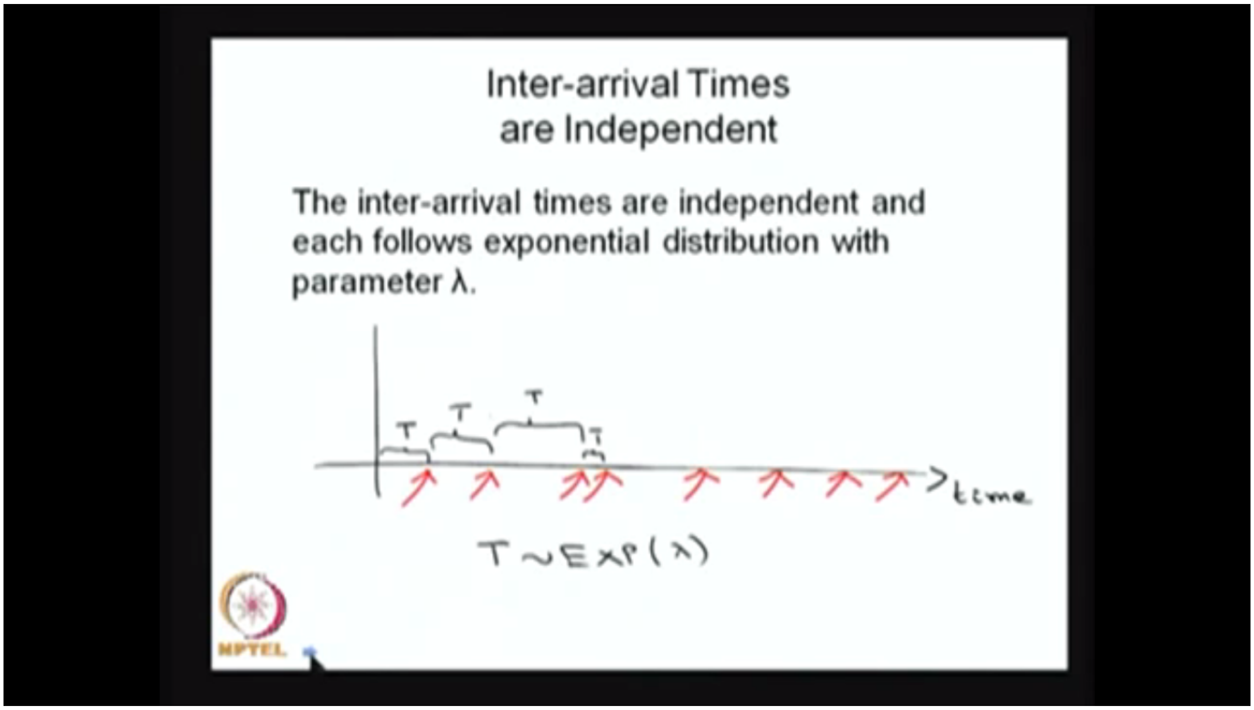
So you can define the Poisson process from the birth death process also. Here birth death process n of t is said to be a Poisson process with the intensity or rate lambda if birth rates are constant for all i and the death rates are 0. You start from the birth death process with all the birth rates are same that means it's a special case of pure birth process in which birth rates are constant for all the states and the death rates are 0 then also you will get the Poisson process.



Here I am giving a sample path for the Poisson process. So this is created using the MATLAB, write the simple code of Poisson process then you develop the sample path that means at time 0 the system is at 0. At some time 1 arrival takes place therefore the system land up 1 therefore the Y-axis is nothing but the n of t.

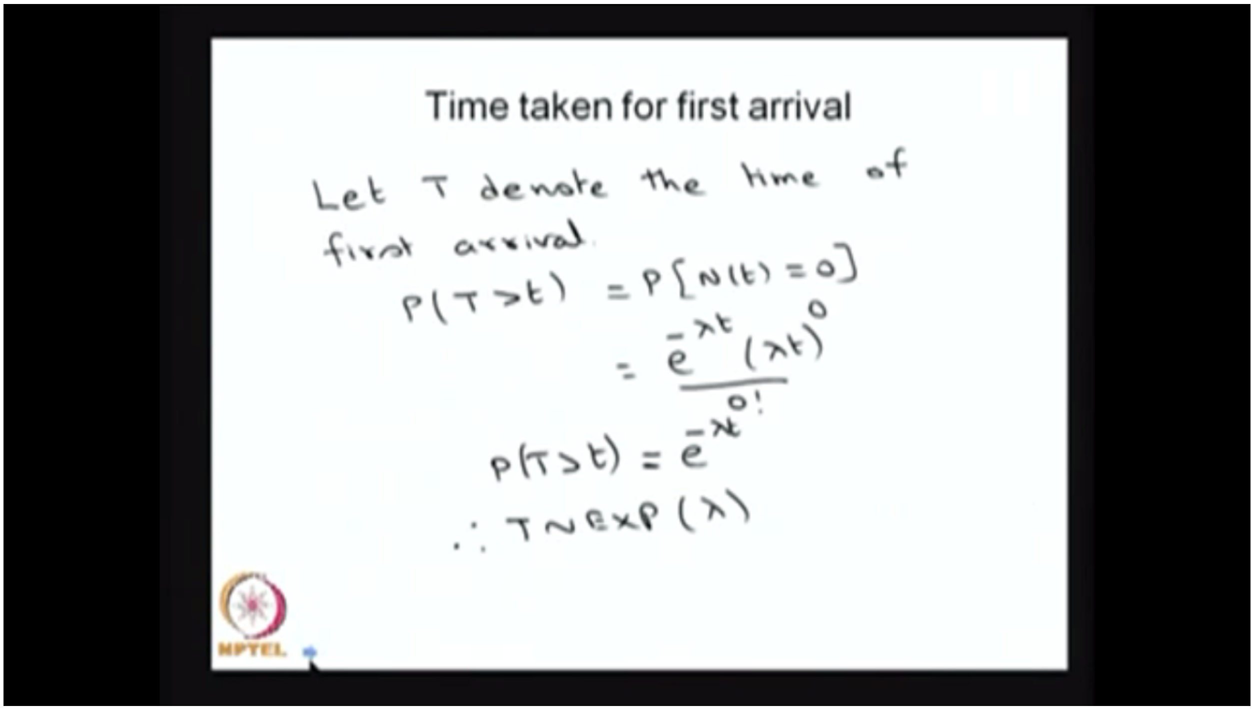
So at this time on arrival takes place therefore the number of customers in the system number of arrivals till this time that is 1 so it's a right continuous function. The value at that point and the right limit is same as both are same which is different from the left limit of the arrival epoch, arrival time of epoch. So the system was in the state 1 till the next arrival takes place. So suppose the arrival takes place here then the n of t value is 2 at this time point in which the arrival epoch and the right limit and so on. So this is the way. Therefore, the system at any time it will be the same value or it will be incremented by only one unit. The Poisson process sample path will be with the 1 unit step increment at any time there is no way the two steps the system can move forward at even in a very small interval of time the system will move into the only one step that you can visualize here.

Therefore, you can go back to the assumptions which we have started in the derivation n of 0 is equal to 0 in a very small interval of time at most one event can takes place and the difference of the random variables defined over non-overlapping intervals are independent and increments are also stationary. So those things you cannot able to visualize in the sample path. So this is just one sample path over that time and n of t.



The second one, inter-arrival times or independent. As well as we can control the inter-arrival times are exponentially distributed also. The inter-arrival times are independent and each follow exponential distribution with the parameter lambda. What is the meaning of inter-arrival times? At time 0 the system is in the state 0. First arrival occurs at this time point. Second arrival occurs this time point and the fourth, third, fourth and so on. The inter-arrival time means what is the time taken for the first arrival. Then what is the interval of time taken for the first arrival to the second arrival and second to the third and so on. So that is the inter-arrival time.

So whenever you have a Poisson process that means that the arrival of event occur over that time that follows a Poisson process then this inter-arrival time suppose I make it as a random variable T and those random variables going to follow exponential distribution with the same parameter lambda and all the inter-arrival times also independent. That means these are all identically distributed random variable. I can go for a different random variable label also X1, X2, X3, X4 and so on so all those random variables are iid random variables and each follows exponential distribution with the parameter lambda.



So this can be proved easy. Let me start giving the proof for the first arrival time that means is the first one from 0 to the first arrival like that you can go for the other arrivals also using the other properties or you can use the multi-dimensional random variable distribution concept and use the function of a random variable and you can get the distributions also but here I am finding the distribution for the first arrival.

So let it T denote the time of first arrival. My interest is to find out what is a distribution of T. I know that this is going to be a continuous random variable because it's a time. So anytime the first arrival can occur. So to find – since it is a continuous random variable I can find out the CDF of the random variable or compliment CDF. So here I am finding first the compliment CDF using that I am going to find out the distribution.

Let me start with the probability that the first arrival is going to takes place after time T. What is the meaning of that? The first arrival is going to occur after time t that means till time t there is no arrival. So both the events are equivalent events. The probability of T greater than small t that is same as the probability of N of t is equal to 0. That means there no even that takes place till time t because N of t denotes the number of arrival of customers during the interval 0 to small t both are closed 0 to 1, 0 to t. Therefore N of t equal to 0 that means till time t nobody turned up that is equal of the first arrival is going to takes place after t.

I don't know the distribution of T but I know what is the probability of N of t equal to 0. Therefore I am writing this relation. So once I substitute the probability mass at 0 for the random variable N of t just now we have proved that N of t for fixed T is a Poisson distribution random variable with the parameter lambda times t therefore I know what is a probability mass at 0.

So substitute the probability mass function with the 0 I will get e power minus lambda t that is a compliment CDF of the random variable T. Once I know the compliment CDF. I can find out the

CDF. From the CDF I can compare the CDF of some standard continuous random variable I can conclude this is nothing but exponential distribution with the parameter lambda because this is a compliment CDF at time T. Therefore it's a lambda times t. So I can prove the distribution of T is exponential distribution in the parameter lambda. That means the first time of forever this random variable that is a continuous random variable and the continuous random variable follows exponential distribution with the parameter lambda. Since I know the increments are independent, increments are stationary and so on I can use a similar logic for inter arrival time of this time also then that is also going to follow in the exponential distribution.

Since the increments are independent so this is the first time and this is a second time therefore the inter-arrival times also going to be independent. That means whenever you have a Poisson process that means the arrival occurs over that time in a very small interval maximum one arrival takes place and the probability of one arrival in that small interval is lambda times delta t from that you get the lambda so you can conclude that's the Poisson distribution, Poisson process. So once the arrival follows a Poisson process the inter-arrival times are exponential and independent.

So from the Poisson process one can get the inter-arrival times are exponential distribution and independent. The converse also true that means if some arrival follows with the inter arrival times exponential and exponential distribution and all the inter arrival times are independent then you can conclude the arrival process is going to form a Poisson process. That means the arrival process Poisson process implies the inter-arrival times are exponentially distributed and are independent. Similarly inter-arrival times are independent as well as exponentially distributed with the parameter lambda then the arrival process is a Poisson process with the parameter lambda.