INDIAN INSTITUTE OF TECHNOLOGY DELHI
NPTEL

National Programme on
Technology Enhanced Learning

Video course on
Stochastic Processes – 1

By

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Module 4: Discrete-time Markov Chain

Lecture # 7
Reducible Markov Chains

Module 4 discrete-time Markov Chain, lecture 7 reducible Markov Chain. The last three lectures we have discussed irreducible Markov Chain and this lecture we are going to discuss reducible Markov Chain.
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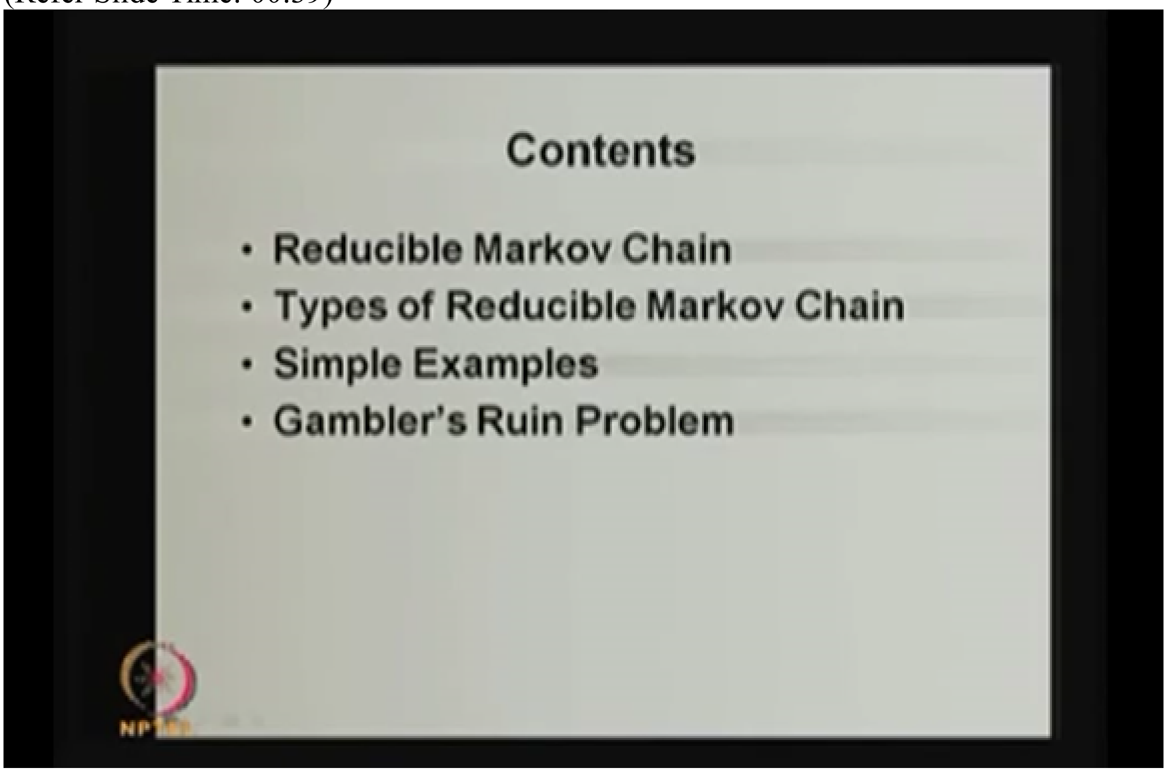

So in this lecture I am going to start with the concept of reducible Markov chain, then I'm going to give the different types of reducible Markov chain, and also I'm going to present some simple

examples, then finally one important application of reducible Markov chain that is Gambler's Ruin chain problem, Gambler's Ruin problem going to be discussed.
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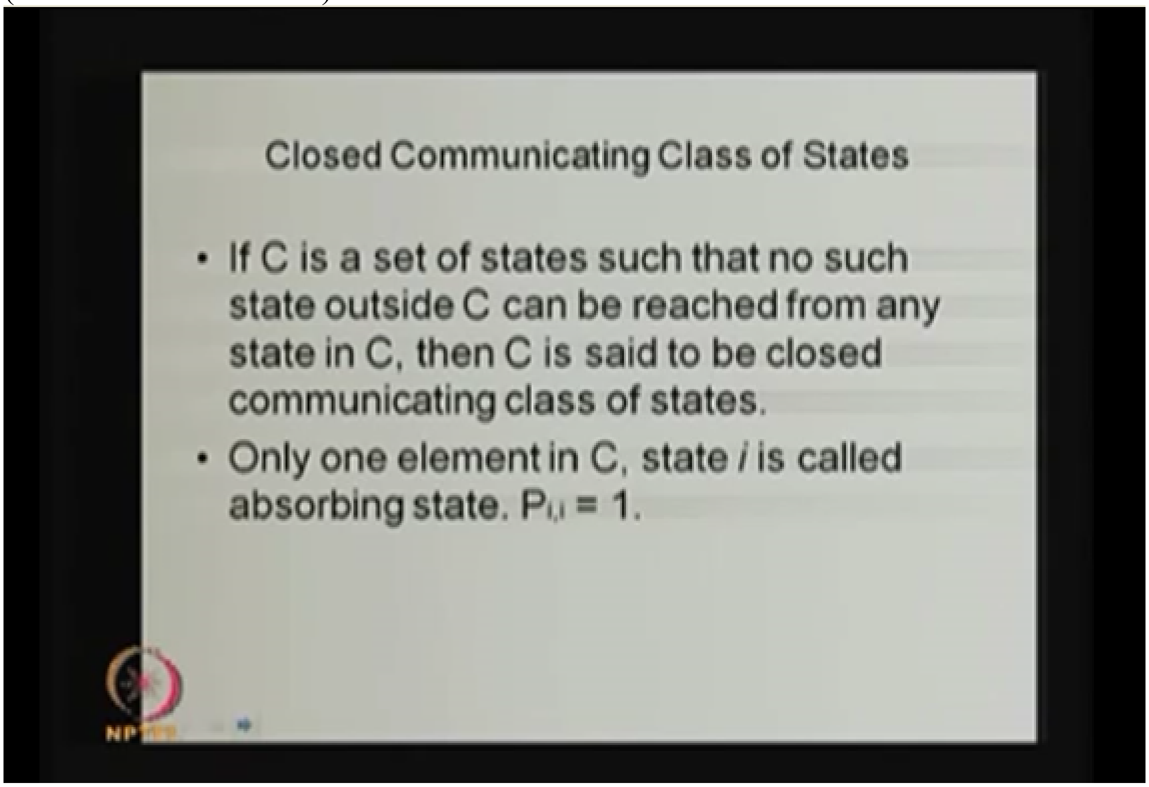

Before we discuss the reducible Markov chain let me explain the closed communicating class of states, this definition we have already given in the few lectures earlier also, again I'm using this, we are going to conclude the Markov chain is reducible Markov chain or irreducible Markov chain.

The closed communicating class, suppose you collect the set of states that dynamic label with the C, if that collection of our set of state is going to be call it as a closed communicating class of states, if it satisfies no such state outside C can be reached from any state in C, then C is set to be closed communicating class of states.

If in a set of states forming a closed communicating class and it has only one element, only one state you cannot include one more state so that it is going to be a closed communicating class of states, then that class is, in that class the state is going to be call it as absorbing state, and you know the definition of absorbing state that means the first step transition probability I to I that's equal to 1.

So there are two ways you can have a absorbing state either PII =1 or the closed communicating class has only one element then that state is going to be absorbing state, so using a closed communicating class of states we are going to distant use or we are going to make the reducible Markov chain and irreducible Markov chain, how? Let me see the definition of a irreducible Markov chain.
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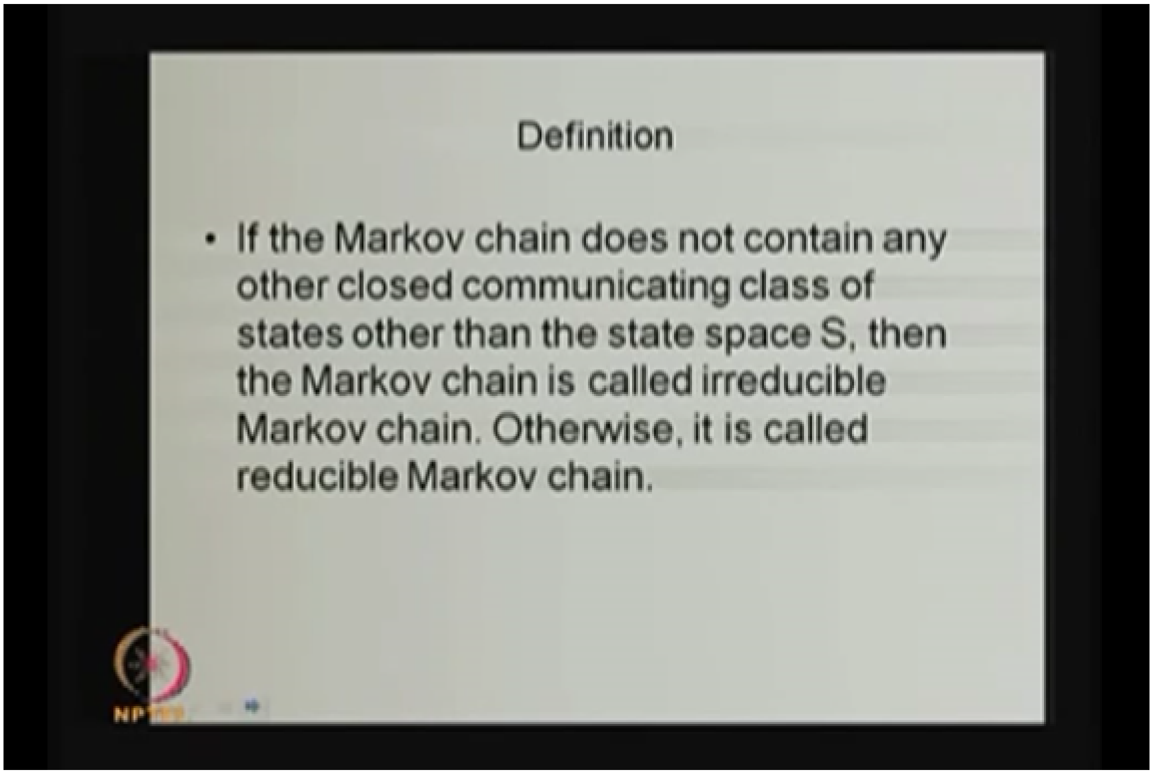

If the Markov chain does not contain any other closed communicating class of states other than the state space S, then the Markov chain is called irreducible Markov chain. Otherwise, it is a reducible Markov chain. That means you have a Markov chain with a state space capital S, we are trying to create the closed communicating class, if that class and the state space S both are one and the same, that means all the states are going to form a one closed communicating class, that means each state is communicating with each other state and that is same as the state space then that Markov chain is a irreducible Markov chain, otherwise that Markov chain is going to be call it as a reducible Markov chain.

Before we go to the various reducible Markov chain I'm going to give few examples,
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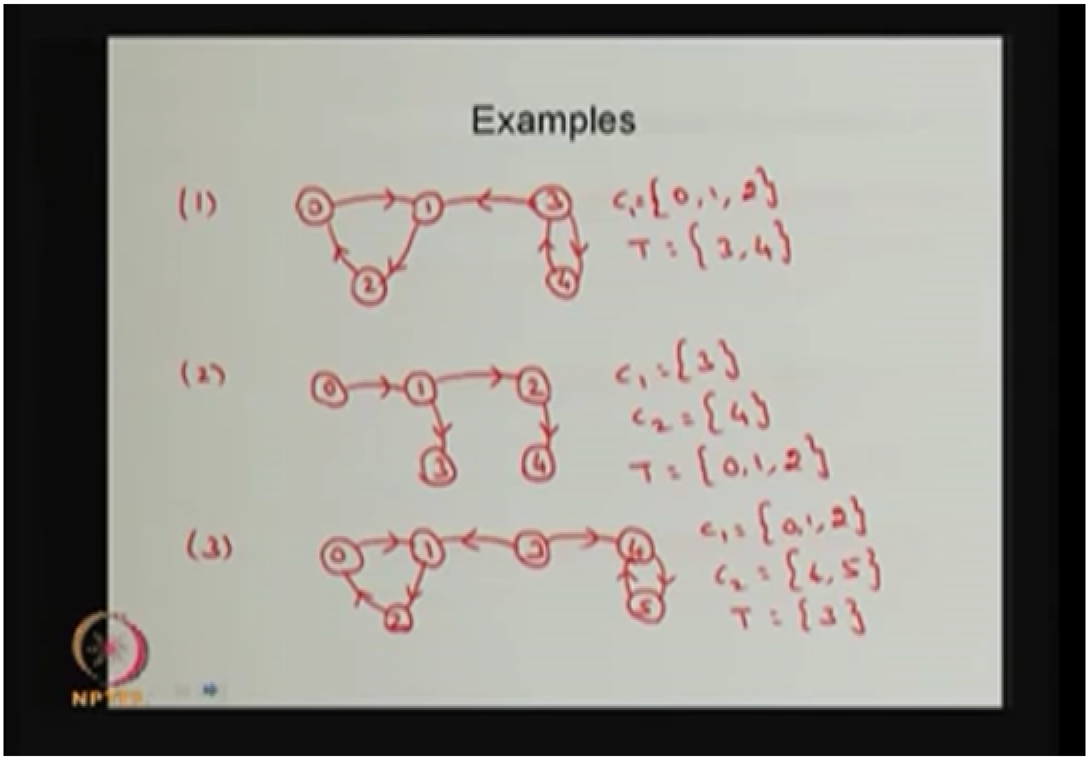

so through this example we can make the classification over the reducible Markov chain. You see the first example it has 5 states instead of the one step transition probability mantra 5 have drawn the state transition diagram, so using these we can easily able to conclude whether it is going to be a reducible Markov chain or irreducible Markov chain.

If you see the arc from 3 to 1 and the states 0, 1 and 2 all three are connected therefore you can conclude 0, 1 and 2 is going to form a closed communicating class because all the states inside that class or communicating each other, there is no state going away from this collection to outside, that's why is the closed communicating class definition, whereas the 3 and 4 even though there is a communication between 3 and 4 states, once the system goes from 3 to 1 it won't be back therefore the states 3 and 4 are going to be transient states, the first visit if you find out FII, capital FII for state 3 and 4 it is going to be less than 1, whatever be the probability here I have not assigned the probability, you can assign the probability positive 0 to 1 and you will get the conclusion, the states 3 and 4 are going to be the transient states.

So since it satisfies the definition of reducible Markov chain that means you have a close communicating class which is other than the state space, that means you have a closed communicating class with the fewer elements then the state space 0, 1 and 2 and few transient states, therefore this Markov chain has a reducible Markov chain of some type I'm going to discuss later.

See the second example, this also has the 5 states, if you observe you will conclude the states 0, 1 and 2 are going to be the transient states, whereas the state that 3 as well as 4 are going to form a two different closed communicating class, but it consists of only one element in it, only one state in it, you cannot include the state 1 along with 3 or you cannot include the state 2 along with 4 to create a closed communicating class, therefore the states 3 and 4 will form a close communicating class with a one single states in each in it respectively, and these two states are absorbing states also, so this is also going to form a reducible Markov chain.

See the third example, this has 3 + 3, 6 states, out of 6 states there is no backyard arch to the states, to the state 3 therefore state 3 will be a transient state, whereas the states 0, 1 and 2 form a closed communicating class. Similarly the states 4 and 5 will form another closed communicating class of states.

So in the third example we have a two closed communicating classes of states, whereas the first example you have one closed communicating class and transient states, you see that all these three examples you have a collection of transient states and closed communicating class either 1 or many or the closed communicating class consist of only one element, but all the states, all the model has fewer transient states therefore you can easily find out the reducible Markov chain whenever it is not going to form a only one closed communicating class with all the states, if that is not there then all other things are going to be a reducible Markov chain.

So based on this three examples there are some more example I can create to the infinite state, but here I have not made it, but based on this three examples you can have some idea how one can have a various types of reducible Markov chain.

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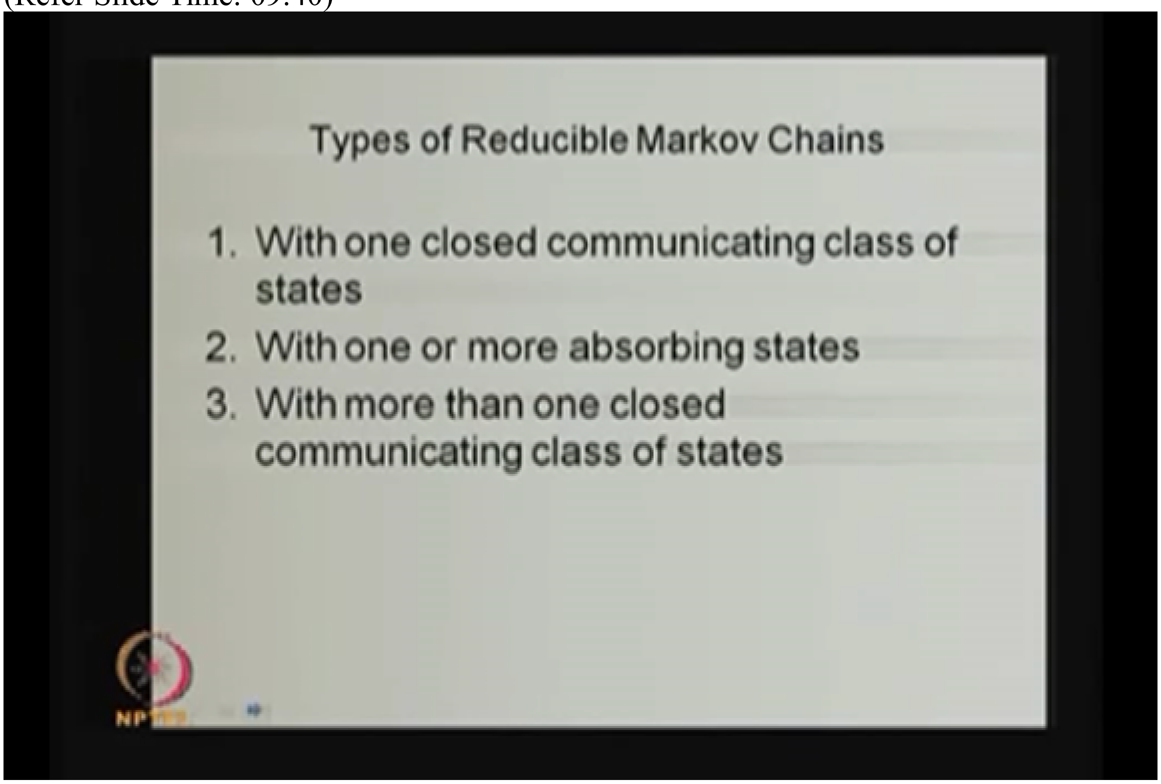

I'm listing here in these the default is all the types has the few transient states, along with that it has a one closed communicating class of states, that's a one type, one or more absorbing states that is similar to the example 2, the first one is similar to the first example, the third one is with more than 1 closed communicating class of states that is a related to the third example, but here I have not specified whether it is a finite state or infinite state Markov chain, so in material of that the reducible Markov chain can be classified into these three in general, so we are going to discuss out of these three the first two we are going to discuss in detail, and the third I'm not going to discuss, so the way we are discussing the first model the similar logic can be used to study the third type also.

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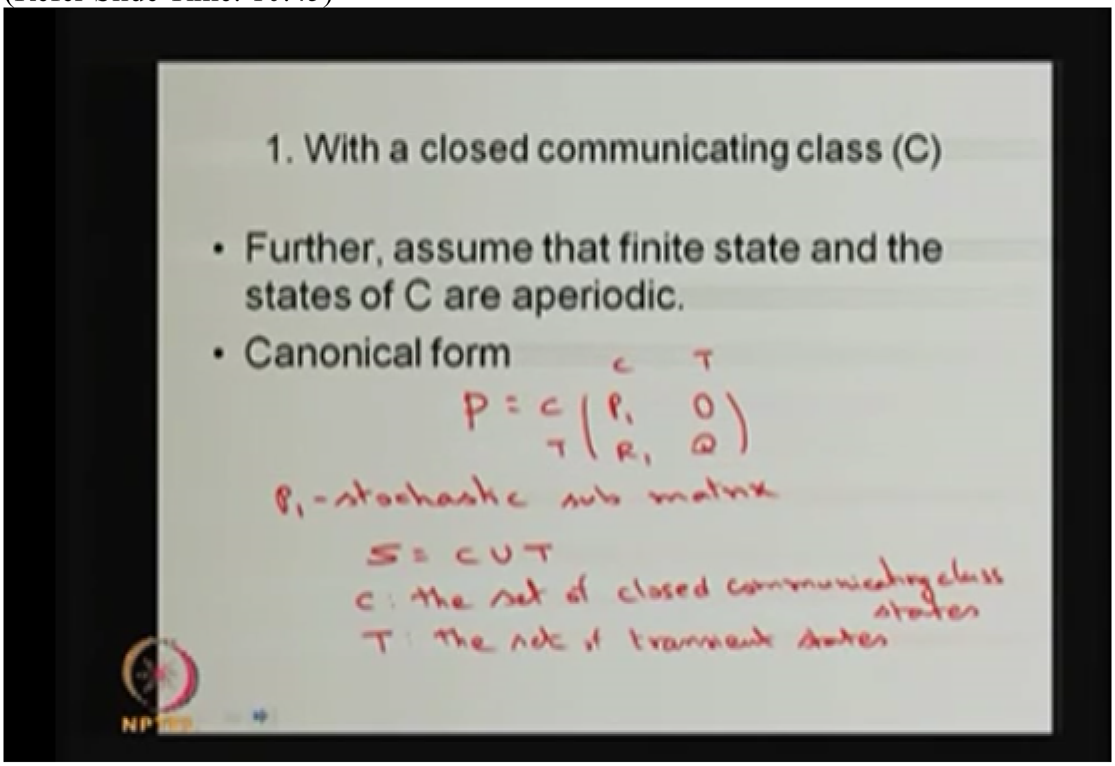

The first type, it's a reducible Markov chain that means it has the fewer transient states and one closed communicating class, my interest is to study the stationary distribution therefore I'm making further assumption so that I can grow for studying the stationary distribution, for that I'm making the first assumption it's a finite state model, state space is finite, and also this model state space has the one closed communicating class and the set of transient states, so whatever the states in the closed communicating class that state I'm making at aperiodic. Aperiodic is important to study the stationary distribution, therefore I'm making the aperiodic.

So this state space is the collection of the transient states as well as one closed communicating class therefore I'm making it a two notation C and T, C for the set of closed communicating class only one, the T is set of all transient states, therefore the state space S is going to be the C union capital T.

Since it has one closed communicating class and set of transient states, I'm reordering the one step transition probability matrix such a way that the first few rows are corresponding to the states of the closed communicating class, therefore I make it a C, but inside suppose the state space then a number of a states in these reducible Markov chain is capital N there is a possibility, some fewer elements, fewer states maybe in the capital C, therefore fewer rows that will make a sub matrix that is P1, that means a C to C that sub matrix is a one-step transition probability sub matrix is P1. Whereas the one step transition probability going from close communicating class that states to the transient states that probability is 0, therefore all the entities are 0 therefore these 0 is nothing but a matrix, sub matrix with the number of rows is number of states in the closed communicating class and the number of column that is the number of transient states, this is the way we reorder the one step transition probability matrix therefore C2 capital T that is a sub-matrix of 0s.

The remaining elements are capital T that you reorder it in the other remaining rows therefore T to C will be a some nonzero fewer elements that is a R1 matrix, R1 sub matrix. And similarly T to T there is possibility of possibilities therefore the probabilities maybe greater or equal to 0, therefore that matrix is a Q matrix, therefore the whole P matrix is divided partition into 4 sub matrices T1, 0 matrix R1 and Q matrix.

Since it is 0 matrix entries of 0 therefore this P1 is also going to be stochastic matrix, the row sum is going to be 1 and the integers are greater or equal to 0 lies between 0 to 1, so this values are, this sub matrix will form a stochastic, this is called a stochastic sub matrix that means I'm just reordering this P matrix that labeling the states such a way that first I'm collecting all the states corresponding to the closed communicating class of states, then set of transient states and this one is called the canonical form.

For a reducible Markov chain this canonical form is very important because once you are able to make a canonical form then you can study the stationary distribution in a easy way.