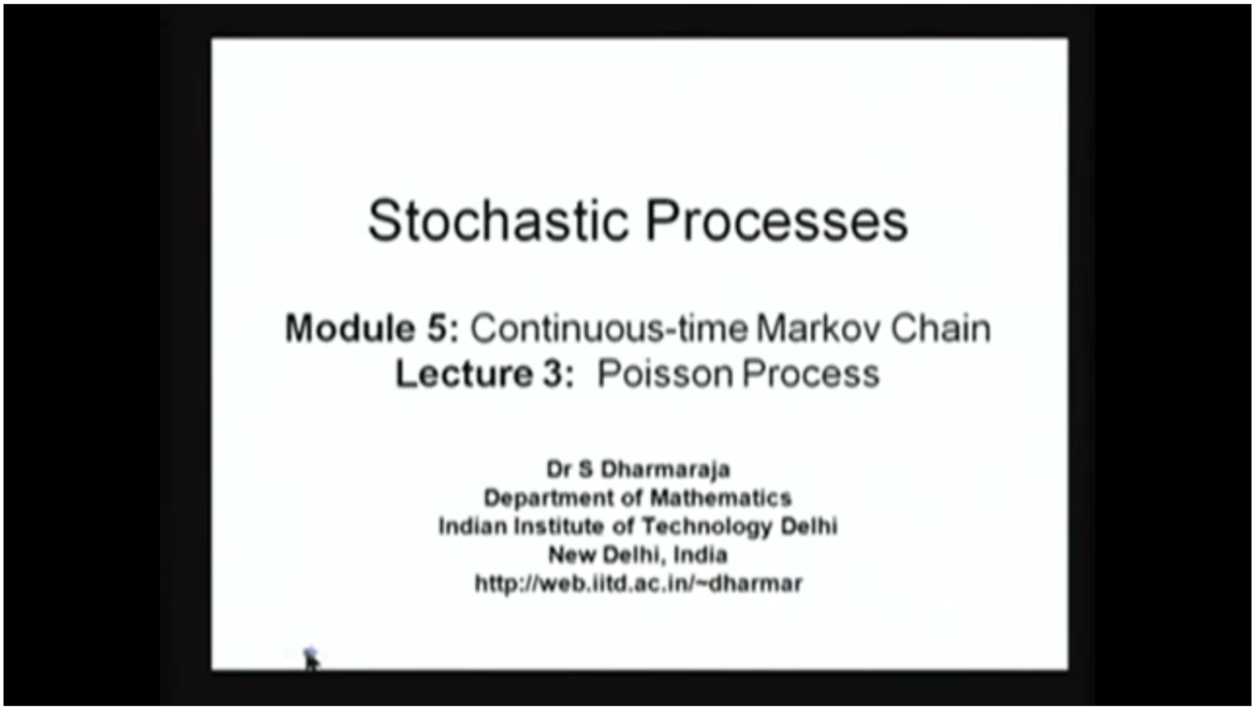


National Programme on

Technology Enhanced Learning

Video Course on

Stochastic Processes - 1

By

Dr. S Dharmaraja

Department of Mathematics, IIT Delhi

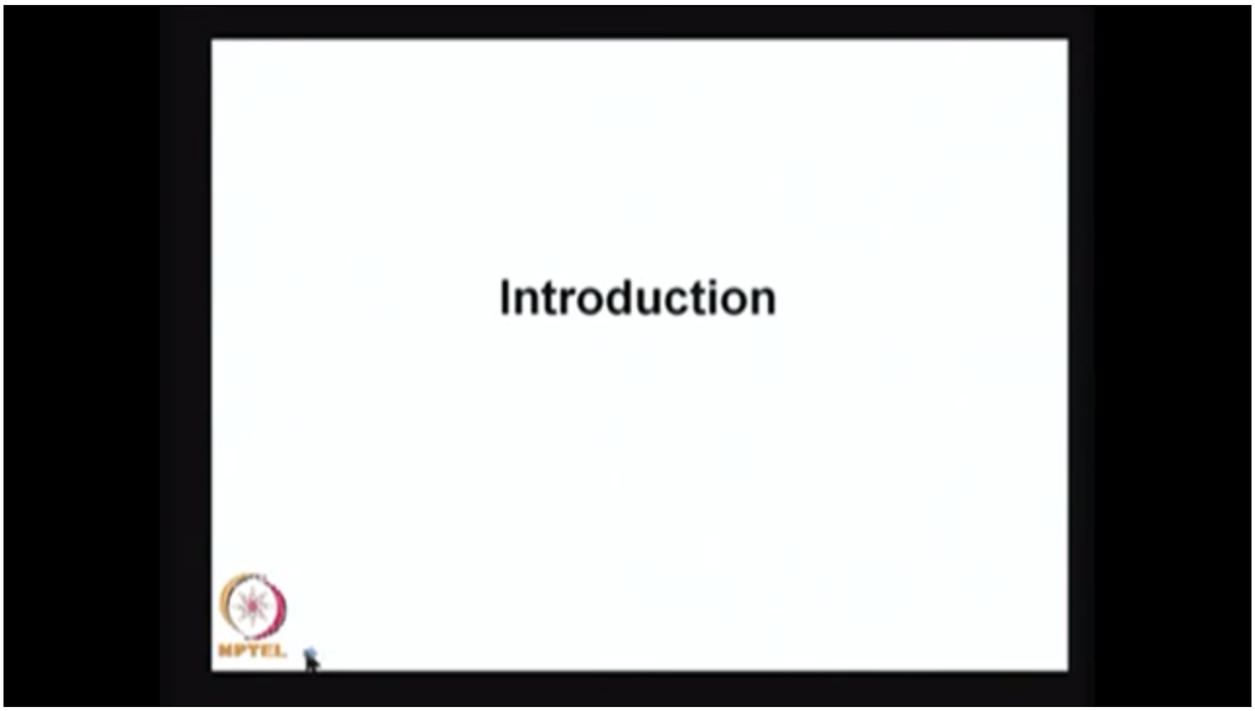
Module 5: Continuous-time Markov Chain

Lecture # 3

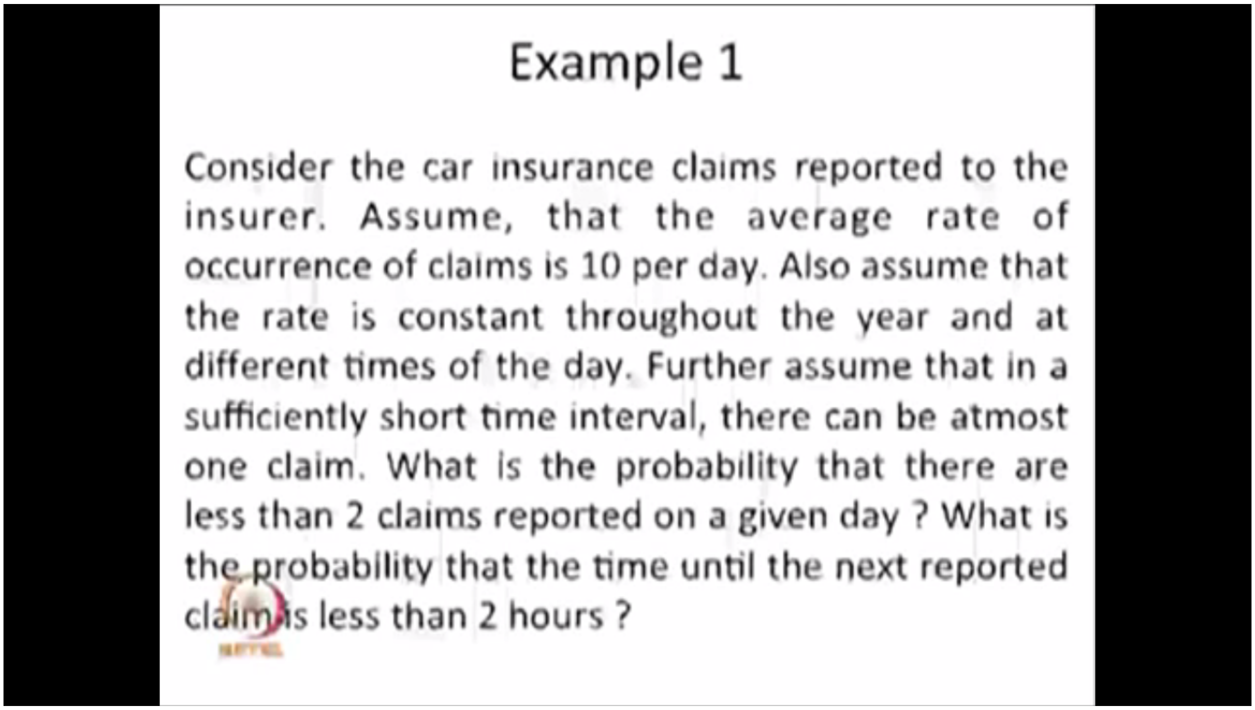
Poisson Processes

This is a model 5 continuous time Markov chain, lecture three Poisson process. In the first two lectures we have discussed the continuous time Markov chain definition, Kolmogorov differential equation, Chapman Kolmogorov equations and infinitesimal generator matrix. Then we have discussed some properties also. In the lecture two we have discussed the birth-death process and their properties and also we have discussed the special cases of birth-death process; pure birth process and death process.

In this lecture we are going to discuss Poisson process and its application. So let me start with the Poisson process definition. Then I give some properties in the Poisson process and I also present some examples.

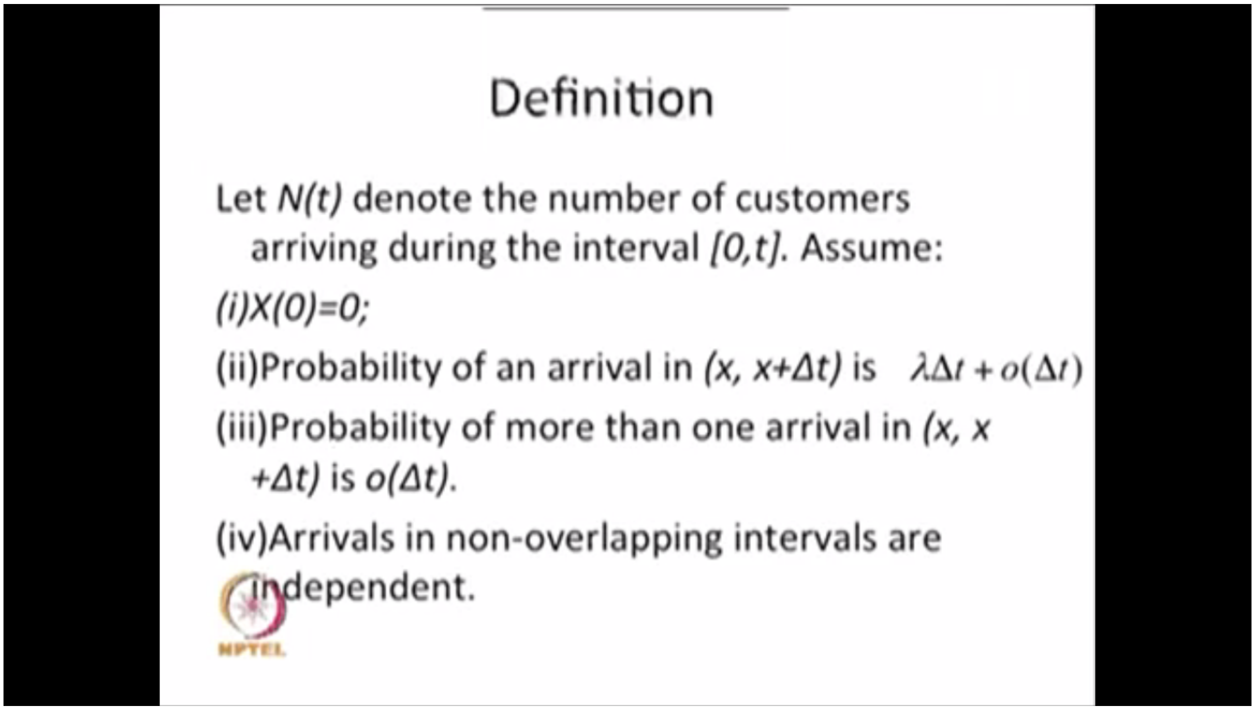


Poisson process is a very important stochastic process. Whenever something happens in some random way occurrence of some event and if it satisfies a few properties then we can model using Poisson process. And a Poisson process has some important properties whereas the other stochastic processes won't be satisfied with those properties. Therefore, the Poisson process is a very important stochastic process for the many modeling in applications like telecommunication or wireless networks or any computer systems or anything any dynamical system in which the arrival comes in some pattern and satisfies a few properties.



So before moving into the actual definition of Poisson process I'm going to give one simple example and through this example I'm going to relate the Poisson process definition. Then later I'm going to solve the same example also. So example number two example one I have something else. Consider a car insurance claims reported to insurer. It may not be car insurance you can think of any motor car or motor insurance or any particular type of vehicle or whatever it is assume that the average rate of occurrence of claims 10 per day. It's average rate per day therefore it's rate. Per day the average rate is 10. Also assumed that this rate is a constant throughout the year and at a different times of a days. So even though this quantity is average quantity there is a possibility someday there is no claim reported at all or there are some day more than some 30-40 claims reported and all the possibilities are there but we make the assumption the average rate is a constant throughout the year at the different times of a day also. Further assume that in a sufficiently short time interval there can be atmost 1 claim. Suppose you think of a very small interval of like 1 minute or 5 minutes or whatever very small quantity comparing to that because here I have given the average rate is a 10 per day. Therefore whatever the time you think of is very negligible. In that the probability of it is sufficiently small interval of time there is a possibility of only maximum one time can be reported. The question is what is the probability that there are less than two claims reported on a given day? What is a probability that less than two claims reported means what's the probability that in a given day either no claim or one claim. Also we are asking the second question what is the probability that the time until the next reported claim is less than two hours. Suppose some time one claim is reported what is the probability that the next time is going to be reported before two hours.

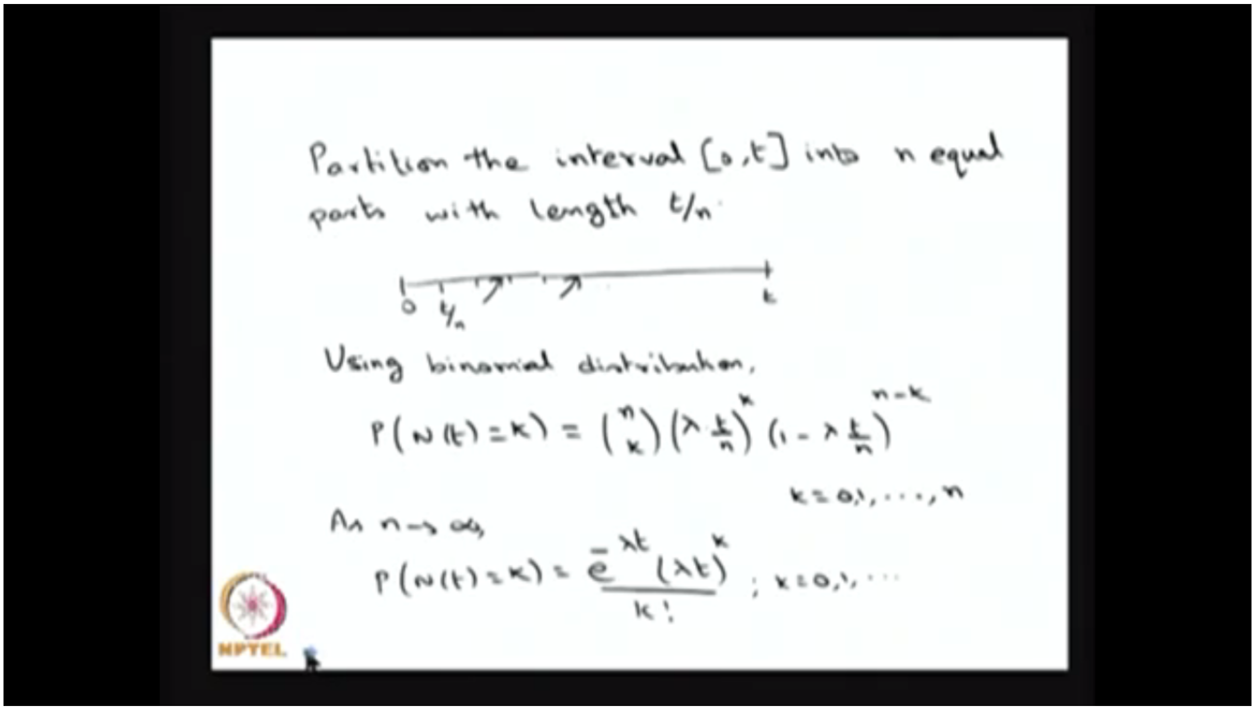
We started with this problem the car insurance claims reported. Therefore the claims is nothing but some event and these events are occurring over the time. Suppose you make the assumption of sufficiently smaller interval of time at most one claim can happen and average rate of occurrence of claims is a constant throughout the time. So with this assumption one we can think of a sort of an arrival process a pure birth process satisfying some condition and that may lead into Poisson process. So the same example we are going to consider it again also.



Now I am going for definition of a Poisson process; how one can derive the Poisson process. Poisson process is a stochastic process with some conditions. So how one can derive the Poisson process. For that let me start with the random variable N of t that denotes the number of customers arriving during the interval zero to time t; that means how many arrivals are takes place in the interval 0 to t. that means F are fixed t N of t is a random variable over the time this is N of t collection that is a stochastic process. I'm making some four assumptions. With these assumptions I am going to conclude the N of t is going to be a stochastic process. The first assumption not X of 0 N of 0 is equal to 0. At time 0 the number of customers is 0, N of 0 is equal to 0. It's wrong enough 0. Second one, the probability of arrival in interval X to X plus delta t that is a lambda times delta t where lambda is strictly greater than 0. That means a probability that only one arrival is going to takes place in the interval of delta t that probability is lambda times delta t for us very very small interval delta t. It's independent of X that means it is increments are stationary; that property I am going to introduce in this assumption.

The probability of more than one arrival in the interval X to X plus delta t is negligible that means atmost maximum on arrival can occur in a very small interval of time that is the assumption that I am specifying in third one. The fourth assumption arrivals in non-overlapping intervals are independent. That means if the arrival occurs in a some interval and another some non-overlapping interval then those are events are going to be from independent. That means there is no dependency over the non-overlapping intervals arrivals going to occur or not.

So with these four assumptions N of 0 is equal to 0 and the probability of one arrival is lambda times delta t in a small interval more than one arrival occurrence in an interval delta t where delta t is very small that is that probability is negligible and the non-overlapping intervals are able or independent.



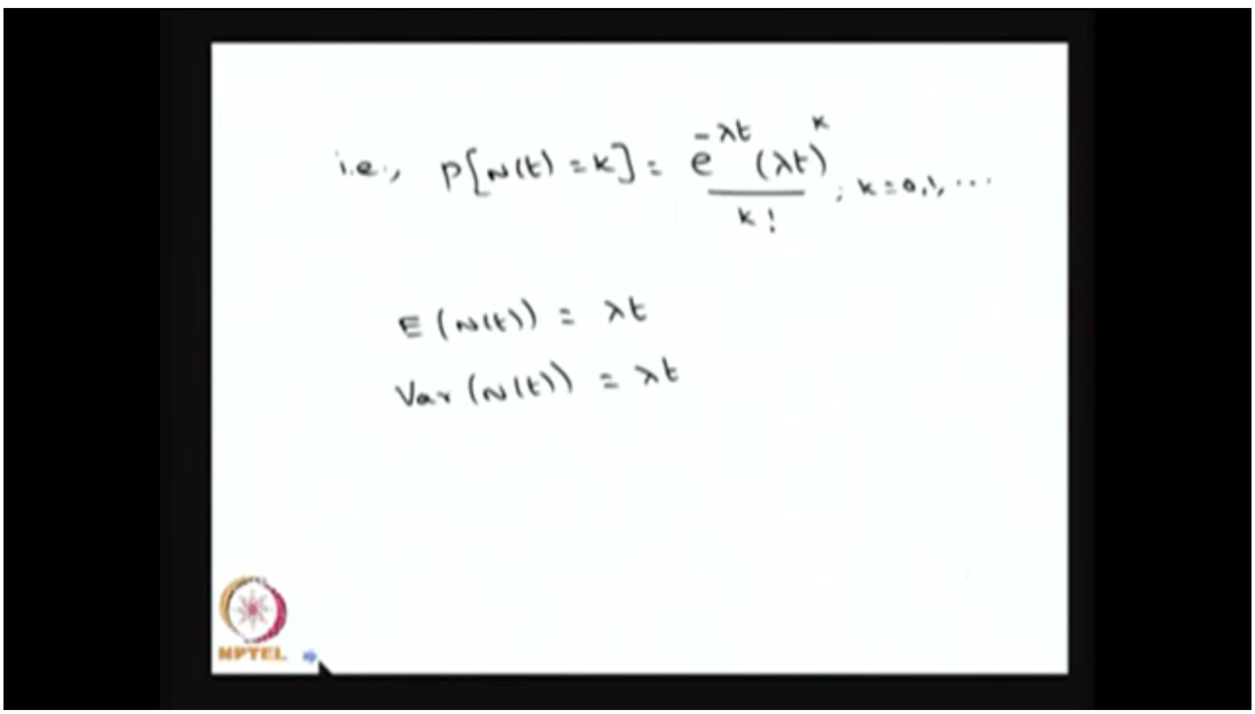
So with this derivation I'm going to find out that distribution of N of t. To find the distribution of N of t first I'm doing I'm partitioning the interval 0 to t into n equal parts with the length t divided by n. The way I use the – the way I partitioned interval 0 to t into n pieces such that t by n is going to be a very small interval so that means I have to partition that interval 0 to t in such a way that the t by n is going to be as small as therefore I can use those assumption of a probability of occurring one arrival in that interval of length t by n that probability is lambda times t by n and the probability of not occurring event in that interval at t by n is 1 minus lambda time t by n. So I can use those concepts, for that I have to partition the interval 0 to t into n parts with a sufficient larger n therefore t by n is going to be smaller.

Now since I partition this interval into n pieces, n parts I can think of yet each parts I can think of a binomial or Bernoulli distribution at each pieces therefore all the non-overlapping intervals occurrences are independent therefore I can think of it is accumulation of n independent Bernoulli trials. Since it is a n independent Bernoulli trials for each interval t by n of length t by n therefore the total number of event occur in the interval 0 to t by partitioning into n equal parts this is sort of what is the probability that K event occurs in the interval 0 to – in the time duration 0 to t as a n partition so out of n equal parts what is the probability that K events occur in the interval 0 to t. That is nothing but since it is a each interval is going to form a Bernoulli distribution with the probability P is lambda times t by n therefore the total number is going to be binomial distribution with the parameters N and P where P is lambda times t by n.

Therefore, this is the probability mass function of K event occurs out of n equal parts. Therefore n is k lambda times t by n power K 1 minus lambda times t by n power n minus K. Now the running index for K goes from 0 to n. That means there is a possibility no event takes place in the interval 0 to t or maximum of an interval n even takes place in all n intervals.

So this is for a sufficiently large n such that the t by n is smaller. We take n tends to infinity to understand the limiting behavior of this scenario as the partition becomes finer. Now I can go for n tends to infinity. What will happen? As n tends to infinity if you do the simplification here as n tends to infinity that simplification I am not doing in this presentation has a limit n tends to infinity the whole thing will land up P power minus lambda t lambda t power K by K factorial. Now the K running index is a 0, 1, 2, and so on. This you can use the concept, the binomial distribution as n tends to infinity and P tends to 0 your n into P becomes a lambda so that will give the Poisson distribution. The limiting case of a binomial distribution is a Poisson distribution. So using that logic this binomial distribution mass has n tends to infinity this becomes a Poisson distribution mass function.

So this is nothing but the right hand side is the probability mass function for a Poisson distribution with the parameter lambda times t and this is a random variable n of t for fixed T. Therefore for fixed T n of t is a Poisson distributed random variable with the parameter lambda times t where lambda is greater than 0.



Therefore we can conclude the stochastic processes related to the N of t for fixed T N of t is a Poisson distribution therefore the stochastic process N of t over the t greater than or equal to 0 that is nothing but a Poisson process. So from the Poisson distribution we are getting Poisson process because each random variable is a Poisson distributed with the parameter lambda times t. Therefore that collection of random variable is a Poisson process with the parameter lambda t. Since it is a Poisson distributed random variable for fixed T, you can get the mean and variance and all other moments also by using the probability mass function of N of t.