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Chapman-Kolmogorov Equation

Lecture-01

With

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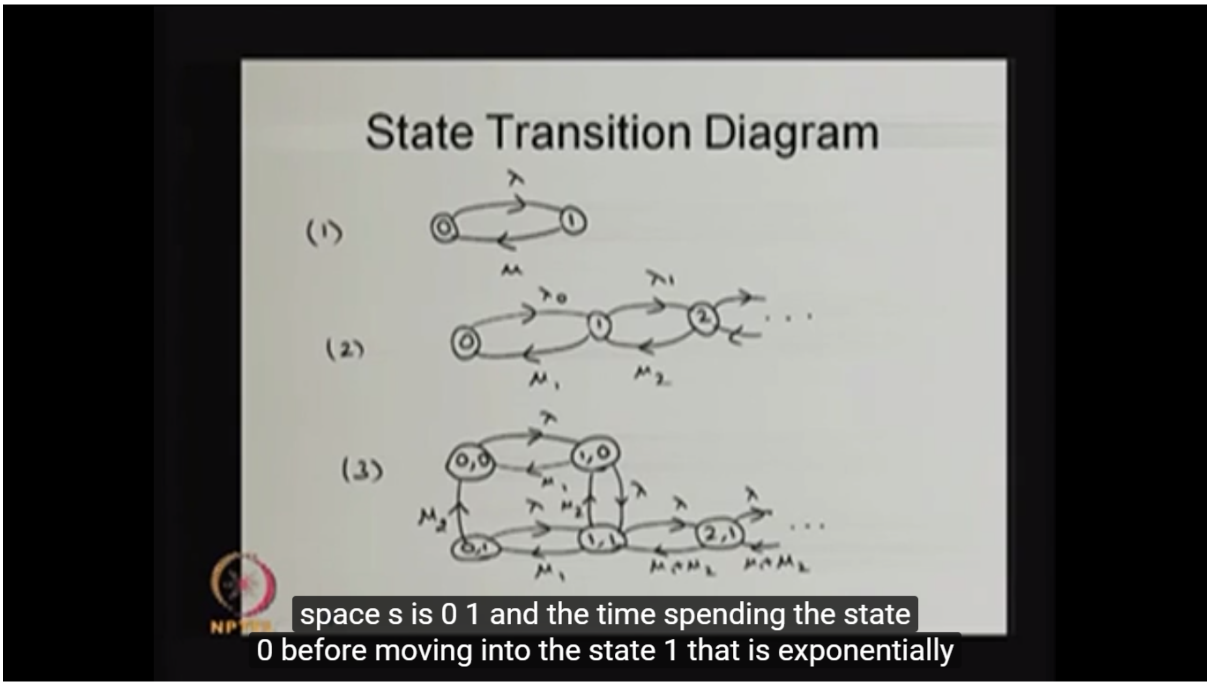
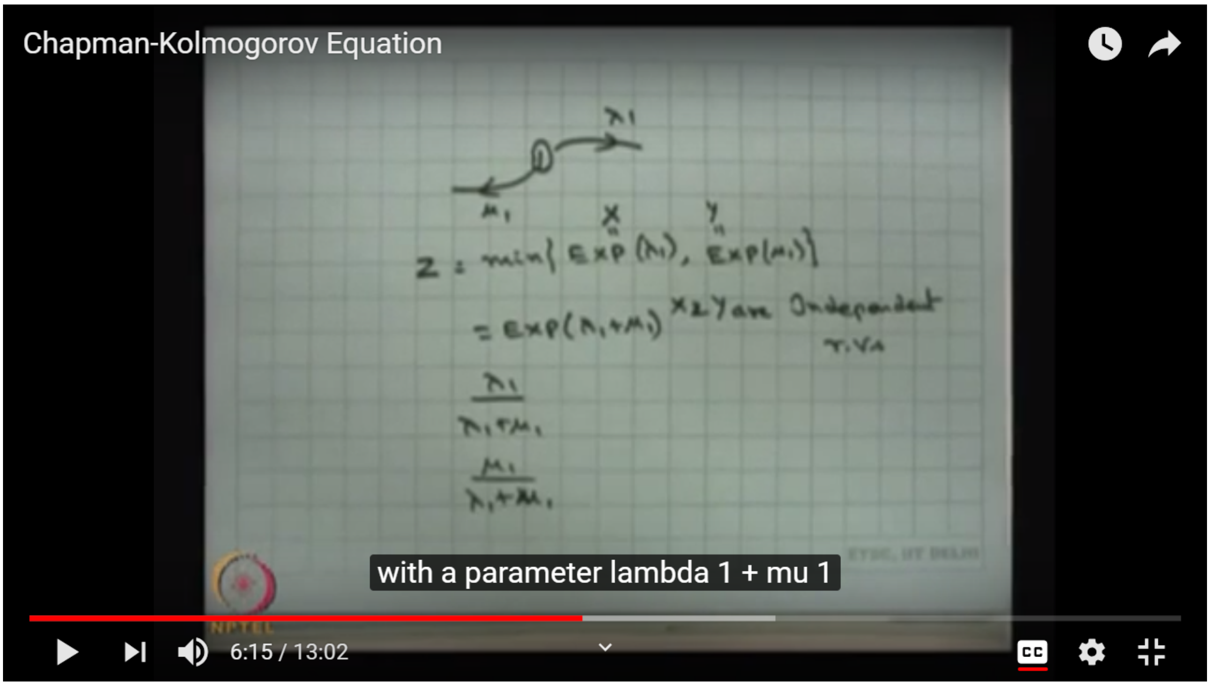


Now I'm going to give few state transition diagrams for the time homogeneous continuous time Markov chain. You see the first example, it has only two states; 0 & 1. So the state space s is 0,1 and the time spent in the state 0 before moving into the state 1 that is exponentially distributed with the parameter, λ. Once the system come to the state 1, the time spent in the state 1 before moving into the state 0 that is exponentially distributed with the parameter, μ. Λ is strictly greater than 0 and the μ is also strictly greater than 0. That means you know the exponential distribution has the mean, 1 divided by the parameter. Therefore, the average time spent in the state 0 before moving into the state 1 that is 1 divided by λ. The average time spent in in the state 1 before moving into the state 0 that is 1 divided by μ. Since it is a two state, so over the time, the system will be in the state 0 or 1 and you can classify the states also the way we have discussed in the continued discrete-time Markov chain. Since both the states are communicating, both the states are accessible from each other direction. Therefore, both the states are communicating each other. Since the state space is 0 and 1, and both the states are communicating each other, therefore this is an irreducible Markov chain. For irreducible Markov chain, all the states are of the same type.

For a finite Markov chain, we have at least one positive recurrent state. Therefore, both the states are going to be a positive recurrent state. But here, there is no periodicity for the continuous-time Markov chain. Therefore, we can conclude the first example, both the states are positive recurrent and the Markov chain is irreducible Markov chain. So the continuous amount of time system spending in state 0 and 1 that is exponentially distributed with the parameters which I discussed earlier.

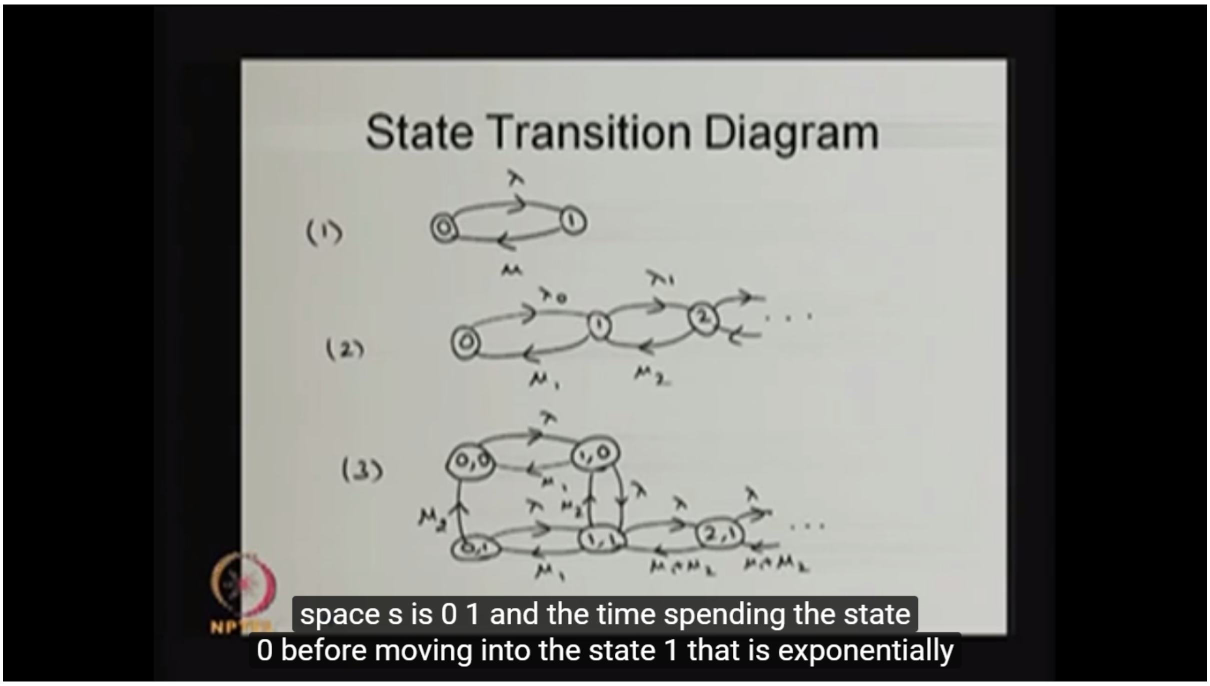
Now I'm moving into the second example. In the second example, we have a state space as a countably infinite and the system spending in this state 0 before moving into the state 1 that is exponentially distributed with the parameter $\lambda_{0,}$ whereas in the state 1, the system can spend exponential amount of time, the amount of time spending in the state 1, before moving into the state 2, that is exponentially distributed with the parameter, λ_1 and similarly, the system spending in the state 1 before moving into the state 0 that is exponentially distributed with the parameter μ_1. Therefore, this is μ_1 and this is λ_1.



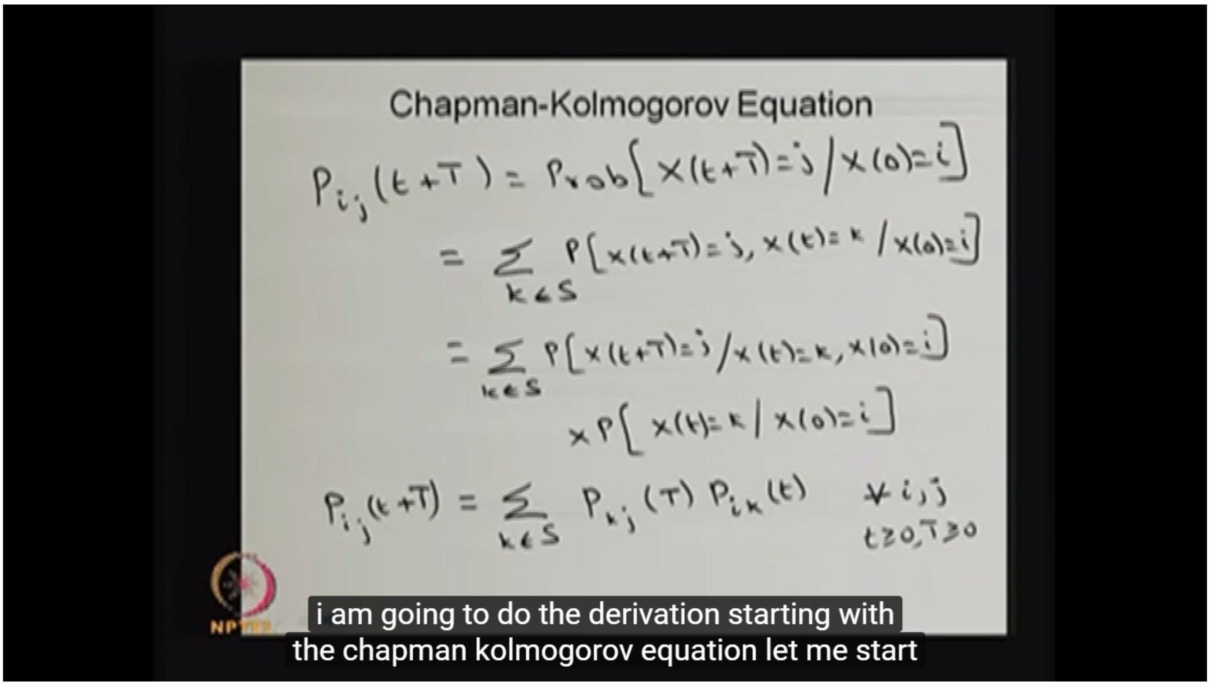
Therefore, the time spending in the state 1 before moving into any other state that is going to be minimum of the exponentially distributed with the parameter, λ_1, one random variable. You can call it as x and you can call it as another random variable that is a exponentially distributed with the parameter, μ_1. Therefore, the amount of time spending in the state 1 before moving into any other state that just now we have concluded that waiting time distribution is exponentially disturbed. That will come from here also.

So here, these two random variables are independent; x and y are independent random variables. Both the random variables are independent. Therefore, the time spending in the state 1 before moving into any other state that is going to be minimum of the random variable with the exponentially distributed parameter, λ_1, and the random variable which follows exponential distribution with the parameter, μ_1. You know that the minimum of two exponential as long as both the random variables are independent random variable, then this is also going to be exponential distribution with the parameters $\lambda_1+\mu_1$ as long as both the random variables are independent and both are exponential. You can do it as a homework; minimum of two exponential are going to be exponential with the parameter $\lambda_1+\mu_1$.

Therefore, the time spending in the state 1 that is exponential distribution with the parameter, λ_1 and μ_1. Also, one can discuss what is the probability that the system moving into the state 2 before moving into the state 1 that is a $\lambda_1/\lambda_1+\mu_1$. Similarly, what is the probability that the system moving into the state 0 before moving into the state 2 that is $\mu_1/\lambda_1+\mu_1$. That also one can find out. So what is the conclusion here is the time spending in the state 1 that is exponential distribution with the parameter, $\lambda_1+\mu_1$.



Similarly, the time spending in the state 2 that suppose if it is λ_2, then $\lambda_2+\mu_2$. So this is one type of continuous-time Markov chain. The third example this is also continuous-time Markov chain, this is sort of a two-dimensional Markov chain with the labelling with the (0,0), (1,0), (2,0) and so on. So all the labelling which is parameters for the exponential distribution. So the change from the discrete-time Markov chain state transition diagram and the state transition diagram of a continuous-time Markov chain, here there is no self-loop, and the labels are the parameters for exponential distribution, whereas the discrete-time Markov chain, it is a one-step transition probability going from one state to other states. Here the labels, the arrow gives the time spending in the state exponential distribution with the parameter λ_0 and moving into the state 1 and so on.



Now, I'm going to find out how the $P_{ij}(T)$. For that I am going to do the derivation starting with the Chapman-Kolmogorov equation. Let me start with what is the transition probability of system is moving from i to j during the time 0 to t+T. That is nothing but what is a transition probability system will be in the state j at that time point at t+T given that it was in the state i at time 0. That is same as I can in between make some other state. I can make one more state k at time point t. For all possible values of k also I will get the same result. That is same as I can make a summation over k, k belonging to s, s is a state's phase. That is same as what is the conditional probability of system will be in the state j at the time, t+T given that it was in the state i at time 0 as well as it was in the state k at t also, multiplied by what is the transition probability of system moving from 0 to T from the state i to k.

That is same as the first conditional probability; you see this is same as the Markov property which we have discussed in the definition of a continuous-time Markov chain. There I have discussed the CDF, cumulative distribution function; here it is the probability mass function where this is a conditional probability mass function. What is the conditional probability mass function of system will be in the state j at time point, t+T given that it was in the state i at the time point 0 as well as it was in the state k at the time point t. And you know that $0 < t < t+T$ because the way we made it is all these values are greater than 0.

Therefore, by using the Markov property of a continuous-time Markov chain, so this is same as what is the probability that the system was in the state k at time t and move into the state j at the time point t+T. Again, you use the time homogeneous property. First, we use the Markov property, therefore this is a transition probability of t to t+T moving from the state k to j, then use the time homogeneous property. Therefore, only the length matters, therefore t to t+T that is same as 0 to T. Therefore, the system is moving from the state of k to j from 0 to T that is $P_{kj}(T)$.

The second one, it's a transition probability; system is moving from state I to k during the interval 0 to T. Therefore, this is $P_{ik}(T)$. So this is valid for all i,j with the $t \geq 0$ and $T \geq 0$. Therefore, the left-hand side is the transition probability of system is moving from the state, i to j from 0 to t+T that is same as the summation over, I can rewrite it in a different way; i to k in the interval 0 to t. k to j instead of t to t+T because of the time homogeneous, I am just making 0 to T. Therefore, this is valid for all values of k summation.

This equation is called the Chapman-Kolmogorov equation for a time homogeneous continuous time Markov chain, because here for this transition probability, we have used Markov property as well as the time homogeneous property also. Therefore, this is a Chapman-Kolmogorov equation of the transition probability of system moving from i to j in t+T can be broken into product of these for all possible values of k.

So like this, you can break it many more ways with the summation for different state of k. Using these, we are going to find out the transition probability of $P_{ij}(T)$. You remember to find out the distribution of X(t), you need the initial state probability vector as well as the transition probability, $P_{ij}(T)$. The initial state probability vector is always given. You have to find out $P_{ij}(T)$. Once you know the $P_{ij}(T)$, you can find out the distribution of X(t) for any time, t.

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