Stationary Distribution and Examples

Dr. S Dharmaraja
Department of Mathematics, IIT Delhi

Now I am going to move, the stationary distribution. The stationary distribution also a very important concept in the Markov chain and as such first I am going to give the definition of a stationary distribution.

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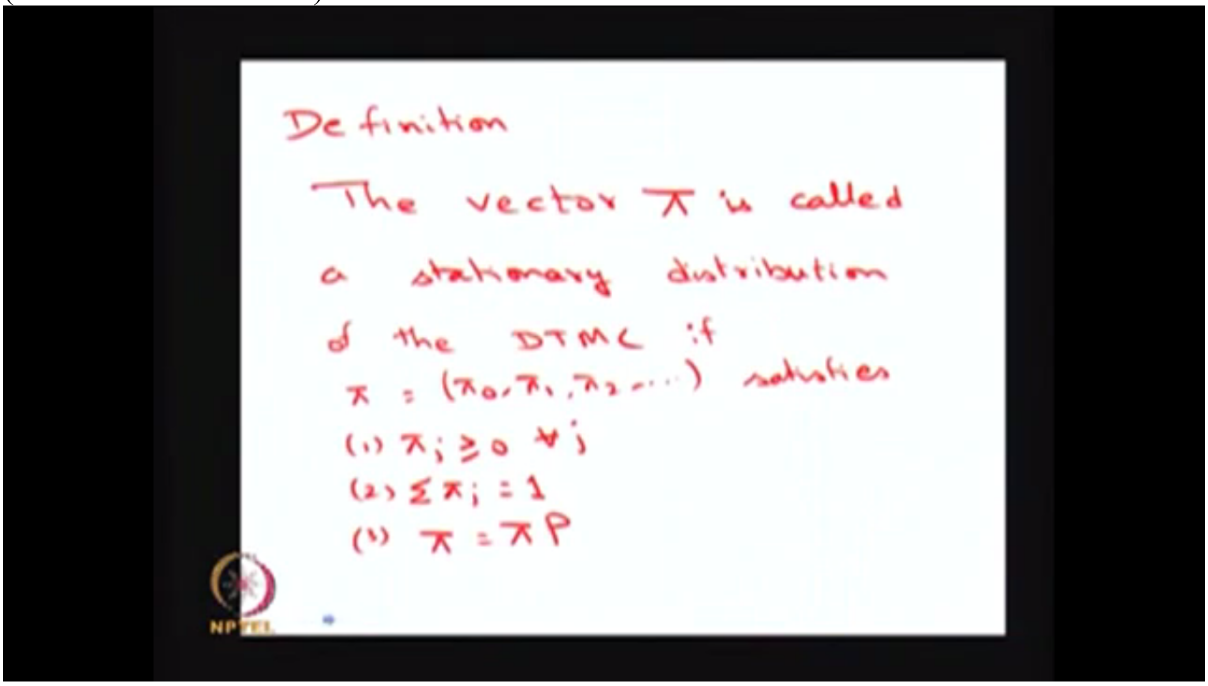

The vector π is called a stationary distribution of a time-homogeneous discrete-time Markov chain if that vector satisfies the first condition. All these values π_i's, sorry, π_j's are greater than or equal to 0 for all j and the summation over the π_j's that is going to be 1 and the third condition π is going to be same as π times P where P is the one-step transition probability matrix.

So any vector π satisfies these three conditions, then that vector is going to be call it as a stationary distribution. This is nothing to do with the limiting distribution, the one I have discussed earlier, but for an irreducible aperiodic Markov chain, the limiting distribution is same as the stationary distribution. That is also going to be same as the equilibrium or a steady-state distribution. All these three distributions are going to be same for an irreducible aperiodic Markov chain, but in general all these three things are going to be different. So here I am giving the definition of a stationary distribution by satisfying these three properties.

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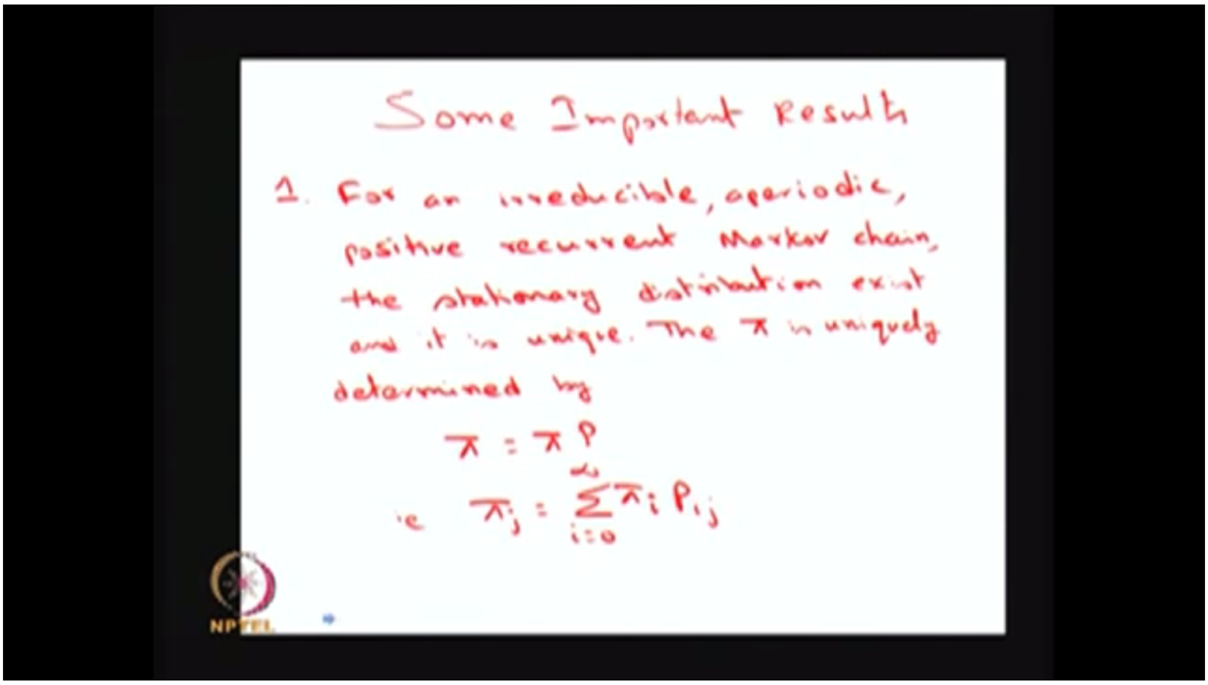

Now I am going to give some important results for that. The first result is for an irreducible aperiodic, a positive recurrent Markov chain, the stationary distribution exists and it is unique. The one definition I have given earlier, I have discussed the aperiodic, irreducible. I have to include the positive recurrent also because these three things are important for an irreducible, aperiodic, a positive recurrent Markov chain, all these three distributions, limiting distribution, stationary distribution, steady state or equilibrium distributions, all three are same. I have to include the positive recurrent also.

So what I am giving in this result, then π is uniquely determined by solving this equation π is equal to πP with the summation of π's are going to be 1. So if I solve π is equal to πP along with the summation of π is equal to 1, that will give a unique π and that π is going to be a stationary distribution for an irreducible, aperiodic, positive recurrent Markov chain.

Irreducible means all the states are communicating with all other states. Aperiodic means the periodicity for a state is 1. The greatest common divisor of a system coming back to the same state, all the possible, possible steps, that greatest common divisor is 1. The positive recurrent means it's a recurrent state. That means with the probability 1, the system start from one state and coming back to the same state, that probability is 1. The positive recurrent means the mean recurrence time that is going to be a finite value. If these three conditions are going to be satisfied by any time-homogeneous discrete-time Markov chain, then the stationary distribution can be computed using π is equal to πP and the summation is equal to 1. That is going to be a unique value.

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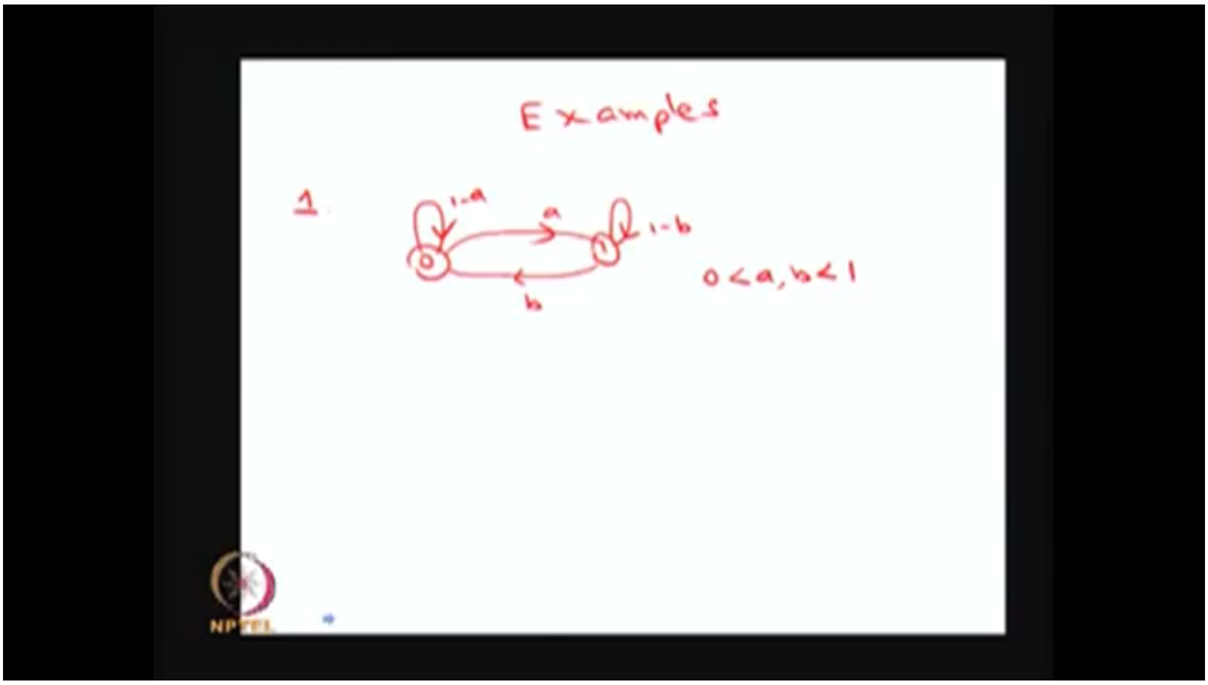

I am giving the same example. I am giving the same example that is the two-state model with states 0 and 1 with the probability is self-loop 1 minus a and self-loop 1 minus b and the system going from the state 0 to 1 in one step that is a and the system is going from the state 1 to 0 that probability is b.

So I am giving a very simple two-state model and you can solve π is equal to πP and the summation is equal to 1 and you will get the probabilities. And these probabilities is same as the probabilities you got it in the limiting state probability.

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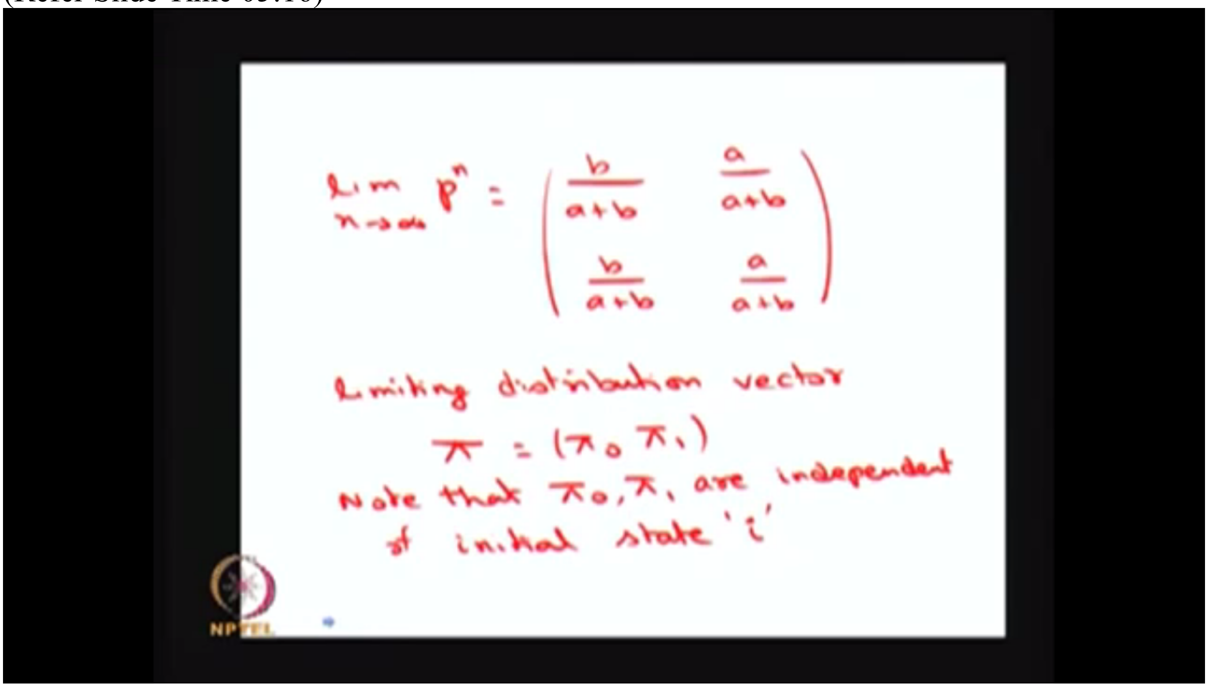

If you solve, yeah, if you solve the two-state model with the π is equal to πP, you will get the probabilities that π_0 is going to be b divided by a+b and π_1 is going to be a divided by a+b and it satisfies the summation of π is equal to 1 and it also satisfied π is equal to πP. That means in this model, it is a irreducible, aperiodic, positive recurrent model. Therefore, the limiting distribution is same as the stationary distributions also.

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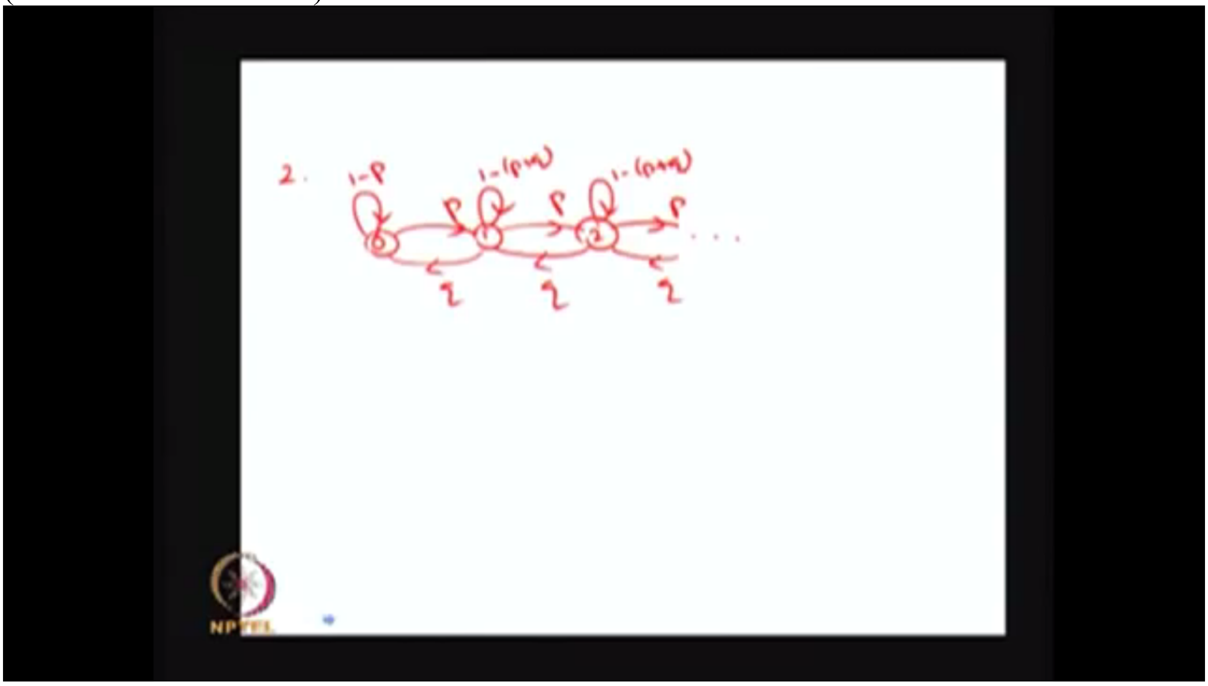

The second example, that is with the infinite state. So here the number of states are going to be countably infinite. I can start with to find out the stationary distributions, before that I have to cross check whether it is going to be a irreducible, aperiodic, positive recurrent Markov chain.

It is irreducible because the way I have given the probabilities, I make the assumption the probabilities are lies between 0 to 1 and the probability of lies -- the q is also lies between 0 to 1. Therefore, each state is communicating with each other state. Therefore, it is going to be a irreducible.

The second one, it is -- it has to be aperiodical. Aperiodic means the periodicity for each state because the greatest common divisor is going to be 1 because the coming back to the state is via self-loop or going to the some other state and coming back and there also has a self-loop, therefore, it is going to be -- all the states are going to be aperiodic. Therefore, the Markov chain is aperiodic.

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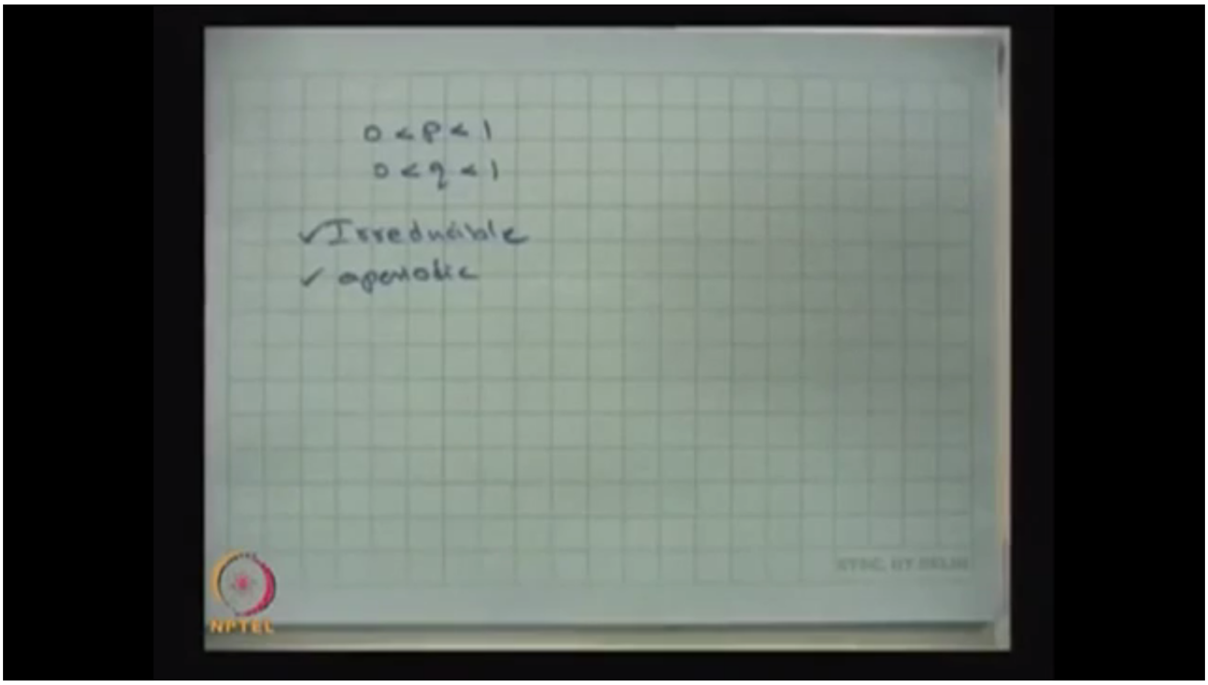

The third one, positive recurrent. Since it is a infinite state model, you cannot get the -- you cannot come to the conclusion whether these μ_{00} is going to be a finite quantity unless otherwise substituting the value of p and q. So what I will do, I will make the assumption, assume that all states are positive recurrent. Then later I will find out what is the condition to be a positive recurrent.

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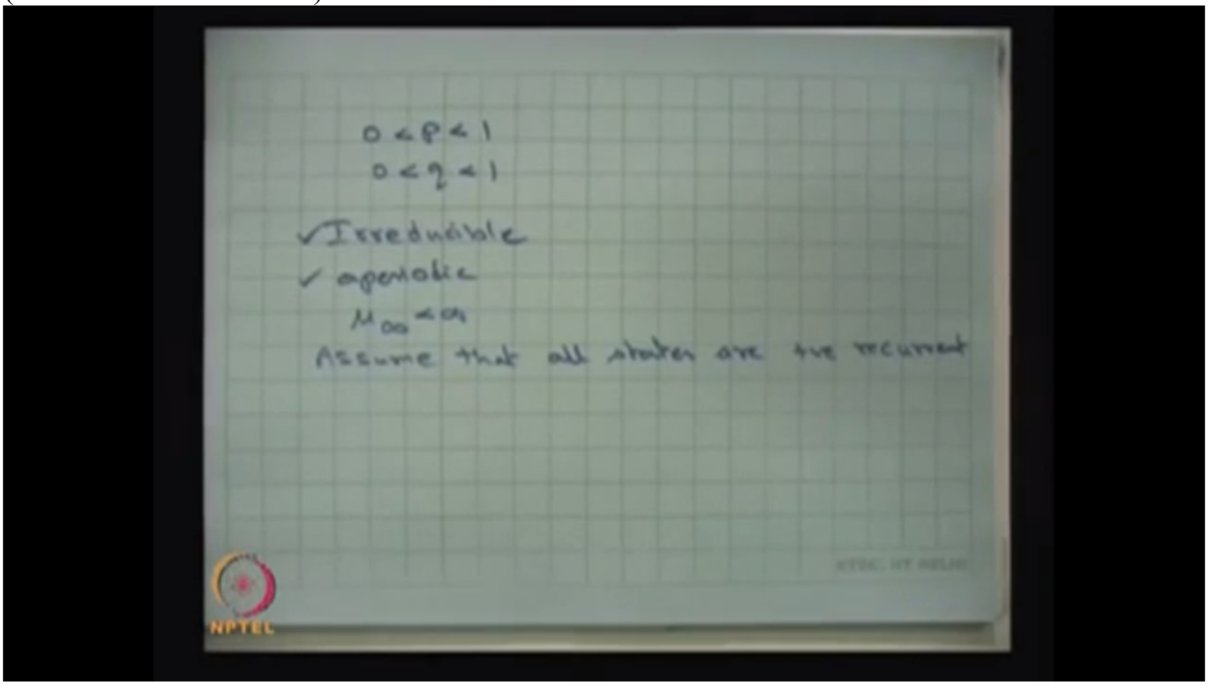

So I make the assumption. Even I don't want to make the -- I don't want to make the assumption for all the states are going to be positive recurrent. I can make the assumption for only one state is going to be a positive recurrent and since it is a irreducible Markov chain and all the states are going to be of the same type, therefore, it will come to the conclusion all the states are going to be a positive recurrent. So I make the assumption one state is going to be a positive recurrent. Therefore, it land up all the states are going to be positive recurrent.

Now once I made a assumption of all the states are positive recurrent, therefore, it satisfies all the results of the first result that is a irreducible, aperiodic, positive recurrent Markov chain with the infinite state space. Therefore, I can find out the -- I can come to the conclusion, the limiting distribution, sorry, the stationary distribution exists and it is going to be unique and that can be computed by solving the equation π is equal to πP with the summation of π_i's is equal to 1 where π is the vector and P is the one-step transition probability matrix. That one-step transition probability matrix can be created using the state transition diagram which I have given.

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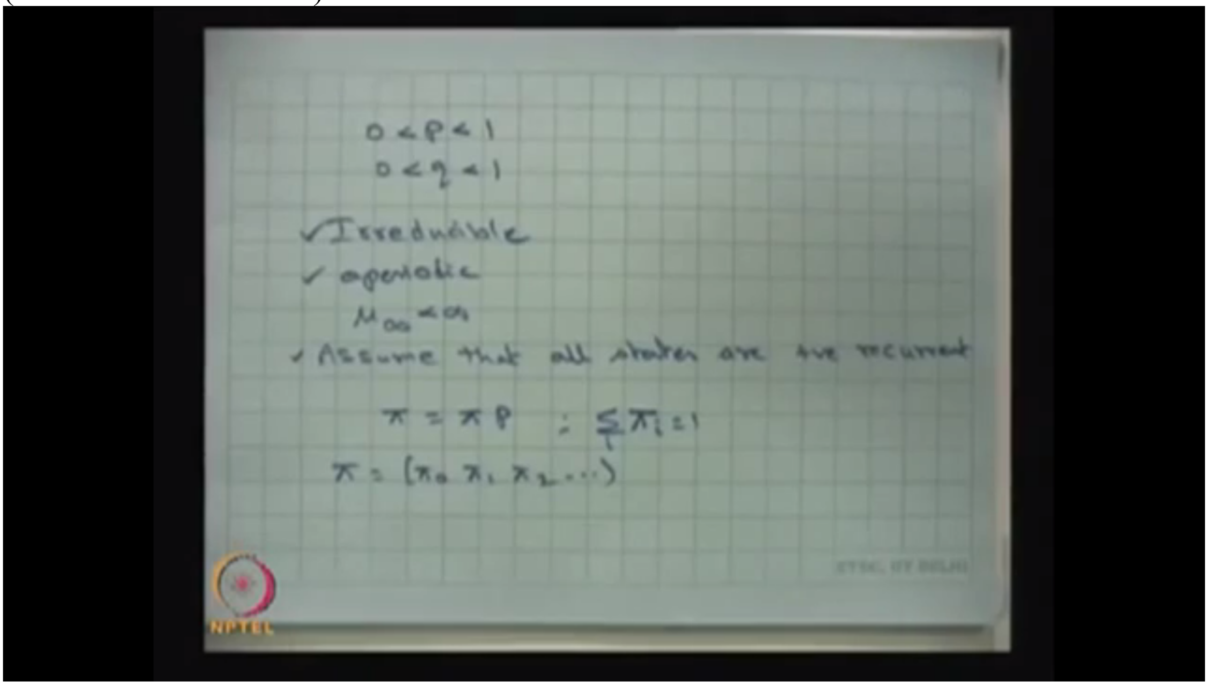

So if I take the -- if I find out what is the first equation from this vector π is equal to π_0, π_1, π_2 and so on, here also this and P is the matrix, therefore, I will get the first equation as π_0 is equal to π_0 times 1 minus P plus π_1 times q. So this is the first equation of from the matrix π is equal to -- in the matrix form π is equal to πP. So the first equation is $\pi_0 = \pi_0 * (1-P) + \pi_1 q$.

So from this equation, I can get π_1 because I can take this π_0 this side and I can cancel. So I will get π_1 is equal to P divided by q times π_0. From the first equation, we get the relation π_1 in terms of π_0.

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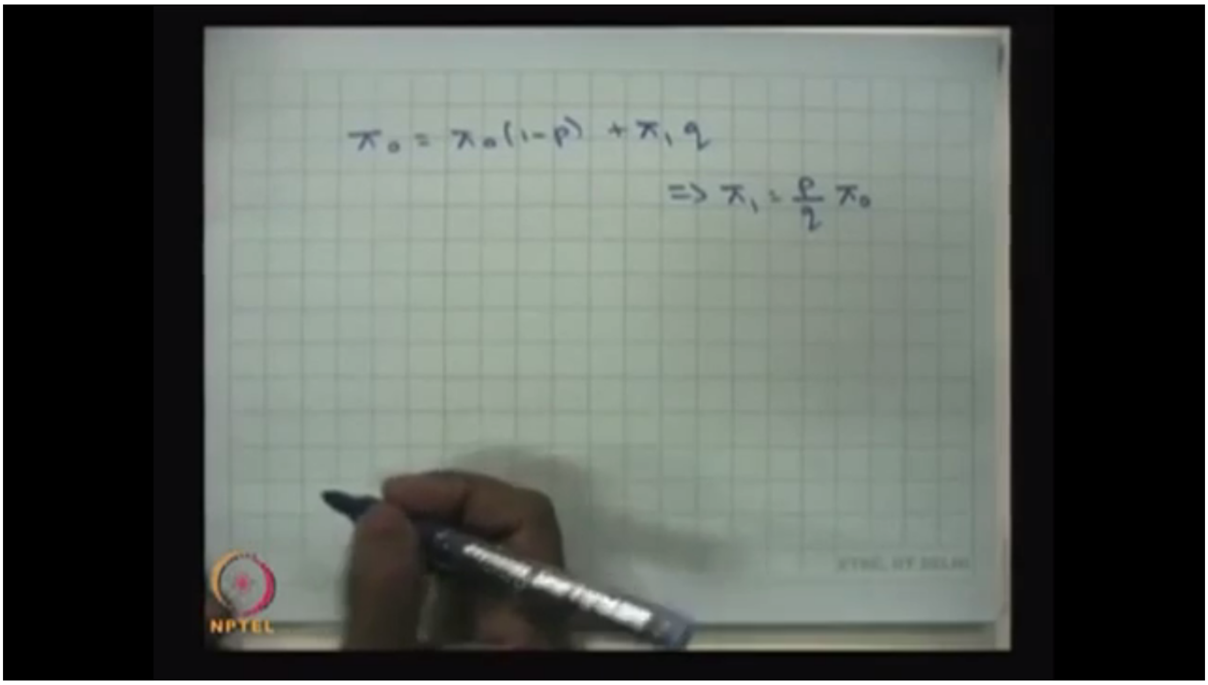

Now I will take a second equation from π is equal to πP. So that will give π_1 is equal to π_0 times P plus π_1 times 1 minus P minus q plus π_2 times q. So this equation I have π_0, π_1 and π_2. So what I can do, I can write π_1 in terms of π_0 and I can simplify this equation. If I simplify, I will get π_2 is same as P^2 by q^2 times π_0.

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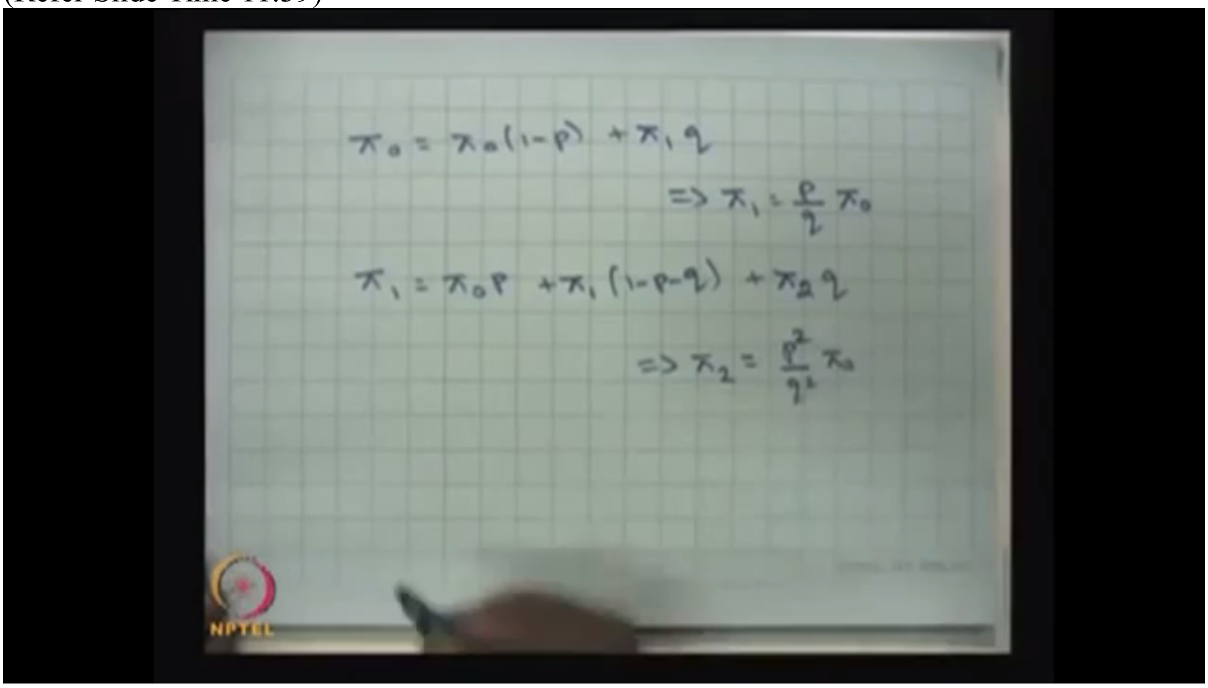

Because I am substituting π_1 in terms of π_0 in this equation, therefore, I get π_2 in terms of π_0. That is π_2 is equal to P^2 by q^2 times π_0.

Similarly, if I take the third equation and do the same thing, finally, I get π_3 is equal to $(P^3/q^3)\pi_0$. The same way I can go further. Therefore, I get π_n in terms of π_0 for n is equal to 1, 2, 3 and so on.

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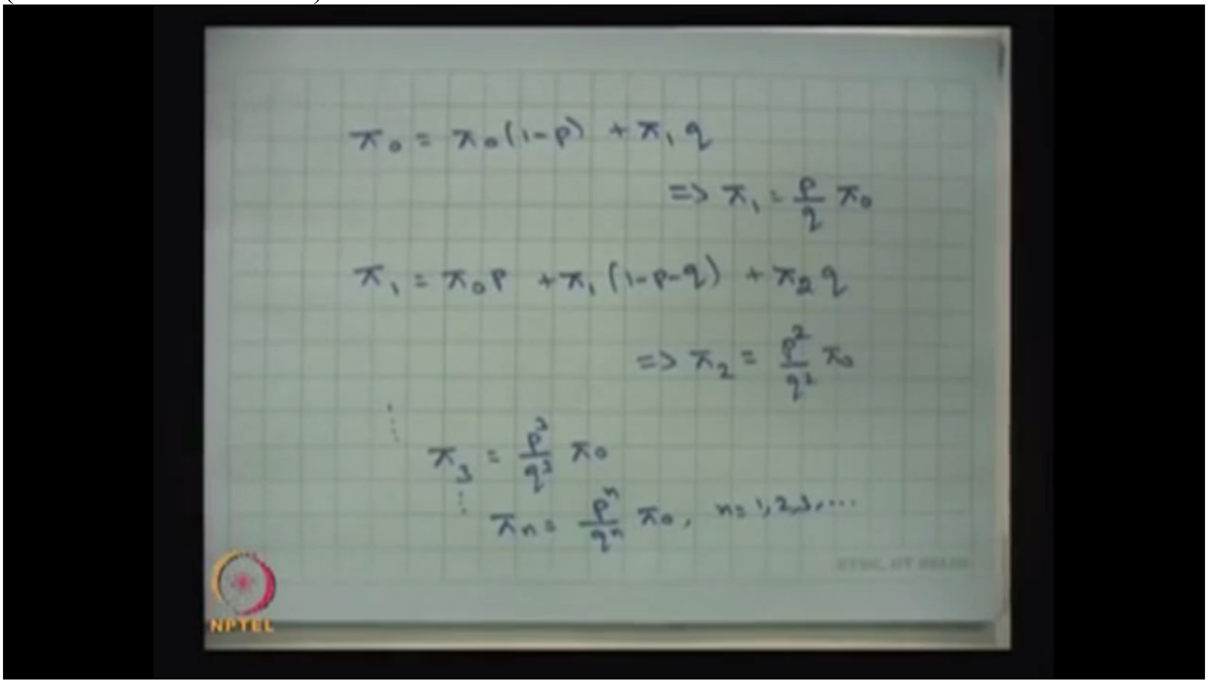

So this is the way I can solve this equation π is equal to πP. That's a homogeneous equation. We have to be very careful with the homogenous equation. So the trivial solutions are going to be 0, but we are trying to find out the non-trivial solution. Therefore, we are using the normalization, that is the summation of π_i is equal to 1. Till now I have not used. So I have -- I have just simplified that π is equal to πP and getting π_n in terms of π_0.

Now I have to use summation of π_i is equal to 1 starting from i is equal to 0 to infinity. Therefore, the π_0 will be out $1 + P/q + P^2/q^2$ and so on that is equal to 1. Therefore, the π_0 is going to be 1 divided by $1 + P/q + P^2/q^2$ and so on that is π_0.

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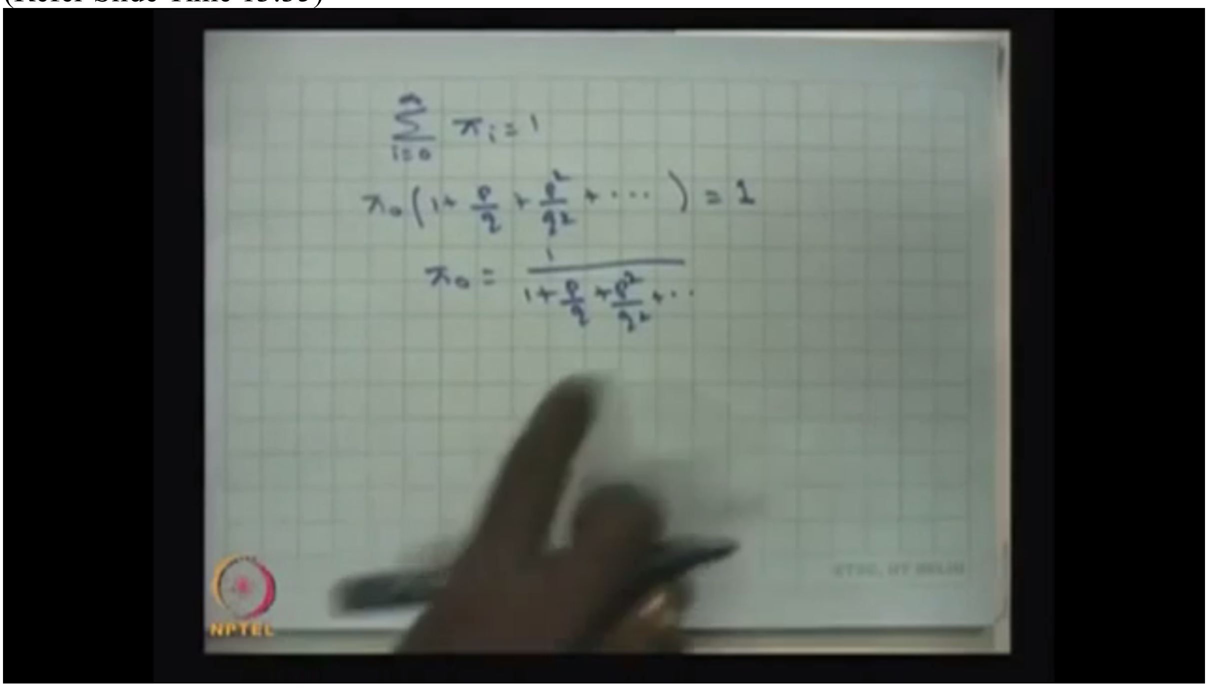

Since it is infinite terms in the denominator, as long as this is converges, you will get a nonzero value for π_0. In turn you will get a π_i is equal to P/q power n times π_0 provided this denominator is going to be converges.

When the denominator is going to be converges, in this situation, as long as the P/q is going to be less than 1 if P/q is less than 1. The earlier condition is P is lies between 0 to 1 and q is lies between 0 to 1. Now I'm making the additional condition P/q is less than 1. That will ensure the denominator converges. Therefore, the π_0 is going to be a nonzero value. Therefore, the π_n's are going to be $(P/q)^n * \pi_0$ where π_0 is written 1 divided by $1 + P/q + (P/q)^2$ and so on. So provided P/q is less than 1.

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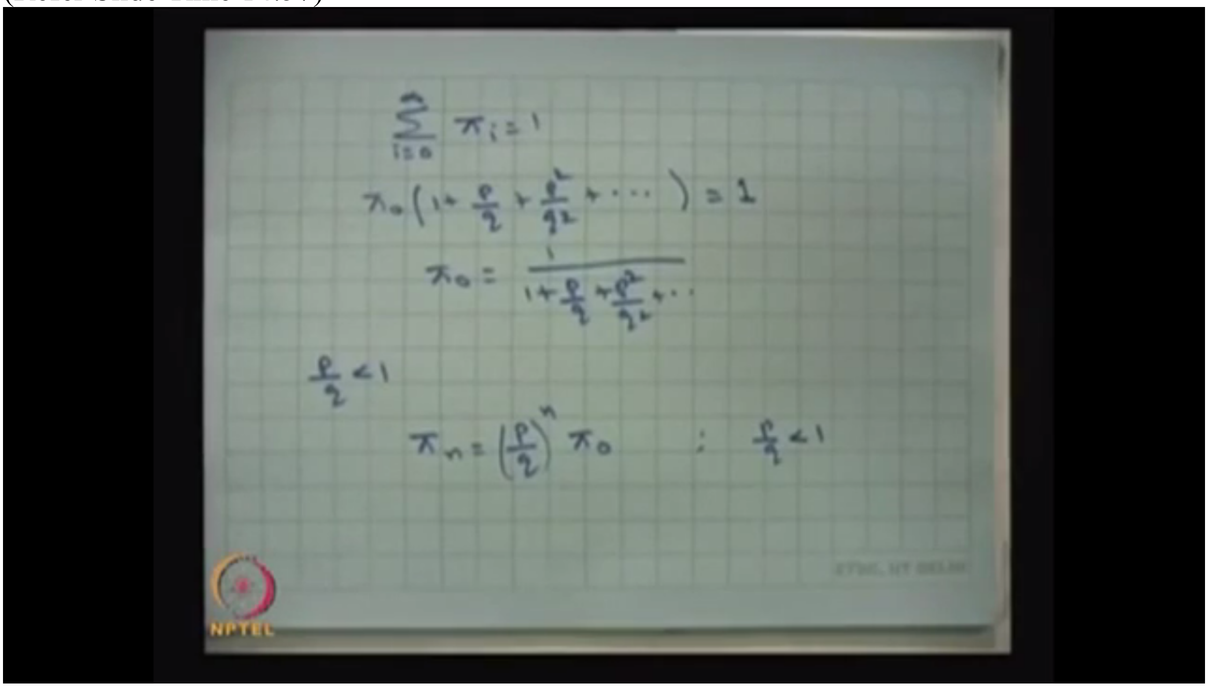

If you recall, we made the assumption the states are going to be a positive recurrent. If this P/q is less than 1, then you can conclude the mean recurrence time is going to be a finite value if you make the assumption P/q is less than 1 that will ensure the mean recurrence time for any state is going to be a finite value. Therefore, all the states are going to be positive recurrent and then the stationary distribution exists.

Therefore, this is the condition for a positive recurrent state for this model and the stationary distribution that is going to be π_n is equal to $(P/q)^n * \pi_0$. This is nothing but in a longer run, what is the probability that the system will be in the state n. That probability is $(P/q)^n$ this π_0 and π_0 is given in this form.

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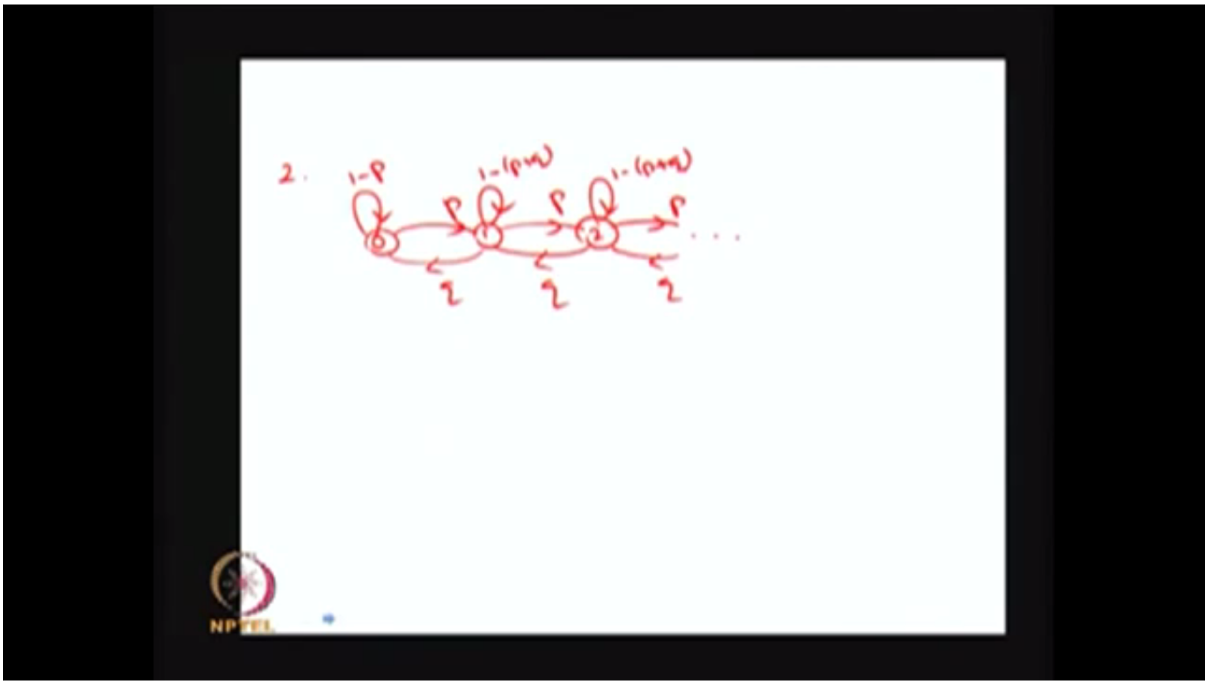

And in this example, we have taken each state for the P/q is the same for all the states. We can go for -- go for in general situation the system going from 0 to 1 could be P_0, system going from the state 1 to 2 may be P_1 and so on. Therefore, it need not all the P_i, P's need not be same and the q's also need not be same.

So you can generalize this model and this model is nothing but a one-dimensional random walk and here the 0 is a -- it's a barrier. The system is not going away from the 0 in the left side. Therefore, 0 is a barrier and this is a one-dimensional random walk in which the system is keep moving into the different states in subsequent steps and there is a possibility the system may be in the same state with the positive probability of 1 minus P plus q in this model. In general, you can go for the P, P_0, P_1, P_2 and so on and similarly q_1, q_2, q_3 and so on also.