INDIAN INSTITUTE OF TECHNOLOGY DELHI

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NPTEL ONLINE CERTIFICATION COURSE

Stochastic Processes - I

Module 5: Continuous-time Markov Chain

Lecture-02

Limiting and Stationary Distributions

Birth Death Processes

With

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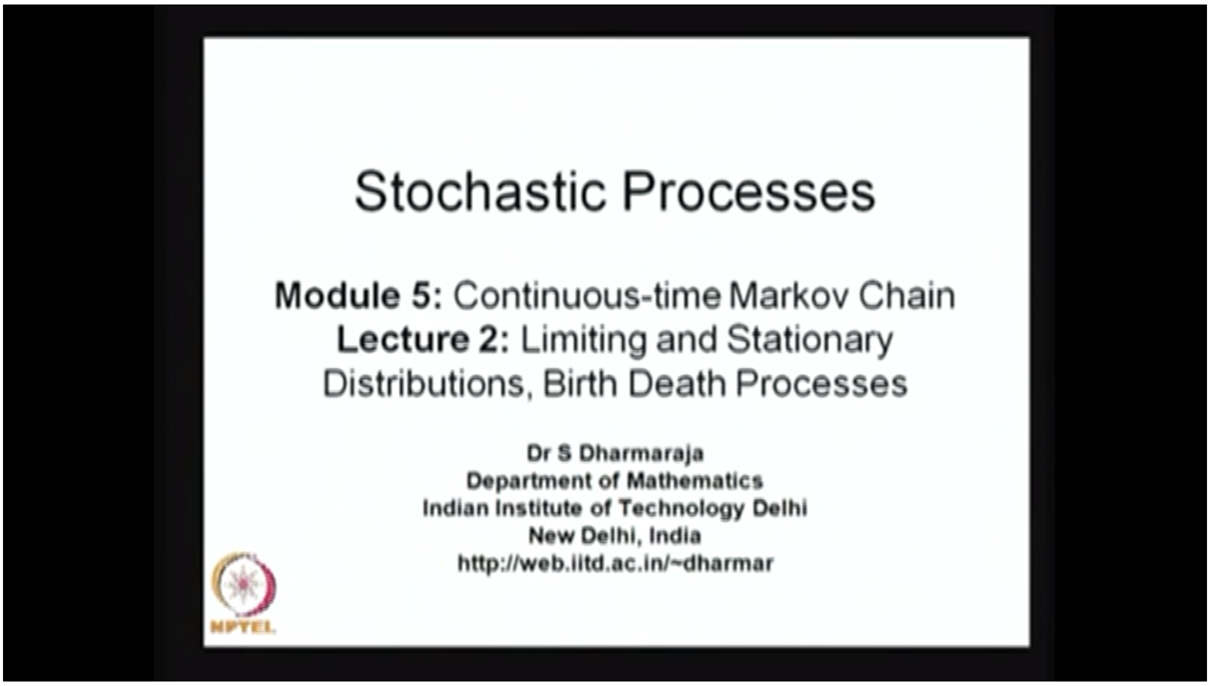
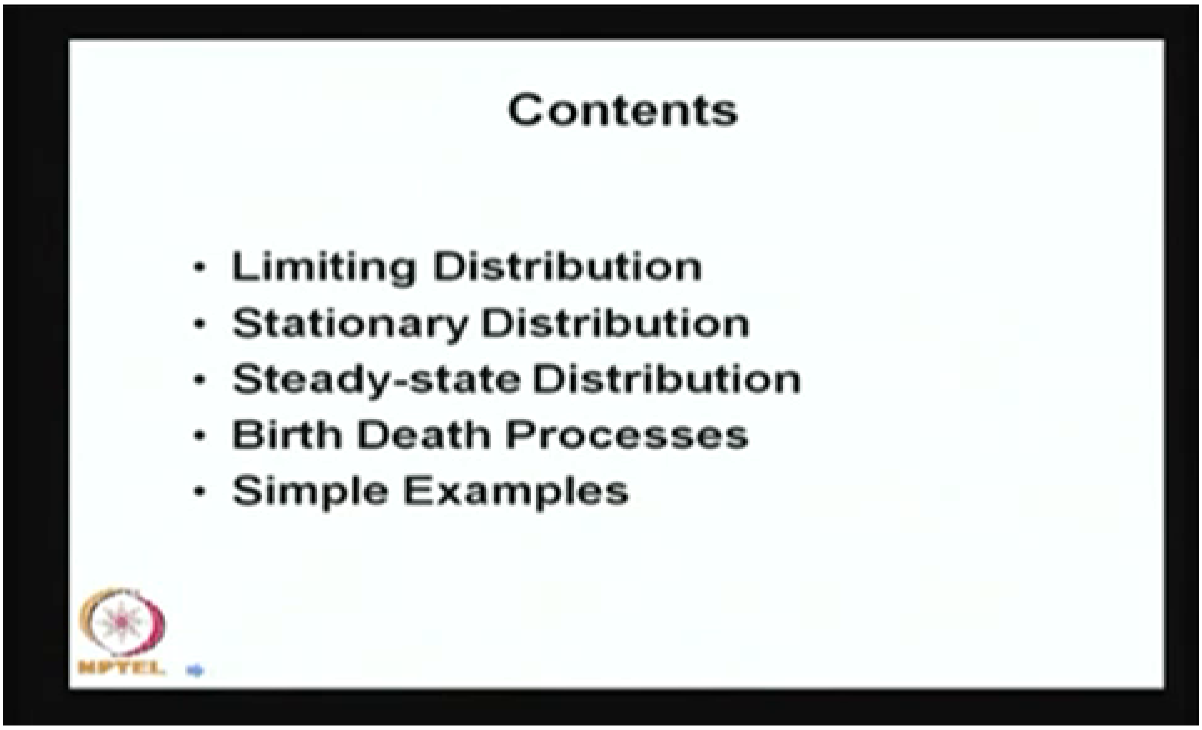
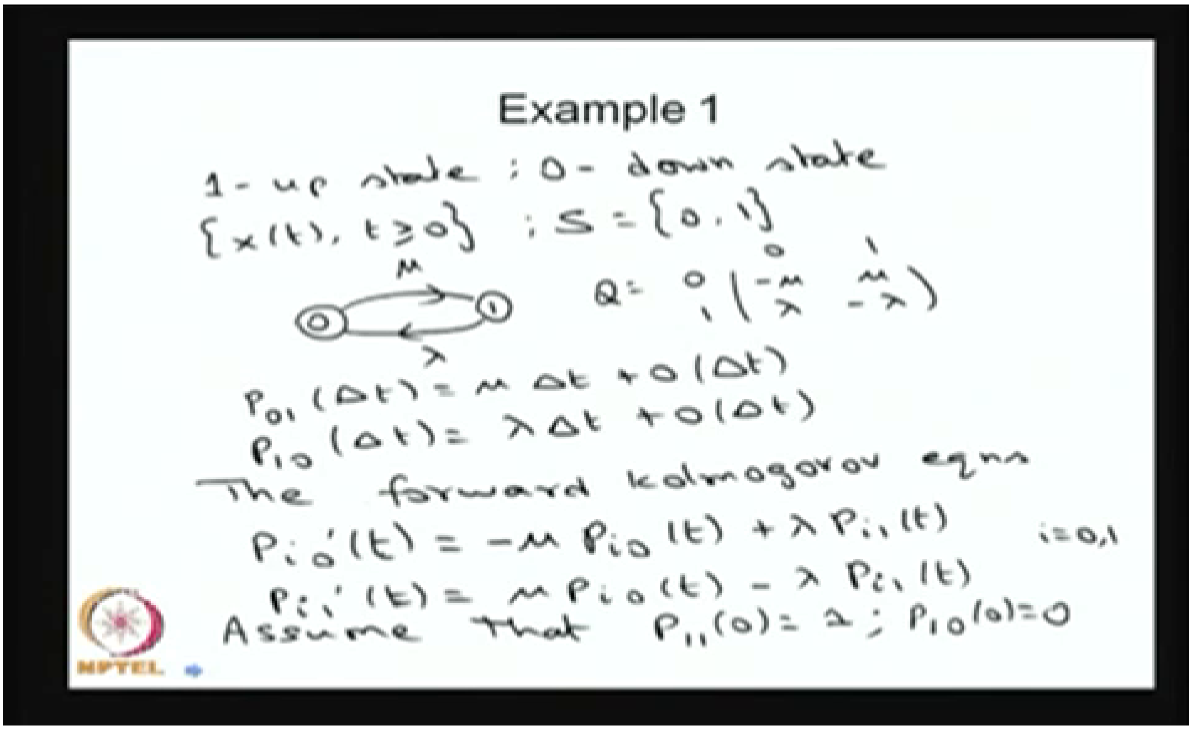


Yeah, this is a Module 5, Continuous-time Markov Chain. In the first lecture, we have discussed the definition of a continuous-time Markov chain. Then we have explained how we can derive the Chapman-Kolmogorov equation, then we have defined infinitesimal generator matrix. Then I have given the Kolmogorov differential equations in the first lecture.



In the lecture 2, I am planning to discuss the limiting distribution, stationary distribution, and steady-state distribution. Followed by that I am planning to give a description about the birth-death processes, and also some simple examples for the limiting distribution, stationary, steady-state distributions and birth death processes.



Before I go to the limiting distribution, let me give the example for the continuous-time Markov chain to get the time dependent solution. This example is the very simplest example that is the two steps continuous-time Markov chain. The default one is the time homogeneous. The state's phase are 1 and 0. 1 you can consider as upstate or operational state and 0 is a downstate non-operation state. So this can be visualized for any model in which the whole dynamics can be described with the two-state and the Markov property is satisfied. The system going from the state 1 to 0 or the time spending in the state 1 before moving into the state 0 that is exponentially distributed with the parameter, λ.

Once it is failed that means the system is in the downstate, the time spent in the repair time that is exponentially distributed with the parameter, μ. So once the repair is over, the system is operation state, therefore it is in the upstate. So 0 is related to the downstate and 1 is related to the upstate and μ is nothing but the mean. 1/μ is the mean time for the repair and 1/λ is the mean time of failure and the failure time is exponentially distributed with the parameter λ and that repair time is exponentially distributed with the parameter μ.

This is a state-transition diagram for the two-state CTMC. The corresponding Q matrix, the infinitesimal generator matrix that it consists of it's a 2x2 matrix. The system going from the state is 0 to 1 that rate is μ. The system is going from the state 1 to 0 that rate is λ and the diagonal values are minus of summation of other values, that row sum. So 0 to 0 is - μ and 1 to 1 is -λ. Therefore, the rates are in the other than diagonal elements and the diagonal elements are minus of sum of the row values, other than that a diagonal element.

So this is nothing but in a very small interval of time, Δt , the system is moving from the state 0 to 1 that probability, the probability of system moving from the state 0 to 1 that is nothing but the downstate to the upstate in a very small interval of time, Δt. Why you are finding the probability of Δt since the model is at time homogeneous; only the interval is matter not the actual time, or you can visualize this as some time t to t+Δt also.

So this is the interval of small, negligible interval Δt, the system is moving from the state 0 to 1 that probability is nothing but the rate, μ is the rate. The rate is nothing but the repair rate. So the mean rate $\mu\Delta t+o(\Delta t)$. It's a small o. Order of Δt d means as $\Delta t \rightarrow 0$, the $o(\Delta t)$ will be 0.

Similarly you can visualize the probability of system moving from the state 1 to 0 in the interval Δt, the small interval Δt that is same as the failure rate, $\lambda\Delta t$ that's a small interval of time plus $o(\Delta t)$. So this $o(\Delta t)$ also tends to 0 as $\Delta t \rightarrow 0$.

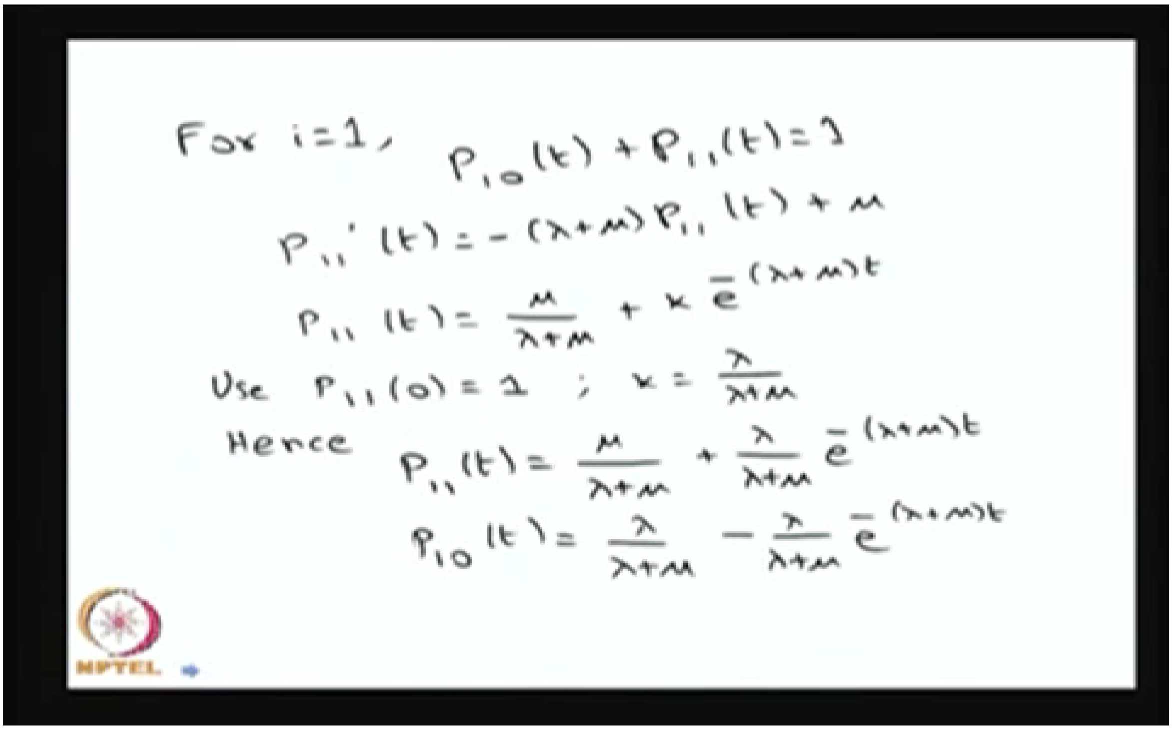
So using this I can make the forward Kolmogorov equation. I can go for writing a forward Kolmogorov equation or backward Kolmogorov equation but forward Kolmogorov equation is easy to make out. So if the system is in the state i at time 0, what is the net rate the system will be the state 1 at the time t? That net rate is nothing but what are all the inflow that probability rate minus what are all the outflows. That's the way you can visualize that right inside. So all the positive terms are related to the incoming rates and all the negative terms related to the outgoing rates. So since it is a two-state model, if the system is in the state 0 at time t, there is a possibility it is not moved anywhere from the state 0 or it would have come from the state 1. Therefore, the incoming will be the state 1, therefore the system will be in the state 1 at time t and given that the starting from the state i that probability multiplied by the rate, sort of inflow, minus, because we are writing the equation for the state 0, therefore it is not moved from the state 0. That is with the rate μ it can move to the state 0 to 1.

Therefore, $-\mu$ times it does not move from the state 0. Therefore, $-\mu$ times the probability of being in the state 0 at time t given that it was in the state i at time 0 that probability multiplied by $-\mu$ that's outflow. And $\lambda P_{i1}(t)$ that's the inflow. Therefore, the left-hand side it's a derivative of the function t. It's a probability function. So $P_{i0}'(t)$ that is nothing but the net rate being in the system at time t in the state 0 given that it was in the state I at time 0. That net rate is same as a inflow minus outflow with the corresponding rates.

Similarly, you can write the equation for the state 1 that means you start from the state 1. Either you would have come from the state 0 to the 1 or you didn't move from the state 1. Therefore, $-\lambda P_{i1}(t) + \mu P_{i0}(t)$ that is the net rate corresponding to the state 1. So now we are able to write the forward Kolmogorov equation.

So this is the interpretation of the forward Kolmogorov equation. You can write easily by making a matrix a $P_{ij}(t)'$ that is equal to $P(t)Q$ where Q is the infinitesimal generator matrix. Then also you will get the same thing. So I am just giving the interpretation.

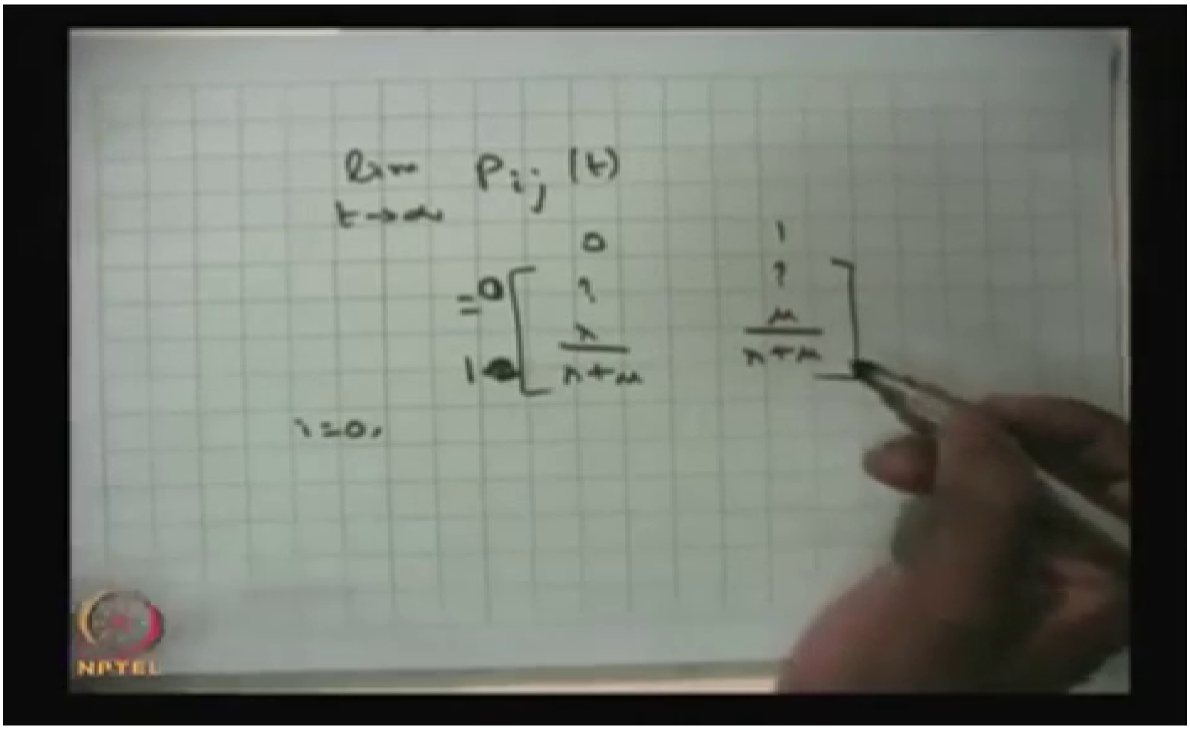
Now my interest is to find out the time dependent or transient solution for these two-state CTMC. For that this is a differential equation. We need initial condition to solve these equations. So I make the assumption at time 0 the system is in the state 1. Therefore, the transition probability of system the $P_{11}(0)$ that is equal to 1. Since I made the assumption the system was in the state 1 at times 0, therefore that being in the state 0 that is going to be 0. So I need both the initial conditions to solve the equation.



So let me start since I made the initial condition state is 1 therefore i is equal to 1. So I'll have the first equation that is I always have the summation of the probability at time t, these transition probabilities are going to be 1, the summation. And also I have two difference of differential equations. So what I can do I can take the second equation in this. Then instead of $P_{10}(t)$ I can use a summation of probability is equal to 1. Therefore, instead of $P_{10}(t)$ I can use the $P_{10}(t)$ is nothing but 1- $P_{11}(t)$. I can substitute in the second equation therefore I get $P_{11}'(t)$ is equal to $-\lambda+\mu P_{11}(t)+ \mu$.

Substituting $P_{10}(t)$ is equal to 1- $P_{11}(t)$ in the second equation, the previous slide. Now I have to solve these differential equations. The unknown is $P_{11}(t)$. Conditional probability, I have to use the initial condition, $P_{11}(0) = 1$ using that I can get $P_{11}(t) = \mu$ divided by $\lambda+ \mu$ plus some constant, $e^{-(\lambda+ \mu)t}$. That constant I can find out using this initial condition. Therefore, k is equal to λ divided by $\lambda+ \mu$. So the $P_{11}(t)$ is equal to substituting $k = \lambda/\lambda+ \mu$ in this equation, I'll get the $P_{11}(t)$. Once I know the $P_{11}(t)$, use the first equation. So I will get $P_{10}(t) = 1- P_{11}(t)$. Therefore, $P_{10}(t)$ that is equal to this expression.

You can cross check now. If you add both the equations, you will get a 1 and if you put $t = 0$ you will get the initial condition also correctly, and if you put $t \to \infty$ that we are going to discuss in the limiting distribution, if you put $t \to \infty$ in this expression, you will get $\mu/ \lambda+\mu$, $\lambda/ \lambda+\mu$. So this is for the $t \to \infty$. Therefore, if you make a matrix, the limit tends to infinity of the element.

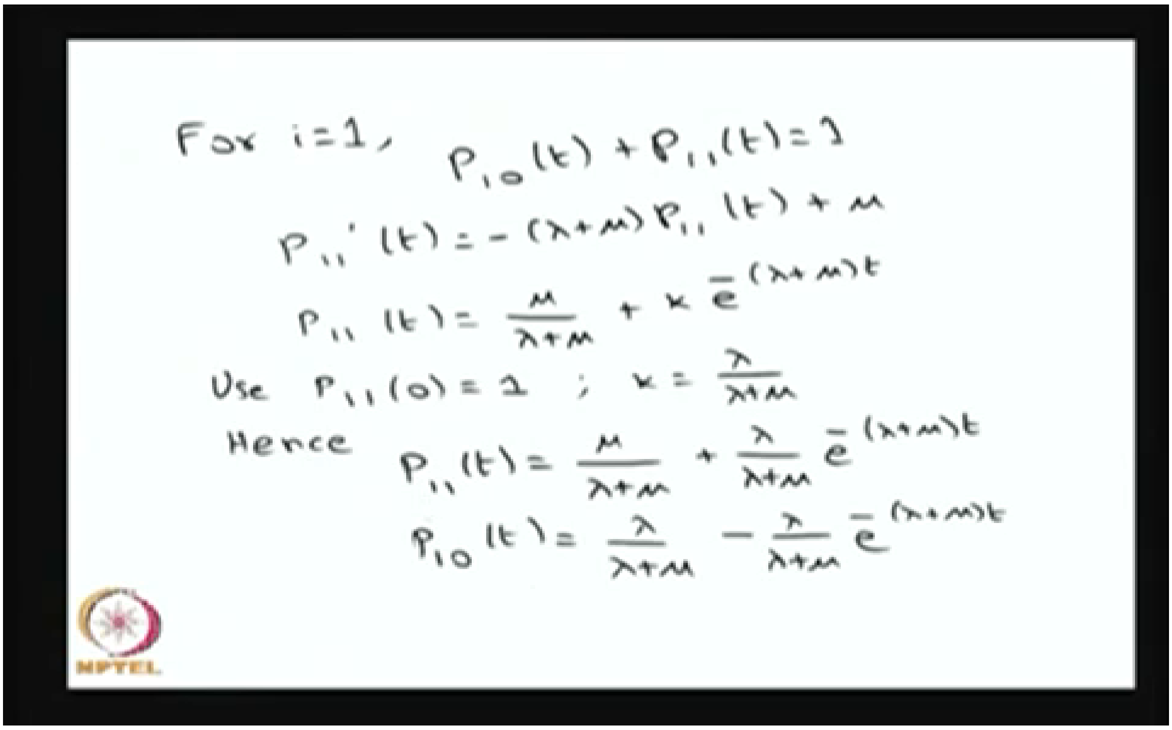


If you find out the limiting distribution of a limit, $t \rightarrow \infty$ of P(t). So you will get the matrix and this matrix has a $t \rightarrow \infty$ for this example. It's a 2x2 matrix and that consists of for different values, you will have -- for now you are doing for the second row therefore, that is equal to λ/ λ+μ, and this is equal to μ/ λ+μ. So if the system start from the state 1 at the $t \rightarrow \infty$ the system will be in the state 0 with the probability λ/ λ+μ and the system will be in the state 1 with the probability μ/ λ+μ

Similarly, if you go for i=0 you will get the same derivation and you can fill up what is the element here.

$$\begin{bmatrix} \frac{\lambda}{\lambda+\mu} & \frac{\mu}{\lambda+\mu} \\ \frac{\lambda}{\lambda+\mu} & \frac{\mu}{\lambda+\mu} \end{bmatrix}$$

So this is the limiting distribution probability matrix and if you see that the rows are going to be identical. So you will have the same identical rows in this row also. So that means you will get the limiting distribution. I will discuss the limiting distribution after giving one more example I will explain in detail.



So this is the transition probability; system starting from the state 1 and being in the state 1 or 0 at the time t.

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