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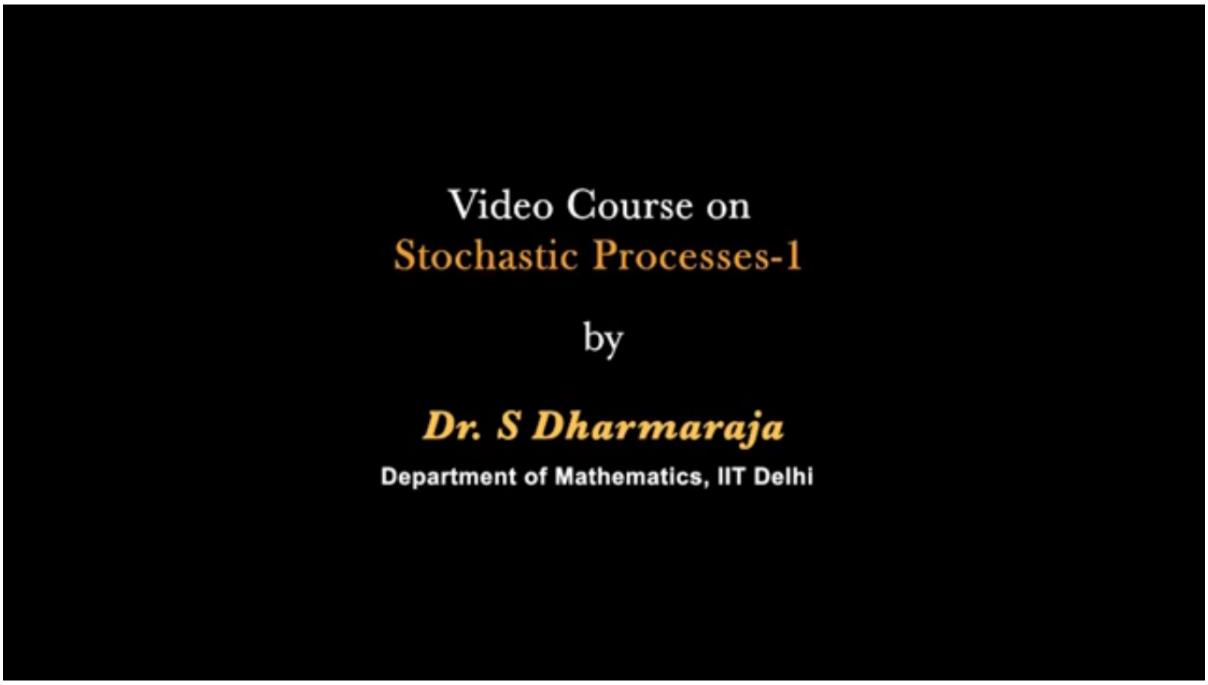

Video Course on
Stochastic Processes-1

by

Dr. S Dharmaraja
Department of Mathematics, IIT Delhi

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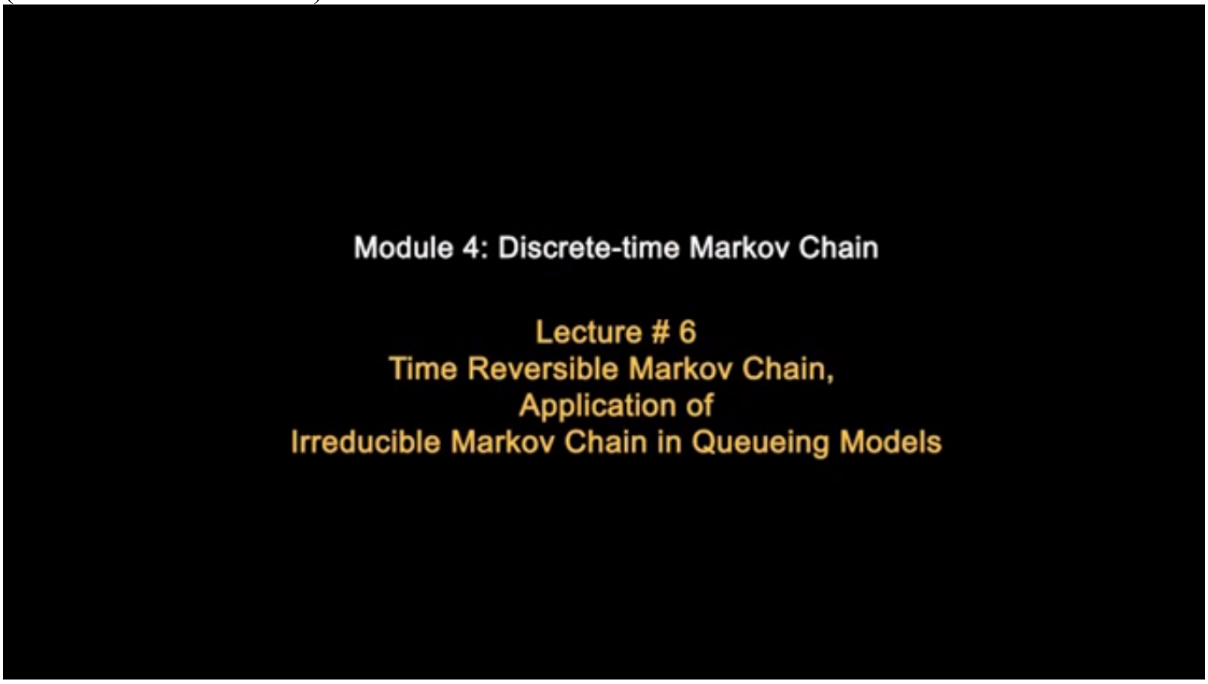

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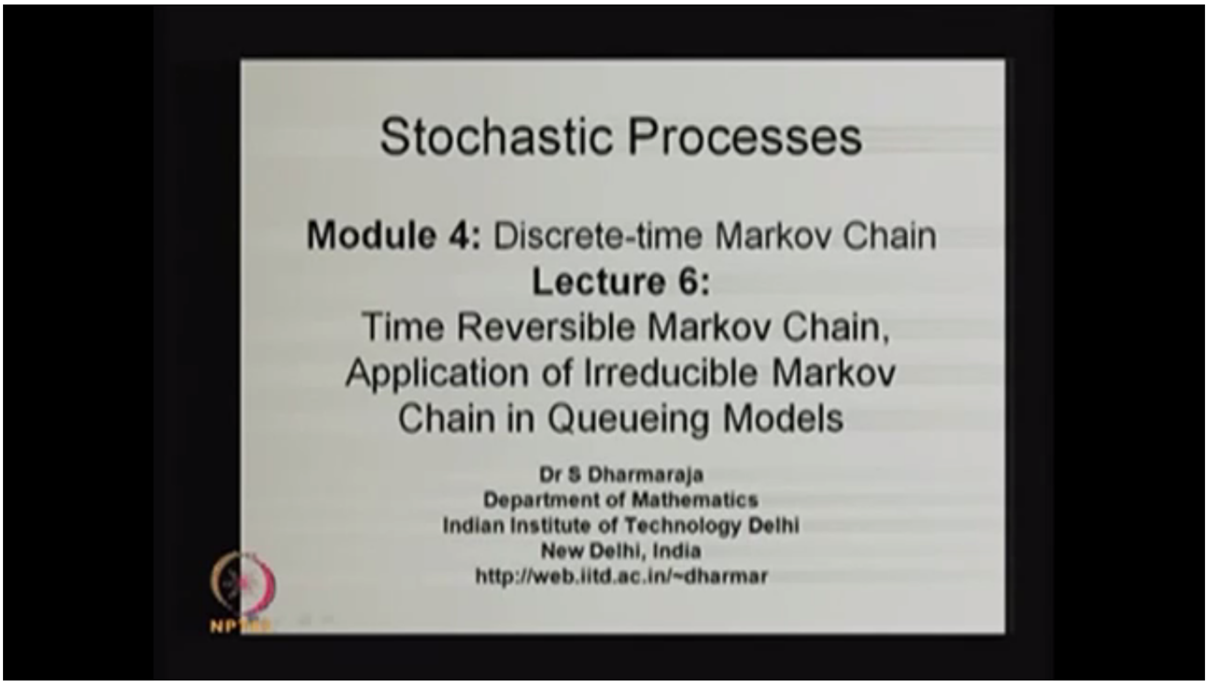

Good morning. This is Module 4: Discrete-time Markov Chain, Lecture 6: Time Reversible Markov Chain and then Application of Irreducible Markov Chain in Queueing Models.

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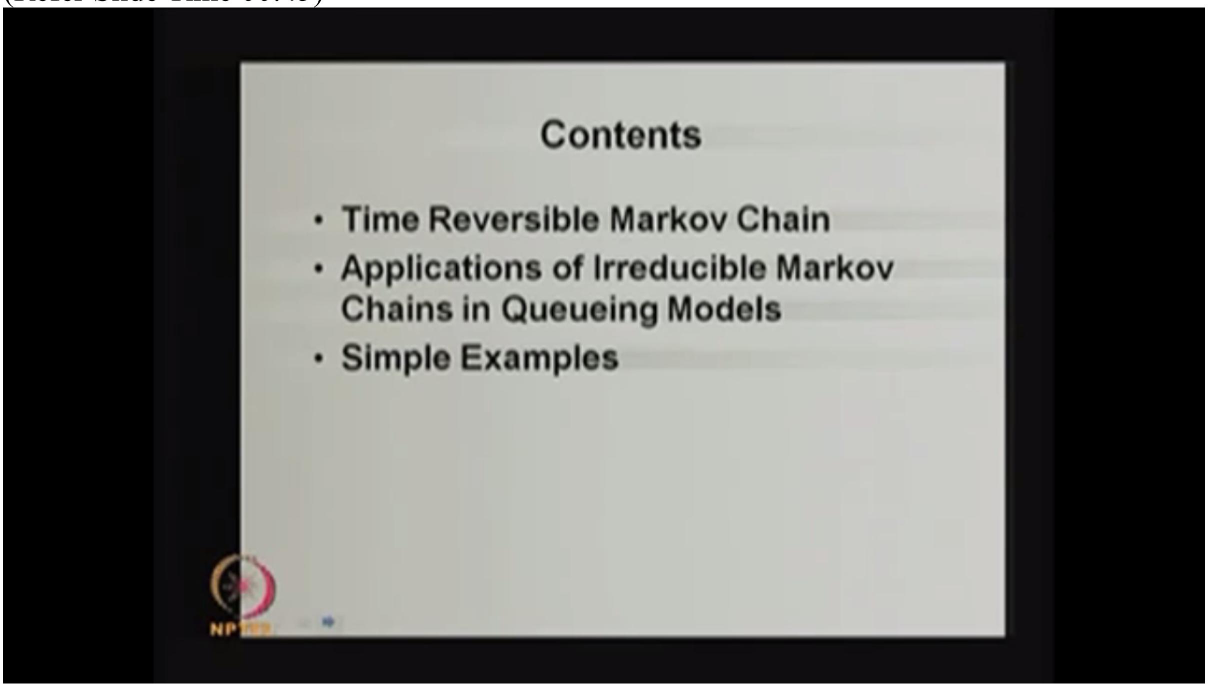

So in this lecture, I am planning to give the time reversible Markov chain and how to compute the stationary distribution in an easy way. Then I am going to give applications of irreducible Markov chain in queueing models. Here irreducible Markov chain means it is a DTMC model because later we are going to give the applications of a irreducible continuous-time Markov chain in queueing models also in the later lectures, and also I am going to give few simple examples.

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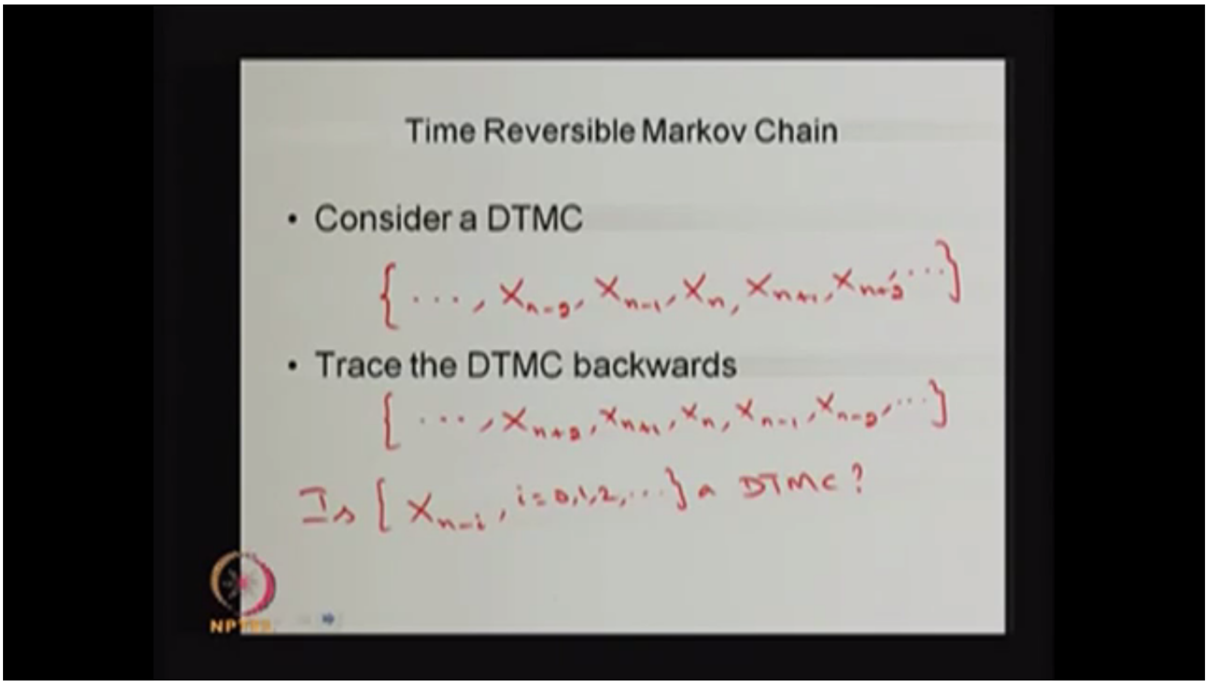

What is the meaning of a time reversible Markov chain? First let me explain how to construct the time reversible Markov chain.

Consider a DTMC. It is of course a time-homogeneous discrete-time Markov chain. You see the collection. It is in the usual way. X_{n-2}, X_{n-1}, X_n, X_{n+1}, X_{n+2} and so on. Now you trace the DTMC backwards. That means you know the first all X_{n+2} then X_{n+1}. Then you know what is X_n. Then you collect X_{n-1}, X_{n-2} and so on.

Now the question is whether if you make a DTMC backwards and that sequence, that sequence of random variable, that's, of course, it's a stochastic process, whether this is going to be a DTMC?

Any stochastic process is going to be a DTMC if it satisfies the Markov property and the state space is discreet and the time-space is also discrete, but if you see these, the DTMC backward, that is also going to satisfy the Markov property the way the given situation. The future depends only on the present, not the past. It's a Markov property. The same thing is going to be satisfied in the reverse also. The Markov property is satisfied by the reverse, the backward DTMC. Therefore, this is also going to be a time-homogenous discrete-time Markov chain.

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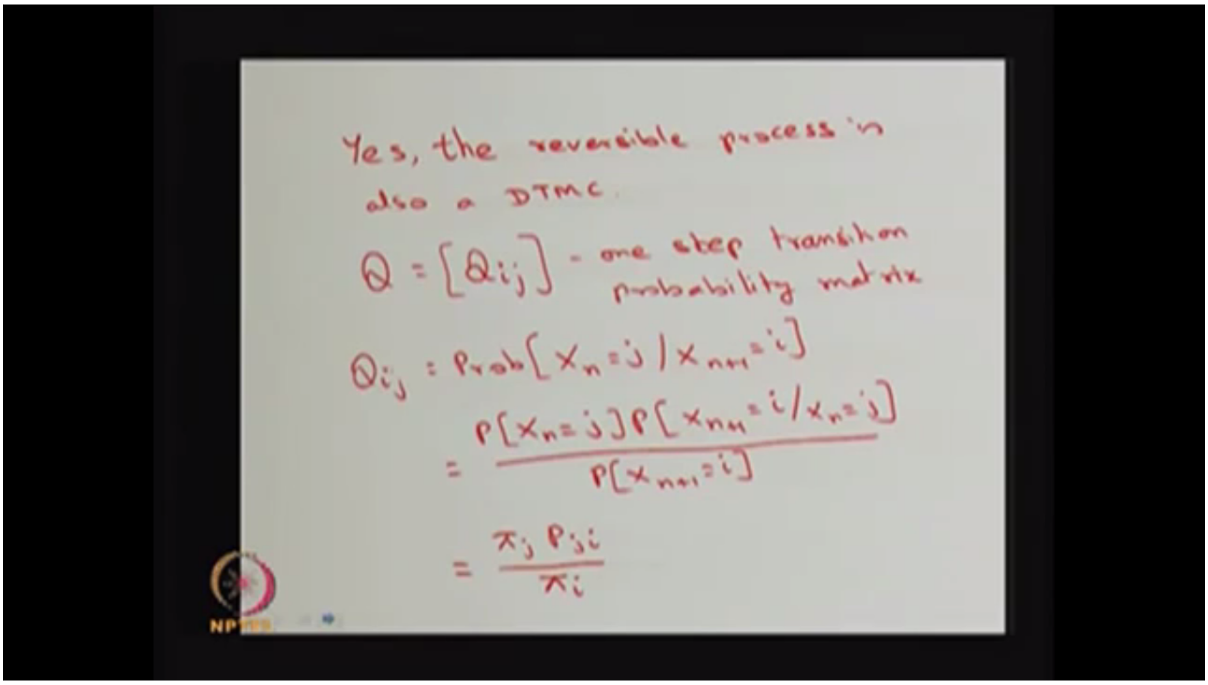

Now I am going to define what -- I'm going to give how to find out the one-step transition probability for the reversible process that you take it as the matrix Q that consists of elements Q_{ij}. You can find out the entries, the Q_{ij} that is nothing but what is the probability that the system will be in the state j at the n^{th} step given that it was in the state i at the $(n+1)^{th}$ step. That is different from the original DTMC.

The original DTMC, it is what is the probability that the system will be in the X_{n+1} in the state j given that X_n was high whereas here it is X_n is equal to j given that X_{n+1} is equal to i. This conditional probability you can compute in this way. The product of probability of X_n is equal to j multiplied by the probability of X_{n+1} is equal to i given that X_n is equal to j divided by what is the probability that X_{n+1} is equal to i.

That is same as the probability of X_n is equal to j is nothing but in a steady state, sorry, in -- what is the probability that in at the n_{th} stage in the system -- in the state that is equal to π_j multiplied by what is the probability that this is a one-step transition probability of system is moving from j to i divided by π_i. That means the Q_{ij} is going to be π_j P_{ji} divided by π_i assuming that the stationary distribution exists. Otherwise, π_j is equal to limit n tends to infinity of P_{ij}^n.

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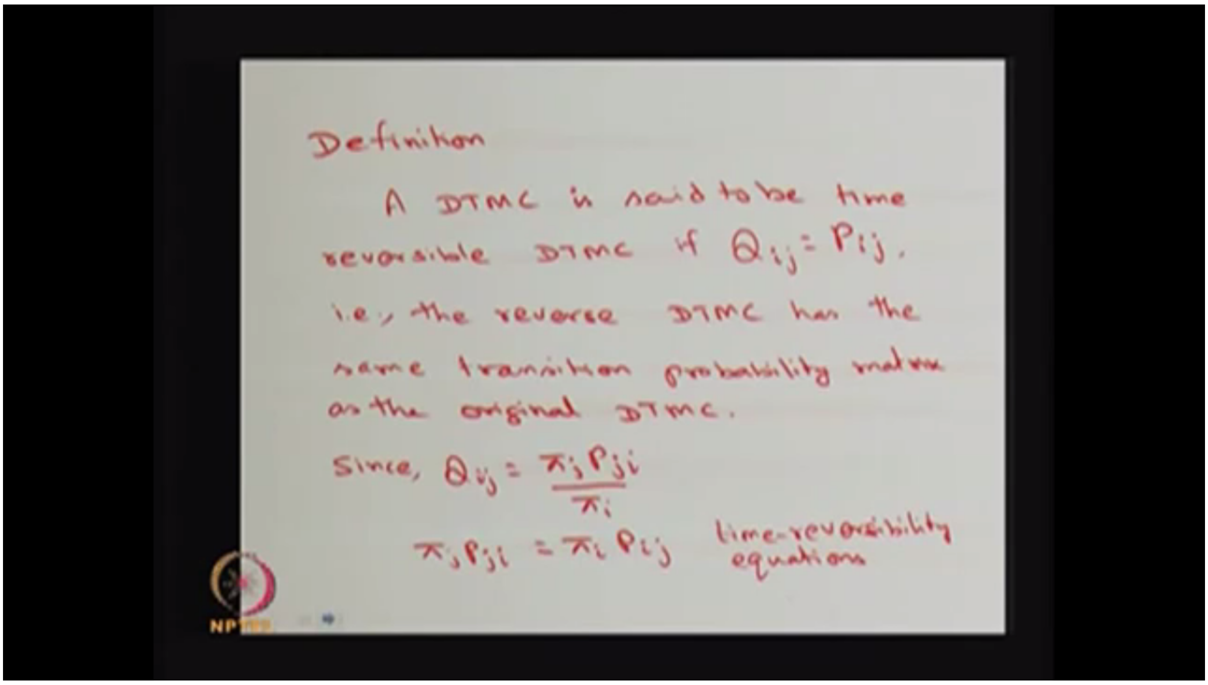

Now I am going to give the definition of a time reversible. A DTMC is said to be a time reversible DTMC if both the transition probabilities are one and the same. That means the one-step transition probability of the new or the time reversible process Q_{ij} is same as what is the one-step transition probability of the original DTMC that is P_{ij}. That is the reverse DTMC has the same transition probability matrix as the original DTMC.

Now I'm going to equate this, the Q_{ij} is equal to this much. Therefore, that is same as $\pi_j P_{ji}$ is equal to $\pi_i P_{ij}$ and if this equation is going to be satisfied, then that DTMC is going to be call it as a time reversible Markov chain and this collection of equation for all P_{ij}, that equation is called the time reversibility equations.

Now I am going to give the few results on time reversible Markov chain.

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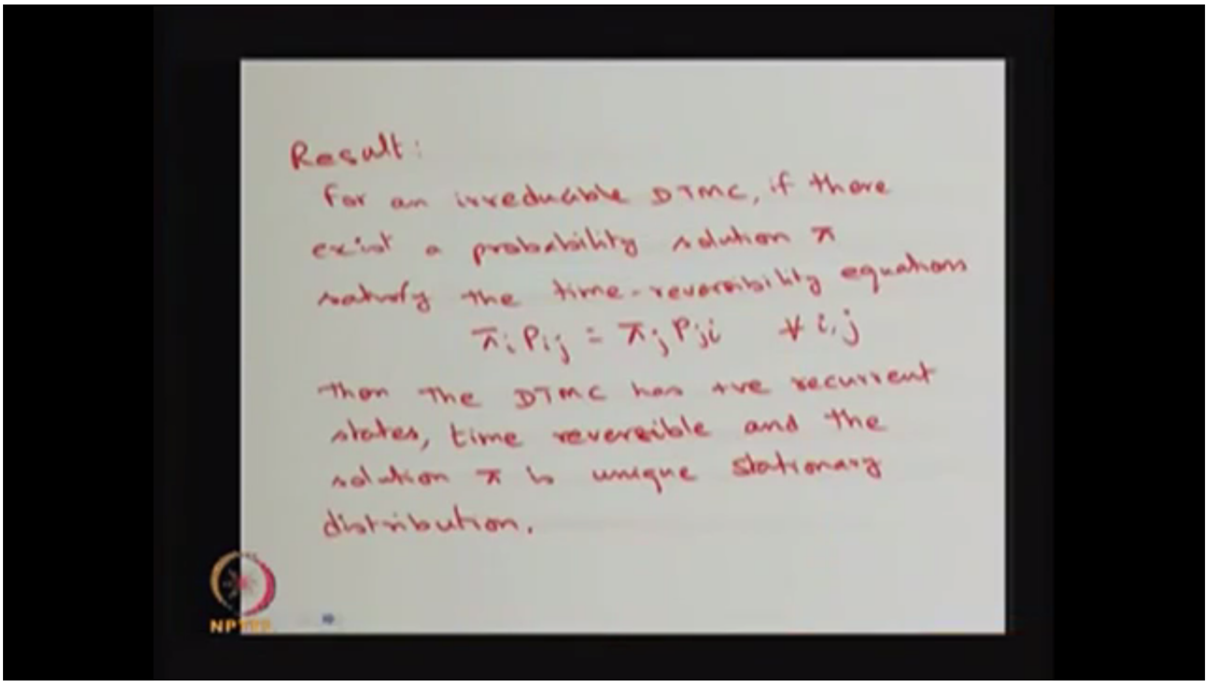

Let me take irreducible DTMC. If there exist a probability solution that is π is a vector satisfies the time reversibility equations, that is the vector π consists of π_1, π_2 and so on, so if that entries satisfies the time reversibility equations, that is π_i is equal to -- $\pi_i P_{ij}$ is equal to $\pi_j P_{ji}$ for all pairs of i, j, then the DTMC has a positive recurrent states and also it is a time reversible and the solution π is a unique stationary distribution.

That means whenever you have a irreducible DTMC and if you have -- there exist a probability solution vector π satisfies the time reversibility equation, then you can conclude the DTMC has positive recurrent states as well as the DTMC is a time reversible Markov chain. Also the vector π that satisfies the time reversibility equations, that vector π is a unique stationary distribution.

So the -- how we can -- one can use the time reversible concept in finding the stationary distribution, that I am going to explain in the next example.

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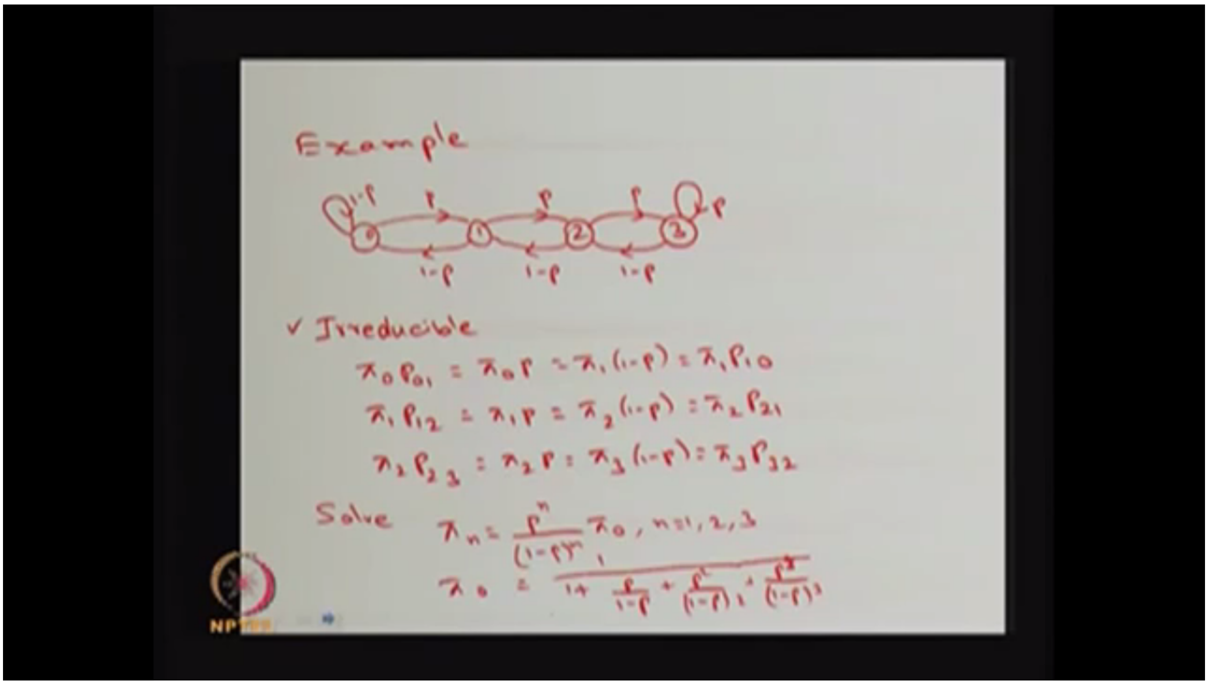

Let me take a simple example, which consists of four states. It's a finite model, finite state space. Also it is irreducible because each state is communicating with each other states and I assume that the probability P that is lies between 0 to 1, therefore, this is going to be a aperiodic states. So this Markov chain is a finite, irreducible and you have the result for a finite, irreducible Markov chain, at least one state is going to be a positive recurrent.

So since it is an irreducible, all the states are of the same type. Therefore, all the states are going to be positive recurrent states and also it is aperiodic. So you can use the result of a irreducible, aperiodic, positive recurrent and also the finite states going to give the unique stationary distribution and that can be computed by solving πP is equal to π where π is the stationary probability vector.

Here we can use the time reversibility concept. Therefore, you don't want to solve actually πP is equal to π, but you can start from the time reversible equation. From that you can get the solution. That's what I have done it in this example.

First I have checked it is a irreducible. Then I check whether the time reversible equation is going to be satisfied by this irreducible Markov chain. So since it has the four states, I am just checking all the states whether -- whether the time reversible -- reversibility equations are going to be satisfied. That is satisfies. Since it is -- you see the previous result.

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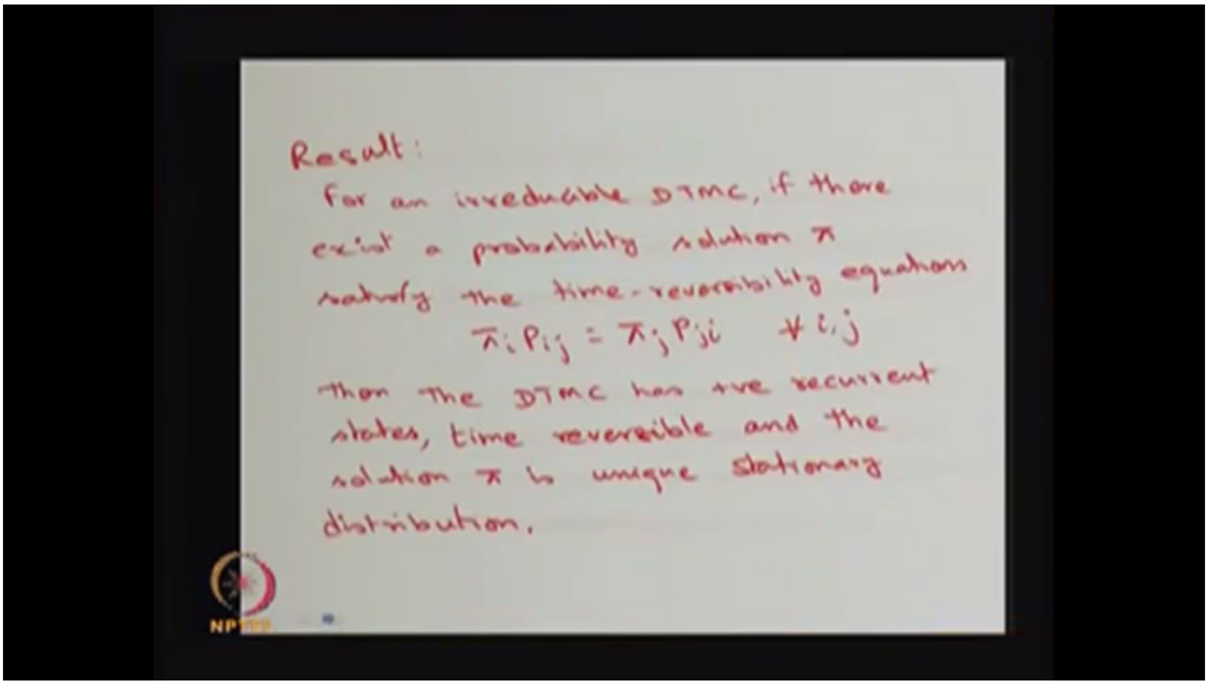

For a irreducible DTMC, if there exists a probability solution, that means I started with there exists a solution, but since I know the result it is a irreducible, aperiodic, positive recurrent and the stationary distribution exists, therefore, I started with the probability solution π and I have checked with the time reversibility equations for the example also. Then I am concluding it is going to have a unique solution.

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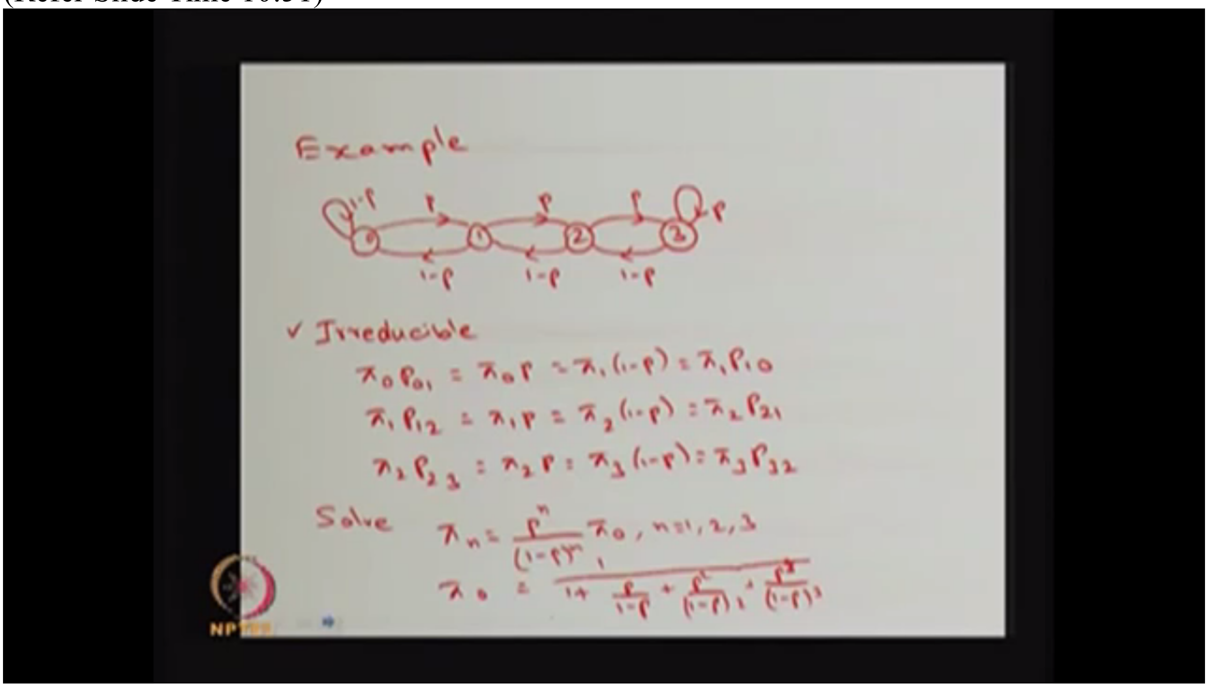

So I have check -- I have verified whether -- I have verified the time reversibility equations. After the time reversibility equations, from that I am getting the π_n in terms of π_0 because the way the recursive relation goes, you can make out π_1 from the first equation you can get π_1 in terms of π_0. Then the second equation π_2 you can get it in terms of π_1. Then in turn you can get π_2 in terms of π_0. Similarly, you can get π_3 in terms of π_0.

Now you have to find out what is π_0? π_0, you can use the normalization equation. That is the summation of π_i is equal to 1. That is a $\pi_0 + \pi_1 + \pi_2 + \pi_3$ that is equal to 1. From that you can get π_0 is equal to 1 divided by $(1 + (P/(1-P)) + (P^2/(1-P)^2) + (P^3/(1-P)^3))$. So this is going to be the π_0. Substitute π_0 in this π_n. Therefore, you got the -- you get the π_n also.

So we are getting the unique stationary distribution because this DTMC is a time reversible. Therefore, without solving the πP is equal to π, you are using the time reversibility equation itself and summation of π_i is equal to 1, you are getting the π_i's. So that is the easy way whenever the DTMC is going to be a time reversible Markov chain.

So in this example, we have used time reversibility property to get the unique stationary distributions.

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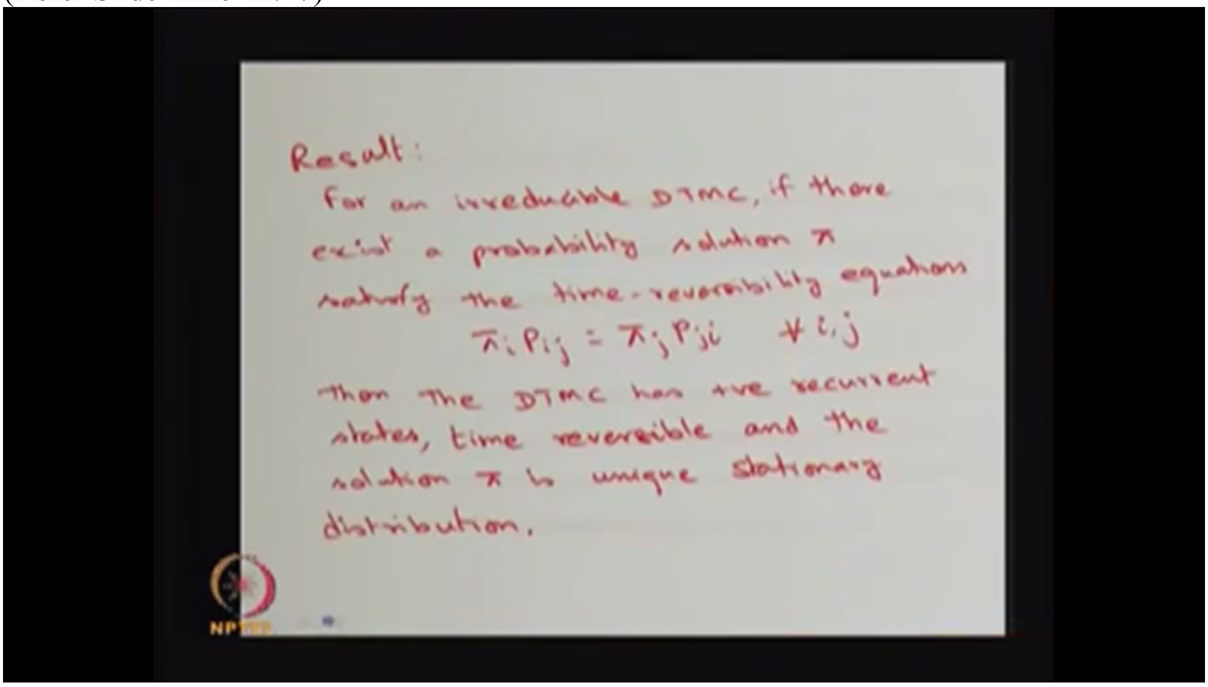

The result we said there is and the solution π is a unique stationary distribution. So in the results means only the proof whether the PI is going to satisfy the equation by πP is equal to π. So that can be easily proved by taking summation of $\pi_i P_{ij}$, that is same as summation over i π_j of P_{ji}. Because it satisfies the time reversibility equation, we can write summation over i π_i times P_{ij} is same as the summation $\pi_j P_{ji}$. That is same as you can take out the π_j outside. That is same as the summation over i the P_{ji} and you know that the summation over i the P_{ji}, that is going to be 1. Therefore, this is going to be a πj.

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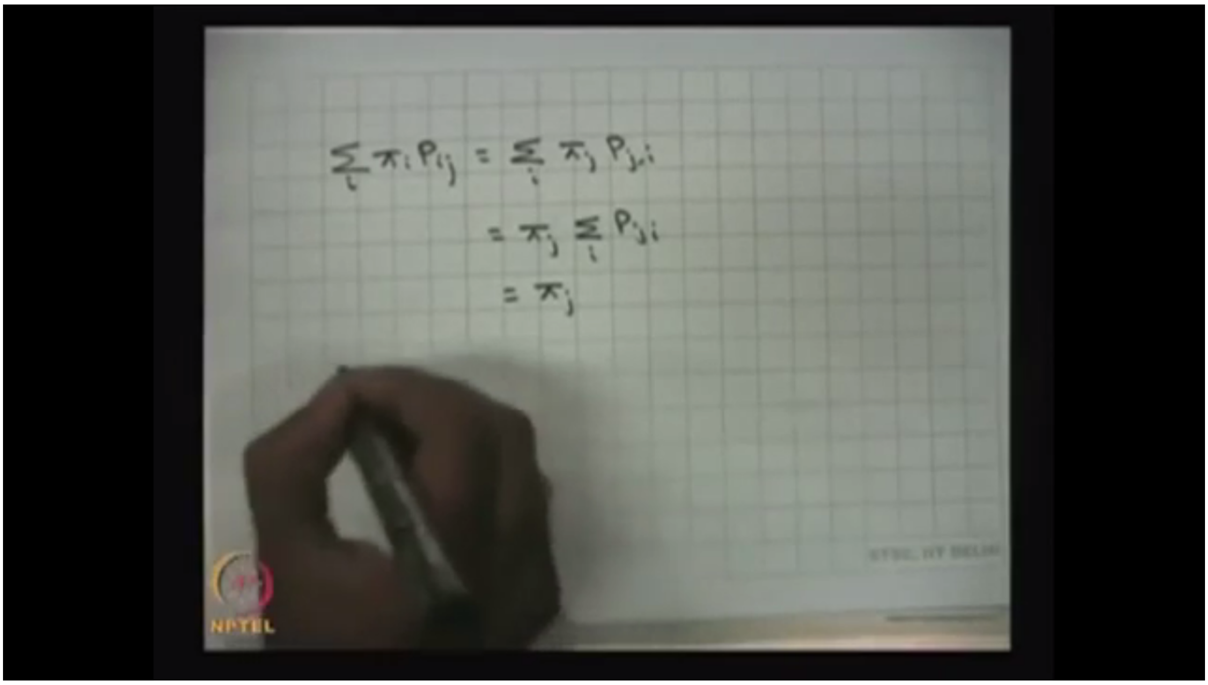

So hence we get summation over i the π_i P_{ij} that is equal to of π_j. So this is nothing but in the matrix form, πP is equal to π.

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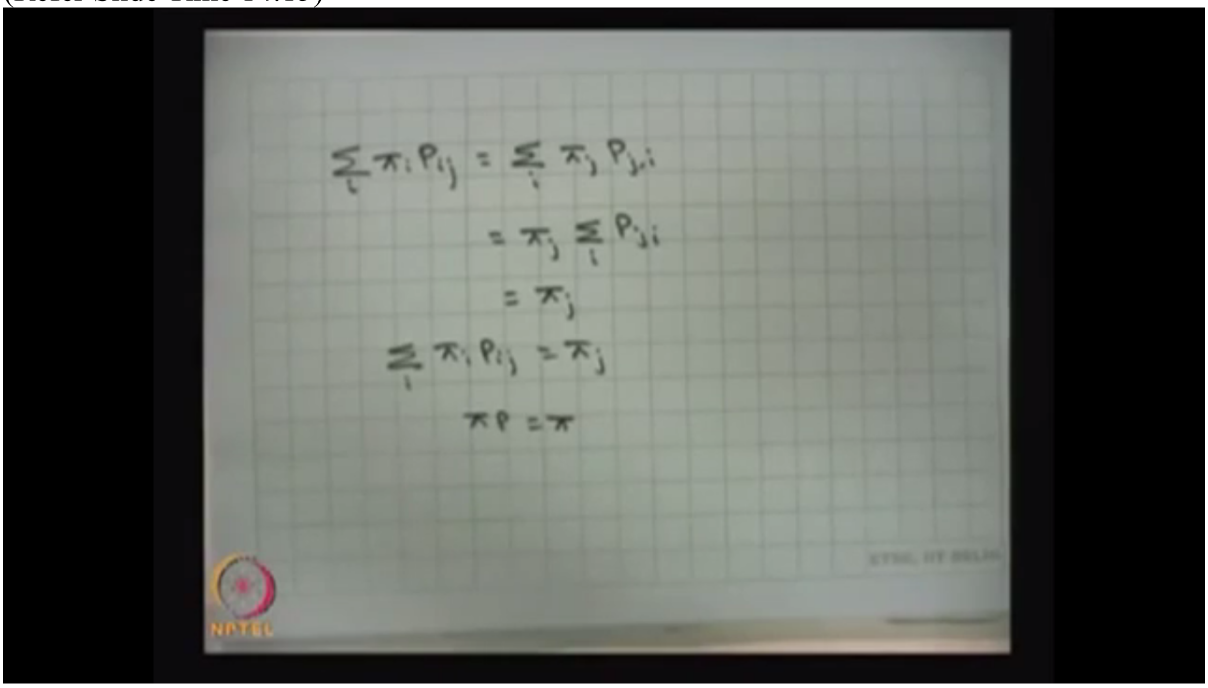

So whenever you have a irreducible DTMC and satisfies the time reversibility equations, then you have a unique solution π and that π unique solution is a stationary distribution. So, and also you can prove easily it has the positive recurrent state and the time reversible Markov chain also.

So with this, with this proof, we have got the result the π is going to be a unique stationary distribution and also I have given the example how to use the time reversibility equations to get the unique stationary distribution.