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NPTEL ONLINE CERTIFICATION COURSE

Infinitesimal Generator Matrix

Lecture-01

With

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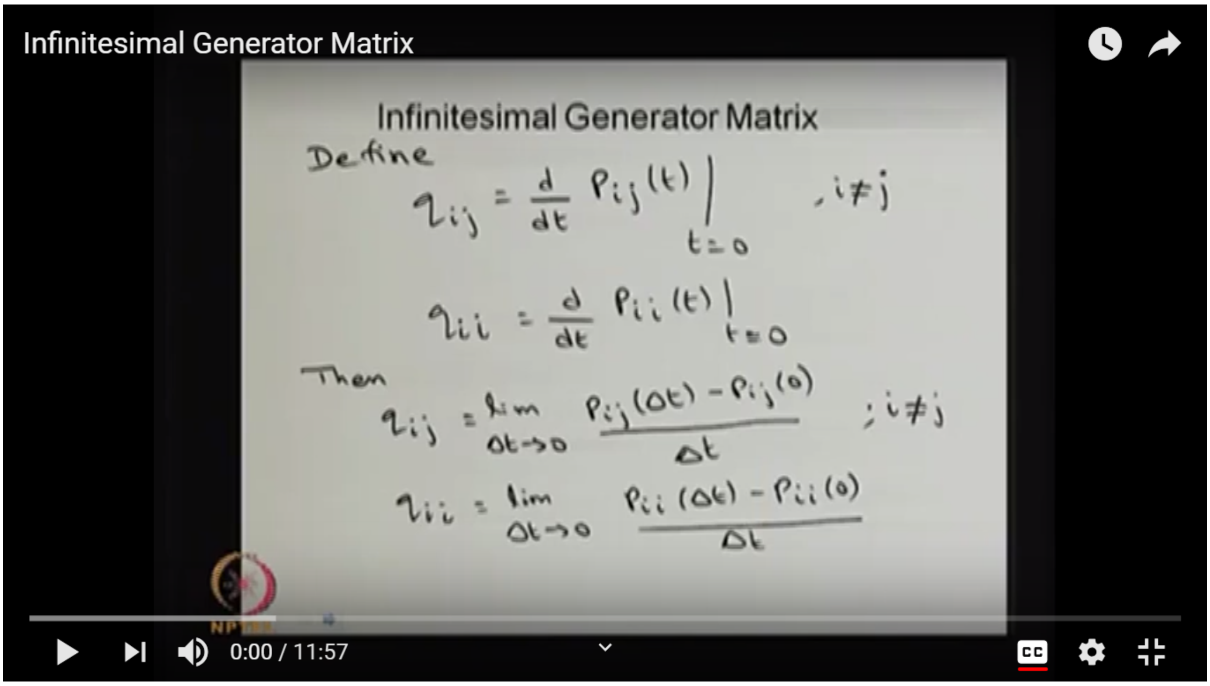
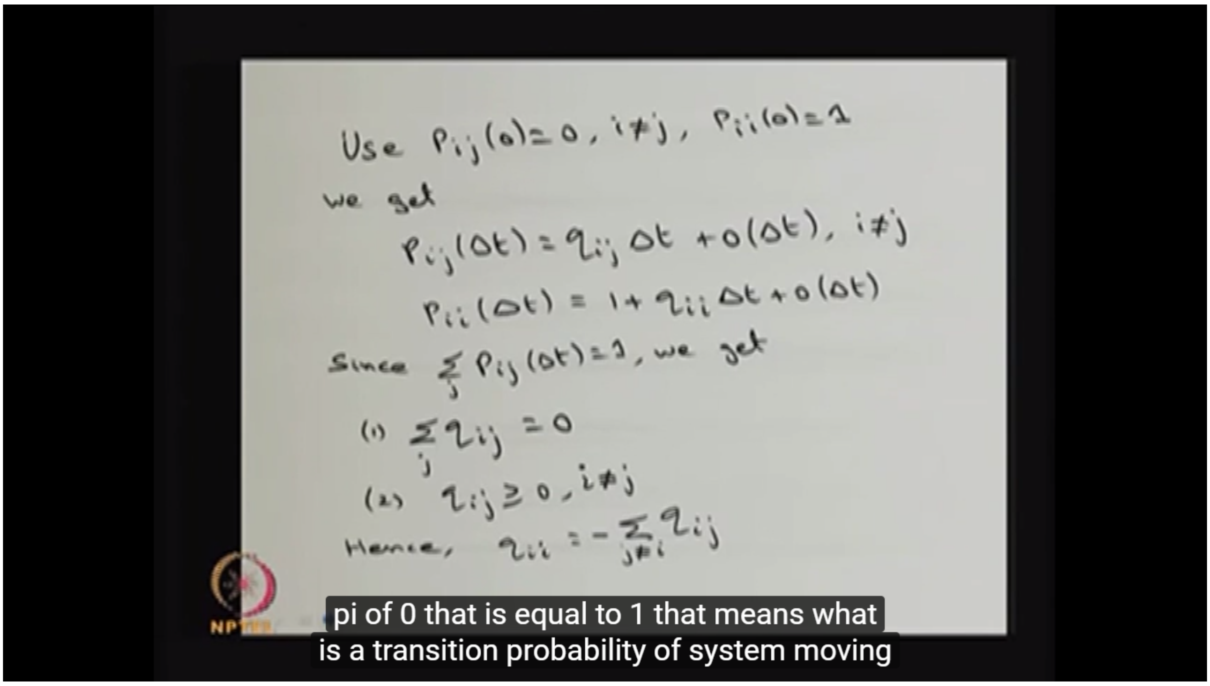


I am going to define the quantity called q_{ij} and later, this is going to form a matrix that is going to be called it as infinite decimal generator matrix. So let me start with the definition, q_{ij} that is nothing but take a derivative of $P_{ij}(t)$. There's a function of t, you can find out the Derivative. It is differentiable function only. So you take a derivative; then substitute $t = 0$ for all $i \neq j$. Then you define q_{ii} that's also in the same way separately, because the q_{ii} the diagonal element is going to be different from all other elements, therefore I am defining separately.

You know how to find out the derivative; derivative of $P_{ij}(t)$ with respect to t that is nothing but the limit, $\Delta t \rightarrow 0$, the difference divided by the Δt. Since $P_{ij}(t)$ is a transition probability of system moving from i to j, you can use $P_{ij}(0) = 0$ for $i \neq j$ for $j = i$ that is $P_{ii}(0)$ that is equal to 1.

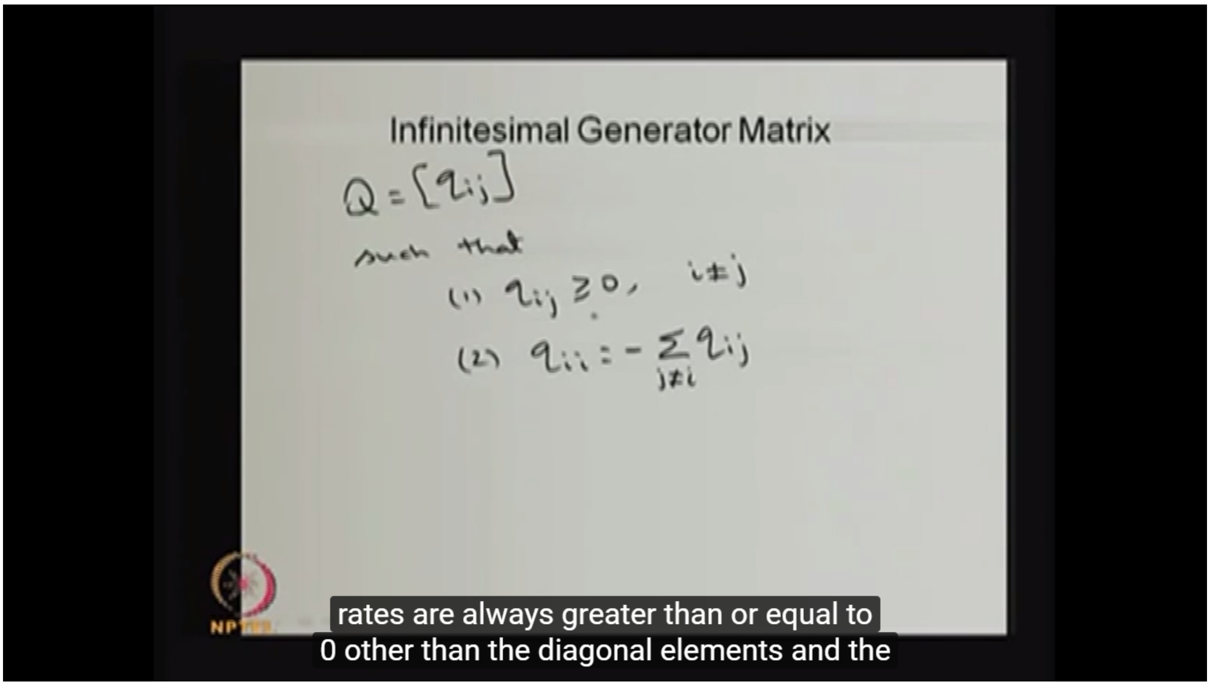


That means what is the transition probability of system moving from the state i to i in the interval 0 that is same as 1. That probability is 1. So use these in the previous limit in this $P_{ij}(0) = 0$ and $P_{ii}(0) = 1$, you substitute then the limit, $\Delta t \to 0$. Therefore, the $P_{ii}(\Delta t)$ this will go to this side. So q_{ij} times Δt, therefore this is going to be $P_{ii}(\Delta t)$ is nothing but the q_{ij} multiplied by Δt plus small order of Δt. That means as Δt tends to 0, this whole quantity will tends to 0.

Similarly you substitute $P_{ii}(0) = 1$ here, therefore $P_{ii}(\Delta t)$ that is same as a this will come to this side, so $1+q_{ii}(\Delta t)$+order of Δt . So this order of Δt this also tends to 0 as Δt tends to 0. You know that the summation of P_{iji} even at the time point Δt , a small negligible time point, Δt at the time also over the I that is equal to 1. Therefore, if you sum it up, you can conclude the left-hand side is a probability. Right-hand side for $i \neq j$ you have q_{ij} whereas that second expression, you have $1+q_{ii}$ therefore using the property of summation of $P_{ii} = 1$, you will get the summation of q_{ij} for all j that is going to be 0.

When you add both the equations for all j, you will get the summation over j, $q_{ii} = 0$ as well as all the q_{ii} quantities are going to be greater or equal to 0 from the first one, because the left-hand side is a probability and this is multiplied by the Δt and Δt is always greater than 0. Therefore, the q_{ii} is going to be greater than 0 for all $i \neq j$ whereas if add over all the j that is going to be 0. Therefore, you will get the q_{ii} that is nothing, but you make the summation for all q_{ij} for j except i then you make a minus sign. So that is going to be the q_{ii} That means the diagonal element is nothing but make the row sum except that the diagonal term and put the minus sign that is going to be the diagonal term.

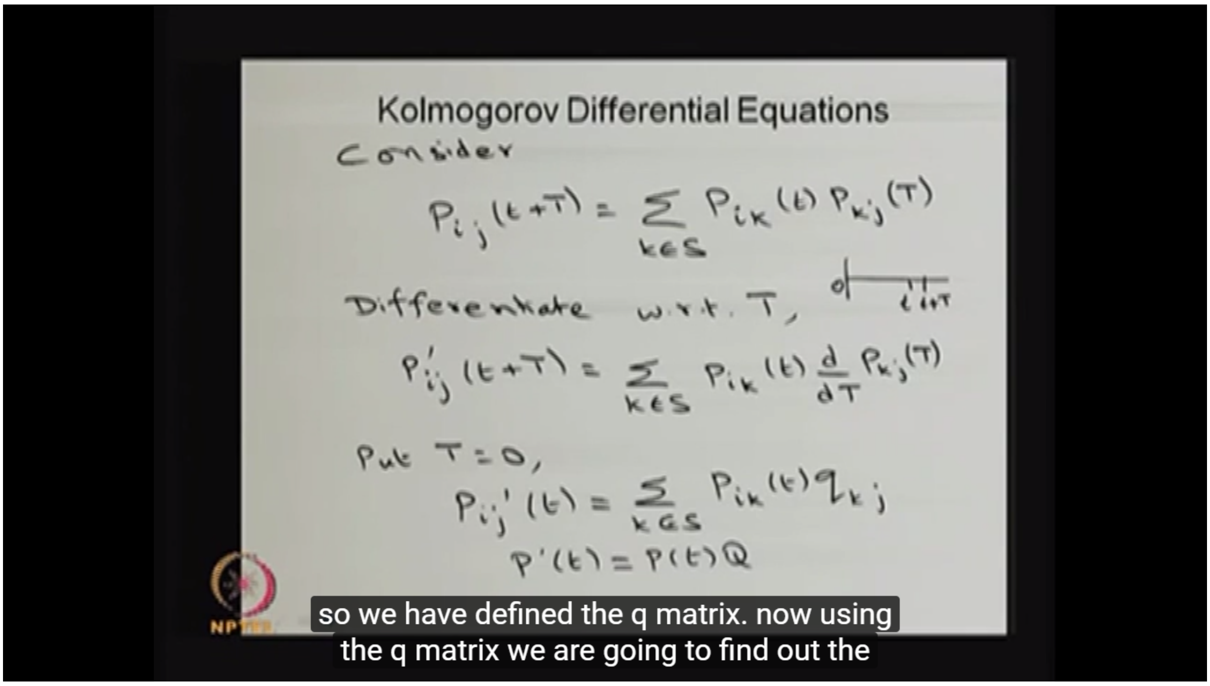
Therefore, when you make a row sum that is going to be 0. The details of the proof can be found in the reference books. so the quantity q_{ij} that has the property, the row sum is going to be 0 and other than the diagonal elements are greater than or equal to 0 therefore the diagonal element is going to be summation of all the other terms with the minus sign.



So using this we can make a matrix and that is going to be Q matrix with the entities q_{ij} such that satisfies the property q_{ij} is always greater or equal to 0 for $i \neq j$ whereas the diagonal element is minus of summation. Therefore, as the property, the row sum is going to be 0.

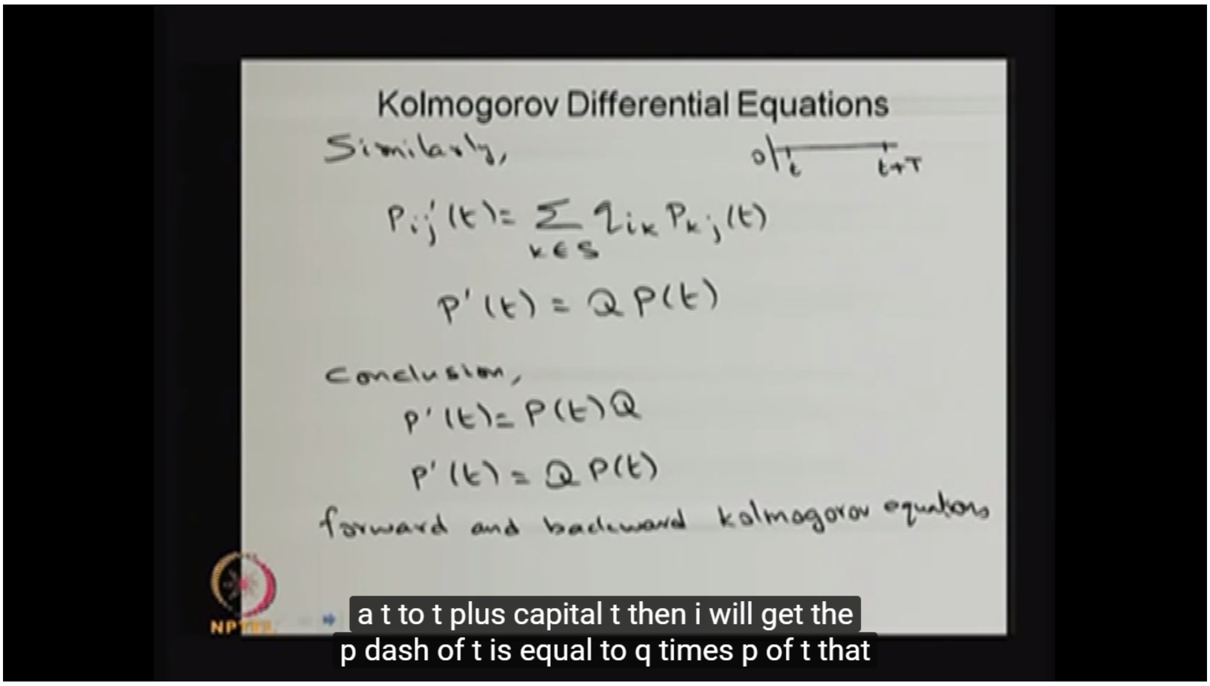
So the difference between this matrix and the one step transition probability matrix in the DTMC that's a probability matrix. So the entries are probability values from 0 to 1 and the summation, row sum is 1, whereas here, because q_{ij}s are obtained by differentiating the P_{ij}s. These are all the rates and these rates are always greater or equal to 0 other than the diagonal elements and the diagonal elements are minus with the summation of all other row elements. So this matrix is called infinitesimal generator matrix. Some books they use the word, rate matrix also whereas here, the rates are placed in the other than the diagonal elements, and the sum of the rates could be 0. That means that the probability of a system moving from that particular state to that particular state is not possible. That probability is 0 or there is in a small interval of time, there is the transition is not possible.

So whenever the rates are greater than 0 that means that there is a positive probability that the system can have a transition of system moving from i to j.

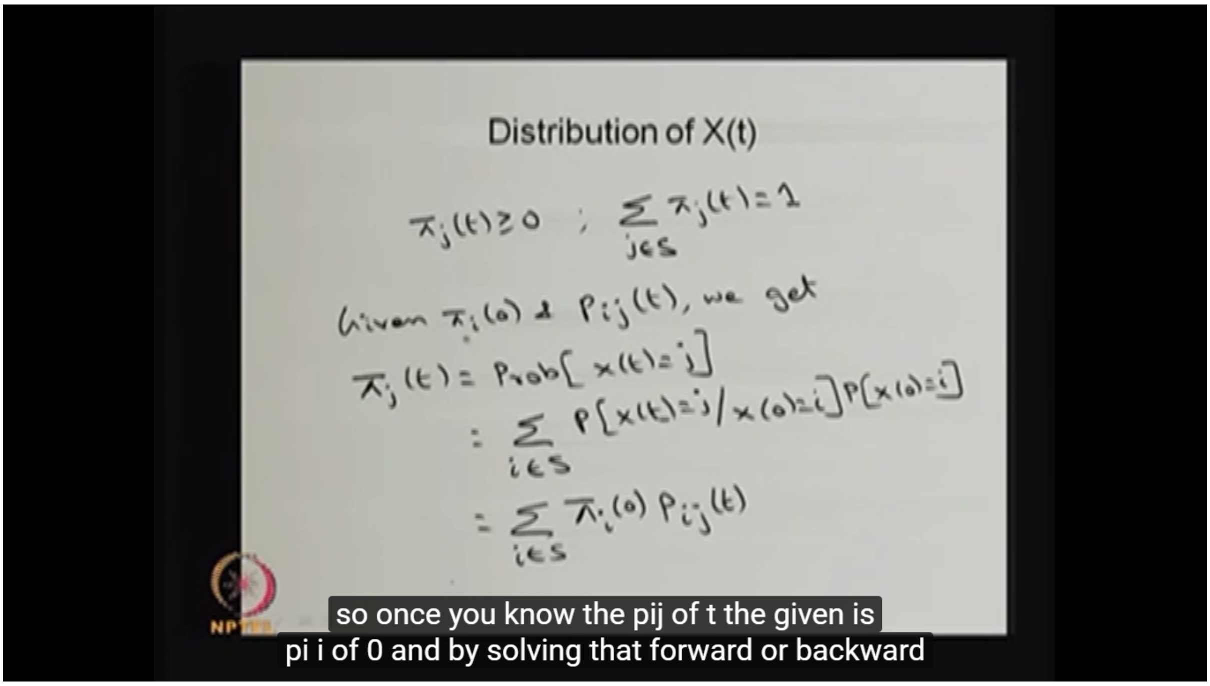


So we have defined the Q matrix, now using the Q matrix we are going to find out the $P_{ij}(T)$. So let me start with the Chapman-Kolmogorov equation. Now I am going to differentiate with respect to T that means I make the interval 0 to t +T as a 0 to t then I make a t to t+T. Differentiate with respect to T, therefore the left-hand side is going to be I have written with a dash (')so the derivative comes inside the $P_{kj}(t)$. Then I am substituting T=0. So basically I am making a system to move from state 0 to small t, then there is a small interval of time from t to t+T. That's an interpretation of this. Then substituting T=0, the left-hand side is going to be $P_{ij}'(t)$ that is same as the summation over this whereas this is nothing but the way we have defined the infinitesimal generator matrix entities. So this is nothing but the q_{kj} that is the rate in which the system is moving from the state k to j.

In a matrix form, I can make it as $P_{ij}(t)$ is going to form a matrix. So the P'(t) that is same as a P(t)Q. So this is the matrix and the P(t) is also matrix and this is the P'(t) means each entities are differentiated with respect to time t. So this is in the matrix form and this equation is called the forward Kolmogorov differential equation because the derivation goes from 0 to T then t to t+T we are considering as a very small interval of time. Therefore, this equation is called a forward Kolmogorov differential equation.

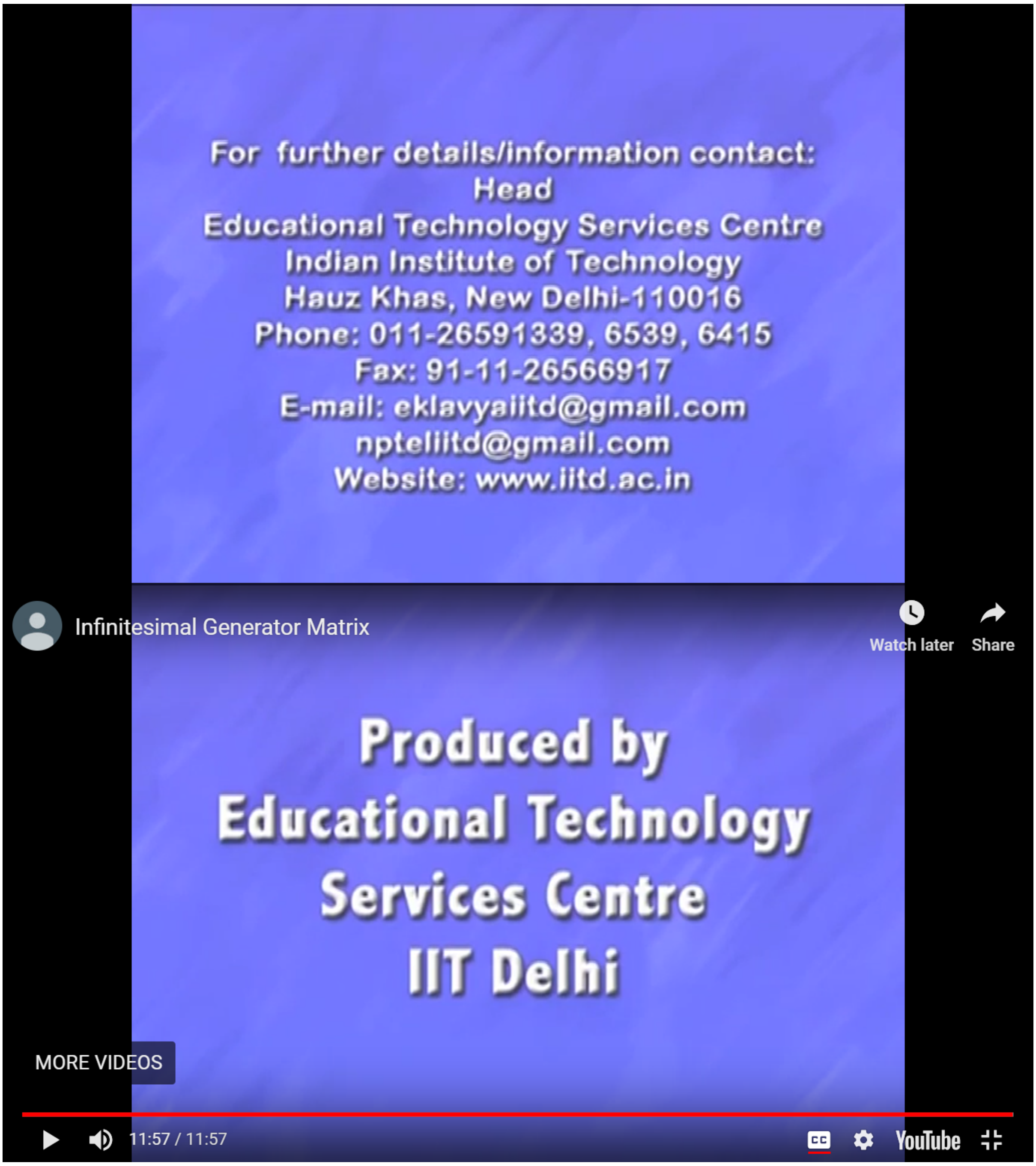


The same way if you do 0 to t that has a small interval of time and t to t+T then you'll get the P'(t)=QP(t) that is called the backward Kolmogorov differential equation. Whether you frame a forward equation or a backward Kolmogorov equation, if you solve that equation you will get the $P_{ij}(t)$. If you solve P'(t)=P(t)Q that's a forward equation. P'(t)=QP(t) that's a backward equation. If you solve the equation with the initial condition because it's a differential equation. So you need initial condition; what is the probability, what is the transition probability of system moving from i to j at time 0. If you know the initial condition by supplying that solving this equation, you will get the $P_{ij}(t)$. Once you know the $P_{ij}(t)$, then you get the distribution of X(t).



So once you know the $P_{ij}(t)$ the given is $\pi_i(0)$ and by solving that forward or backward Kolmogorov differential equation, you will get the $P_{ij}(t)$. Using these two you can get the $P_{ij}(t)$. So for a given $\pi_i(0)$ and $P_{ij}(t)$ that means the transition probability and the initial state probability vector, one can find out the distribution of X(t). So in this lecture, I have started with the Markov process. Then I have discussed the definition of a continuous-time Markov chain and also I have given what is the distribution of time spending in any state before moving to any other state and also I explained the infinitesimal generator matrix and using that how to find out the transition probability of $P_{ij}(t)$ from the Chapman-Kolmogorov equation and we got a forward as well as the backward Kolmogorov differential equations.

By solving a forward or backward Kolmogorov differential equation one can get the $P_{ij}(t)$. That's the transition probability. Using this equation you can get the $\pi j(t)$ that is nothing but the distribution of X(t). With this let me stop this lecture and the next lecture I will go for a simple example of a continuous time Markov chain as well as the stationary, limiting distribution, and the steady state distribution in the next lecture.



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