INDIAN INSTITUTE OF TECHNOLOGY DELHI
NPTEL

National Programme on
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Video course on
Stochastic Processes – 1

By

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Module 5: Continuous-time Markov Chain

Lecture # 1
Definition, Kolmogorov Differential Equations And infinitesimal Generator Matrix

Good morning, this is stochastic process module 5, continuous time Markov chain. I'm planning for 6 to 8 lectures in this module. And I'm going to start the lecture 1 with the definition of continuous time Markov chain then the derivation of Kolmogorov differential equations, and I'm going to give some simple examples further, continuous time Markov chain and also I'm trying to give the stationary and eliminating distributions of continuous time Markov chain in this lecture.

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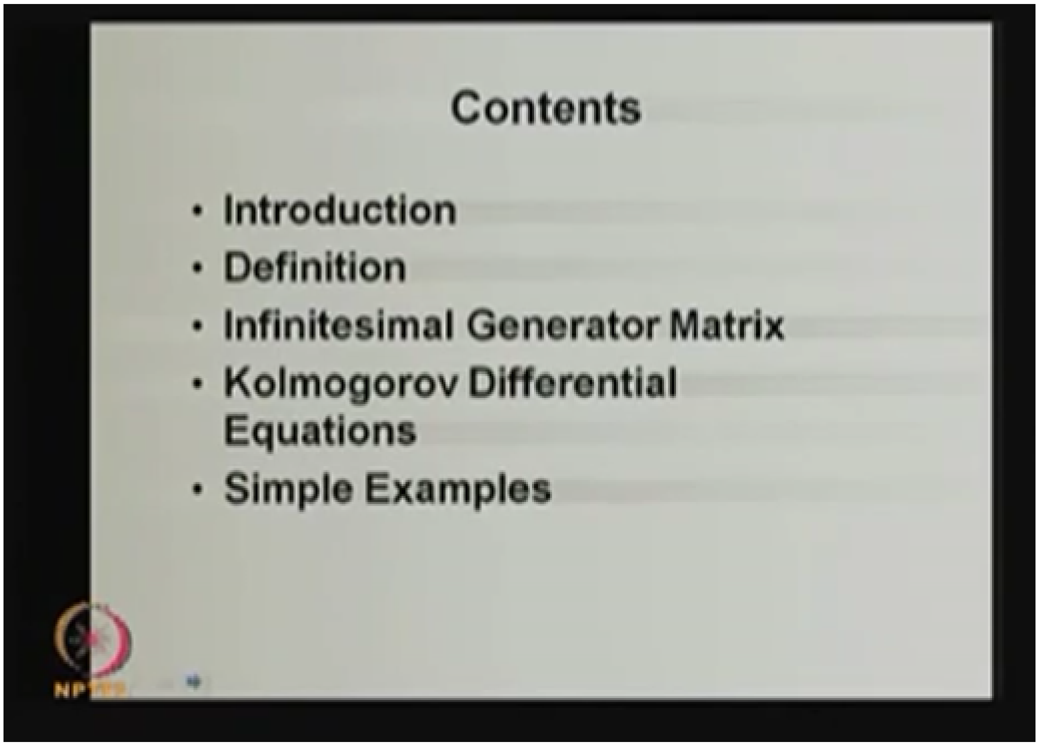

Let me start with the introduction of continuous time Markov chain.
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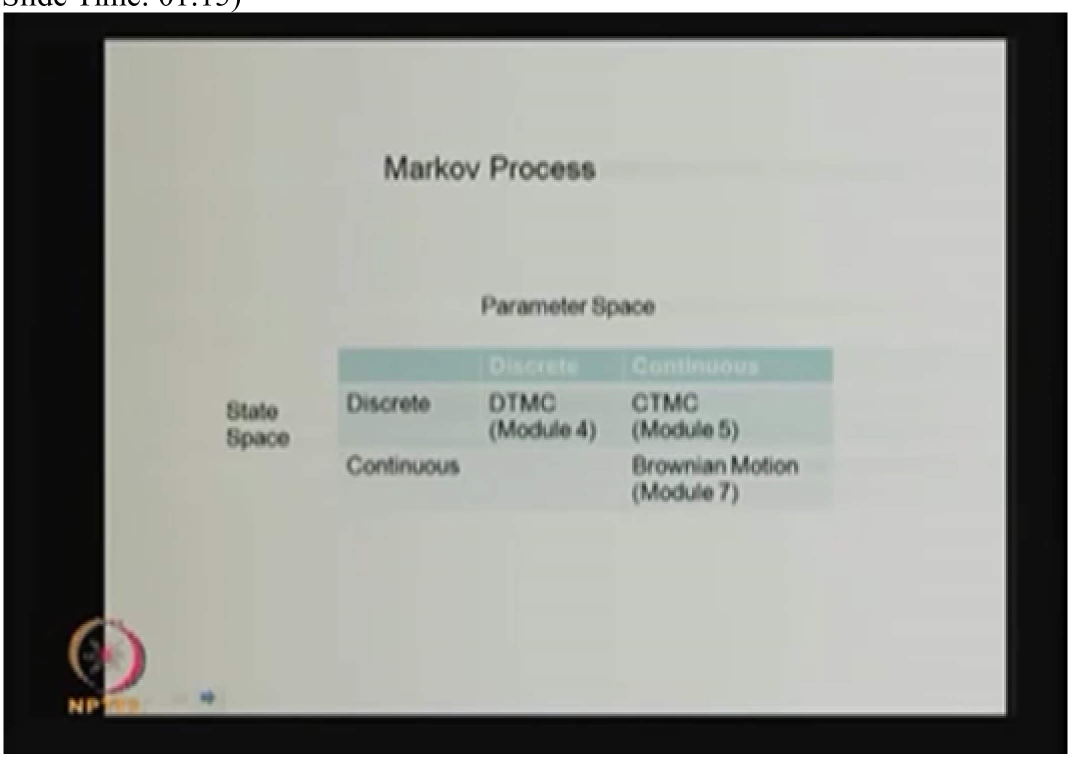

The continuous time of Markov chain is a special case of stochastic process, this is the stochastic process in which the Markov properties satisfied therefore it is called a Markov process. Based on the classification of the state space and parameter space whether it is a discrete or continuous we can classify the Markov process. Suppose the state space is a discrete then we will say that Markov process is a Markov chain.

Along with the state spaces are discrete, if the parameter space is also discrete then we'll say discrete time Markov chain, that means it's stochastic process satisfying the Markov property, that state space is discrete and the parameter space is also discrete, this we have discussed in the module 4.

A stochastic process satisfying the Markov property and state space is discrete and the parameter space is continuous then that stochastic process is called the continuous time Markov chain that we are going to discuss in the module 5.

There are other type of Markov process also which has the state space it is continuous and the parameter space is also continues that is called the Brownian motion or wiener process, that we are going to discuss in the module 7.

Now in this lecture we are going to discuss the continuous time Markov chain under module 5. Let me start with the definition,
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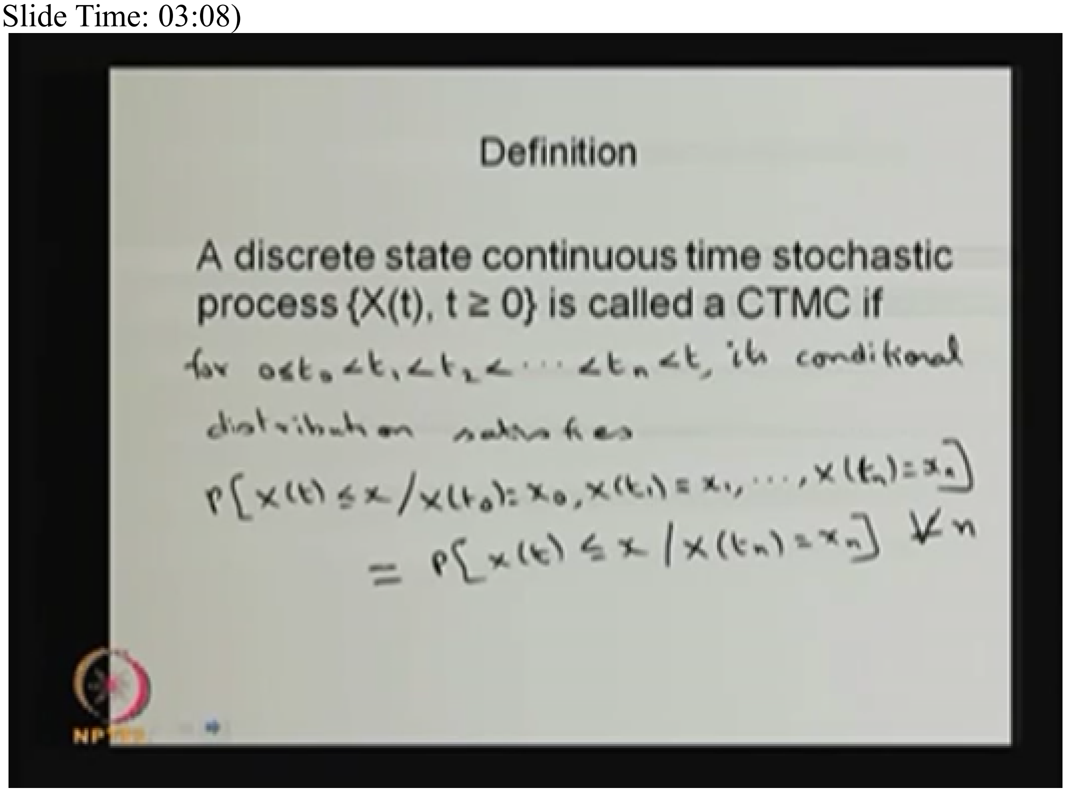

definition of continuous time Markov chain, a discrete state continuous time that means the state space is discrete, that means the possible values of the random variable going to take the value, for possible values of parameter space that is going to be finite or accountably infinite therefore the state space is going to be call it as a discrete.

Continuous time means the parameter space or the possible values of the T that collection is uncountably infinite, therefore it is called a continuous time, that means a parameter space is continuous, so a discrete state continuous time stochastic process X(t) for T greater than or equal to 0 need not be T greater than or equal to 0 also, but here I'm making the very simplest one, so the X(t) for fixed T it's a random variable, for every T that collection that is going to be stochastic process, and the state space is discrete and parameter space is continuous and that stochastic process is going to be call it as a continuous time Markov chain, if it satisfies the following condition.

If you take N time points arbitrary time points, N+1 time points that is T naught to TN you can say the T naught can be 0 also, and with this inequality T naught less than T1, less than T2 and so on TN, and you take the any arbitrary T that is TN less than T with this inequality.

For fixed T that X(t) is going to be a random variable therefore now we are going to find out the conditional distribution for this N+1 random variable with the random variable X(t) that means at T naught you have a X(t naught) that's a random variable, at T1 X(t1) is a random variable, similarly at TN X(tn) is a random variable. You have N+1 random variable, with this N random variable given that means it takes already some values with the X naught, X1, XN so on respectively, and you are finding the conditional CDF for the random variable X(t), so that means you have a N+2 random variables taken at the arbitrary time points T naught to TN as well as small t.

And you are finding the conditional CDF of the random variable X(t) given that already the other N+1 random variables taken at those arbitrary time points you've taken the value X naught to X1 and so on till XN, it is taken already these values that conditional distribution, conditional CDF, if that is same as again it is the conditional CDF of X(t) given the last random variable X(tn) = X, so these N+1 time points are arbitrary time points, so if it's satisfies for all N for every N that means if the conditional distribution of N+1 random variable is same as the conditional distribution of the last random variable, if this property is satisfied by the discrete state, continuous time stochastic process for arbitrary time points, then that stochastic process is called the continuous time Markov chain.

This is a very important concept, this is called the Markov property that means the T is sort of future, so what is the probability that the random variable will be in some state at the future time point T given that you know the present state that is where this system is in time point TN that is small xn and I know the past information starting from X(t naught) T X(tn-1) I know the information, that means what is the probability of that future the random variable X(t) will be in some state given that it was in the states X naught at time point T naught, it was in the state X1 at the time point T1 and so on, latest at the time point TN the system was in the state XN that is same as what is the probability that the future, the random variable will be in some state at time point T given that it is now in the state XN at the time point TN, that means future given present as well as the past information is same as future given only the present and independent of the past information that is called the memory less property or Markov property.

So since this property is satisfied by the stochastic process which has the state spaces are discrete and the parameter space is continuous then that stochastic process is called the continuous time Markov chain.

So this is the definition, now we are going to give some more properties over the continuous time Markov chain and some simple examples as well as the, I'm going to explain the limiting distribution and the stationary distribution for continuous time Markov chain in this lecture.

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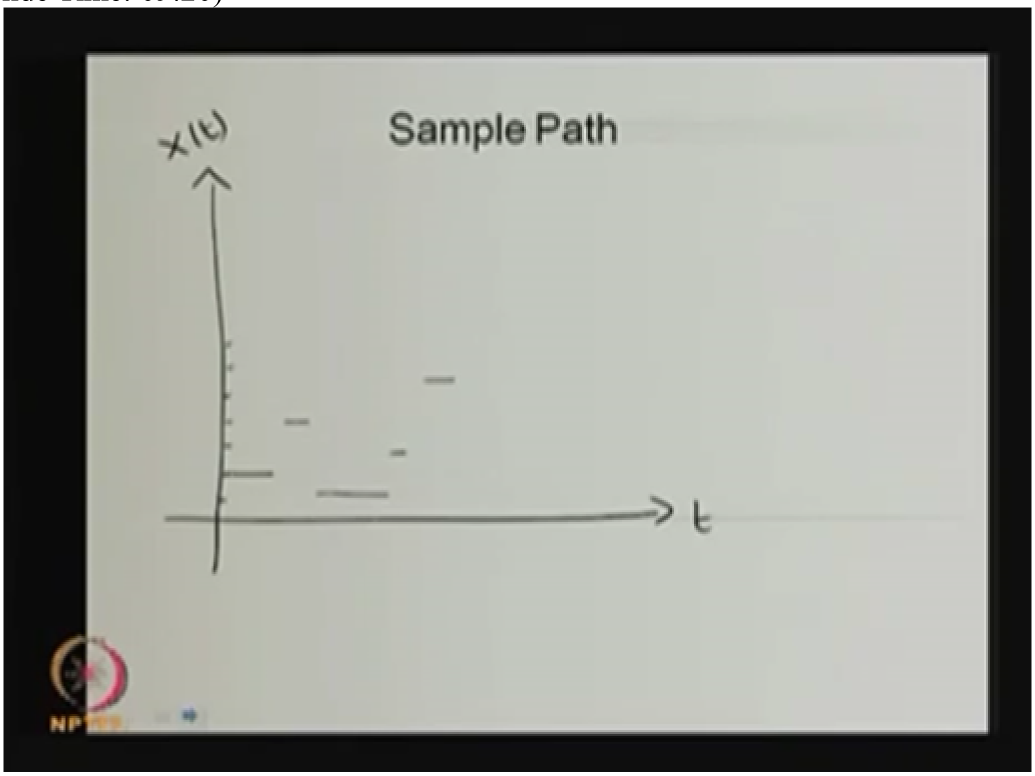

Let me show the sample path over the time T that is X axis, the Y axis is X(t), so the system was in some state at time point 0, it was in the same state for some time, then it moved into the some other state then it was there in that state for some time, then it moved into some other state and so on.

If you see the sample path the following observation, the system can stay in some state for some amount of time after that it will move to the some state, so there is no equal interval of a system going to be in some state also, it can be some positive amount of time the system can be in the some discrete states, so here the observations are there, state space is discrete whereas the parameter space is continuous and the time spent in each state that is going to be a some positive amount of time before moving into any other states, so this is the observation in the sample path which I have drawn.

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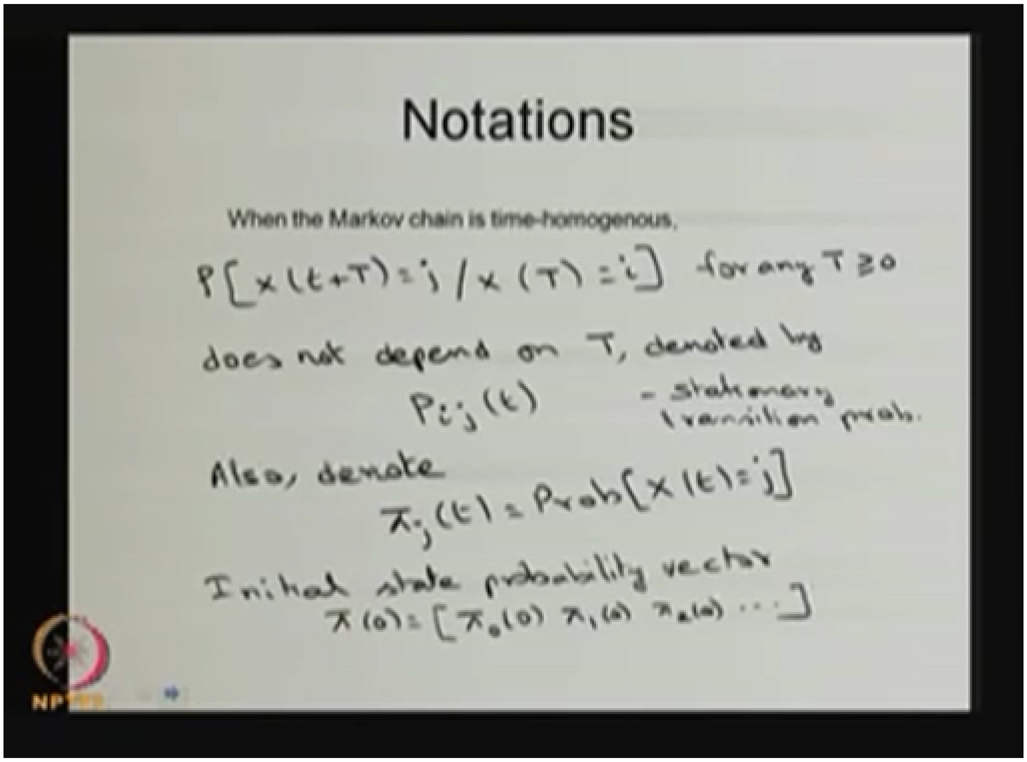

Now I am going for few notations to study or to study the behavior of a continuous time Markov chain, whenever the Markov chain that means here it is the continuous time Markov chain it is the time homogenous then the conditional probability of system being in the state J at time point t+ capital T given that the capital T it was in the state I that does not depend on capital T, here we assume that the state changes from I to J at a future time point t+ capital T, this transition probability says the system was in the state I at the time point T, let me draw the simple diagram, the system was in the state I at the capital T, then what is the probability that the system will be in the state J, what is the probability that the system will be in the state J at the time point t+T? It is independent of capital T whenever the Markov chain is going to be a time homogenous.

For any T greater than or equal to 0 that means the actual time does not matter, only the length matters, the length of the transition time, that means the small t's matters not the capital T whenever it is a time homogenous so that is, that we can denote it as a PIJ(t) because it depends on only the interval not the actual time therefore it is a function of small t, PIJ(t) that means that is the transition probability the system, so the same thing can be written as the PIJ(t) this is the notation.

What is the transition probability that the system was, what is the probability that the system will be in the state J given that it was in the state I at time 0, since it is valid for any interval of T to t+T it is independent of capital T, therefore I can represent in this transition probability as a probability that the system in the state J at time T given that it was in the state I at time 0, this denoted by PIJ(t), so this notation you should remember, it's a transition probability with the suffix two letters I, J(t), this also call it as a stationary transition probability. Stationary means it is a time invariant only the length of the interval is matters.
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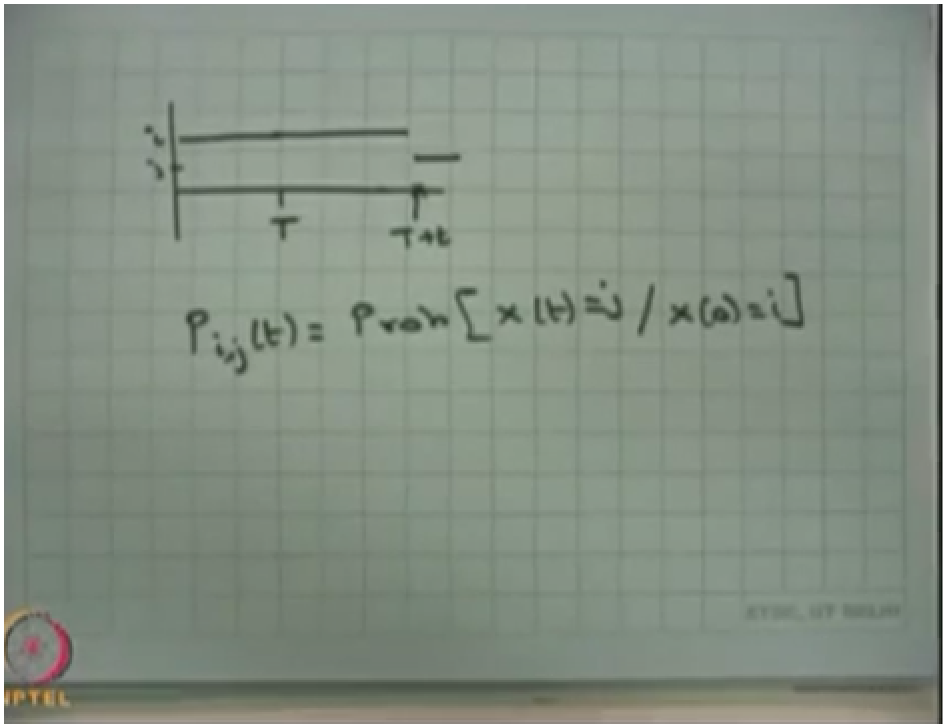

Similarly I'm denoting the next notation pi J(t), the PJ(t) is the conditional probability whereas the pi J(t) that is unconditional one, what is the probability that the system will be in the state J at time T? That is the possibility system would have been coming to the state J before time T for at time 0 itself or it would have come before just before T whatever it is, this probability will give the interpretation, what is the probability that the system will be in the state J at time T? Only it gives the information at the time T, this is the unconditional probability, I need another notation for a initial state probability vector also that is pi naught, pi naught is a vector which consists of entities, what is the probability that the system was in the state 0 at time 0, therefore this I can write it as pi J(0) that is nothing but what is the probability that the system was in state J at time 0, so this is the meaning of pi J(0).

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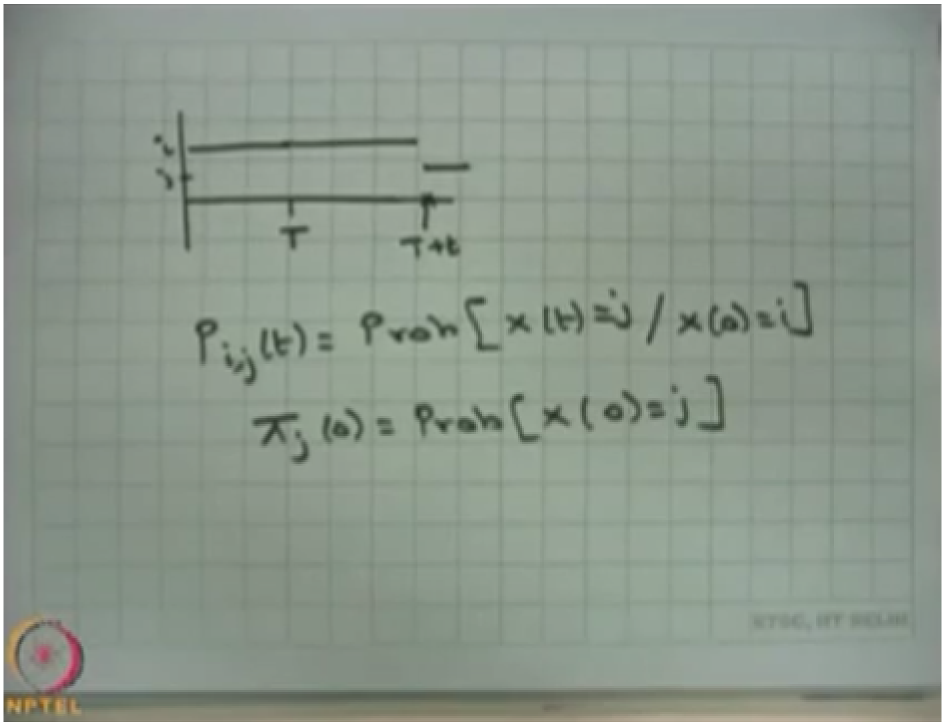

What is the probability that? The system was in the state J at time 0 that is pi J(0), right, with this entities we are framing at arc that is pi naught. So in this we are giving a three notations, one is the transition probability PIJ(t) that is the conditional probability, the other one is unconditional probability that is pi J(t) and initial state probability vector I naught.

Using these I'm trying to find out what is the distribution of X(t) for any time T?
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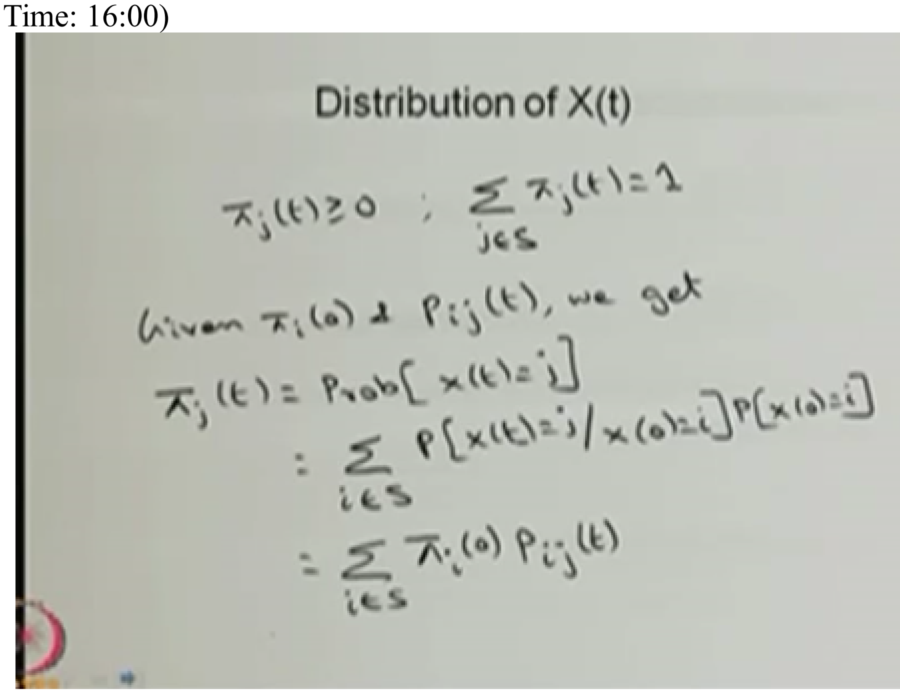

For anytime T X(t) will form, make a stochastic process, here it is a continuous time Markov chain the default one is the time homogenous continuous time Markov chain and our interest is to find out what is the distribution of the random variable X(t).

It has the probability mass function that is pi J(t) and if you make a summation over S, where S is the state space that summation is going to be 1. If I know the initial state probability vector with the entities pi I(0) as well as if I know the transition probability of system moving from the state I to J from 0 to small t, I can able to find out what is the probability mass function of system being in the state J at time T, that is an pi J(t) that is M as probability that X(t) = J, that is same as I can make a summation, I can make a conditional, what is the probability that the system will be in the state J at time T given that it was the state I multiplied by, what is the probability that a system was the state I at time 0, for all possible value of I where S is nothing but the state space.

I know that the probability of X(0) = I that is same as pi I(0), and this transition probability since the Markov chain is a time homogeneous, so 0 to T that is nothing but 0 to, 0 is the time point and it is any time point and I is the state in which the system was in the state, at time 0, so PIJ(t) if I multiply pi I(0), PIJ(t) for all possible values of I, I'll get the probability that the system will be in the state J at time T, that means if you want to find out the distribution of X(t) for any time T I need initial state probability vector as well as the transition probability of system moving from one state to other state, other states.

This is given usually the initial state probability vector is given, so what do we want to find out is PIJ(t), so how to find the PIJ(t), that derivation I'm going to do it in the another two, three slides.