Example of Limiting Distribution and Ergodicity

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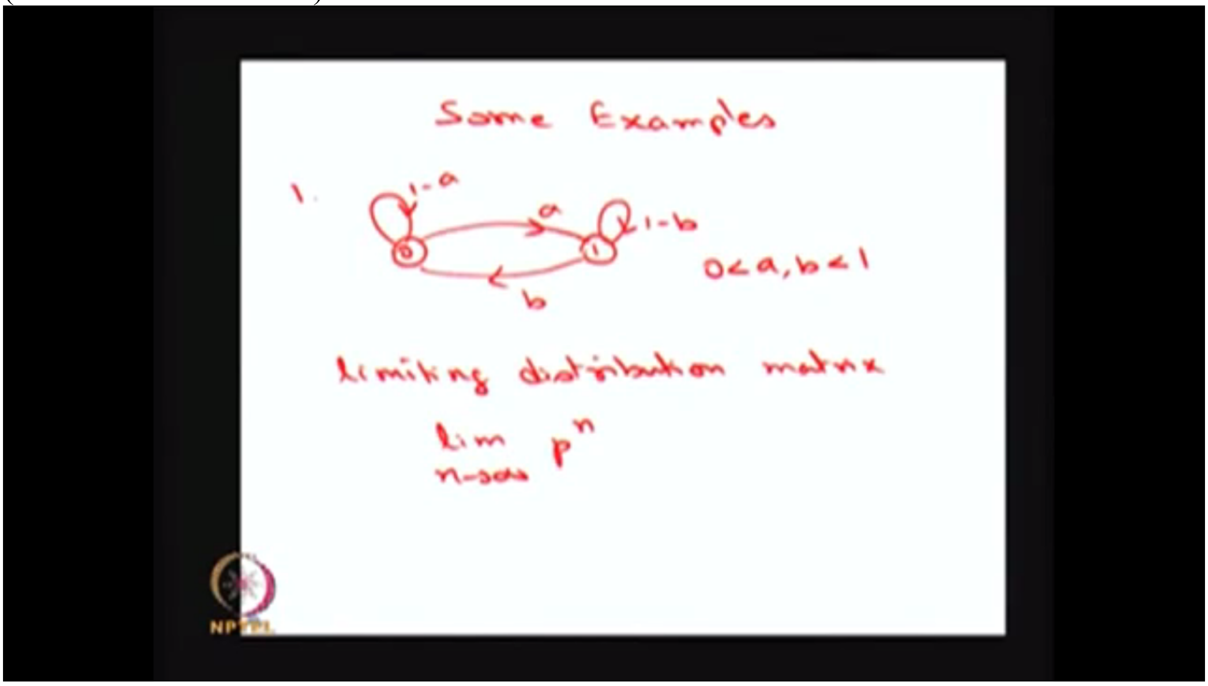

Now I am going to discuss the simple situation in which how we can get the limiting state probabilities. This is a simple model in which we have only two states and this two states model is the very good example in the sense this can be interpreted as the many situation.

For example, you can think of a weather problem in which 0 is for a rainy day and 1 is for the sunny day and what is the probability that the next day is going to be a sunny day from the rainy day that probability is a and from a rainy day to sunny day, it is going to be the probability b, and the next day is going to be the same thing whether it is a rainy day or sunny day according to the probabilities 1 minus a and 1 minus b and you can assume that both the probabilities a and b lies between open interval 0 to 1.

In this case, this is a very simple two-state model. Like this, we can give many more applications can be interpreted with the two-state model with the transition probability. This is a one-step transition probability with the P matrix that is the P matrix is the state 0 and 1, 0 and 1. So 0 to 0, 1 minus a, 0 to 1 that probability a, and 1 to 0 the probability is b and 1 to 1 that is probability 1 minus b.

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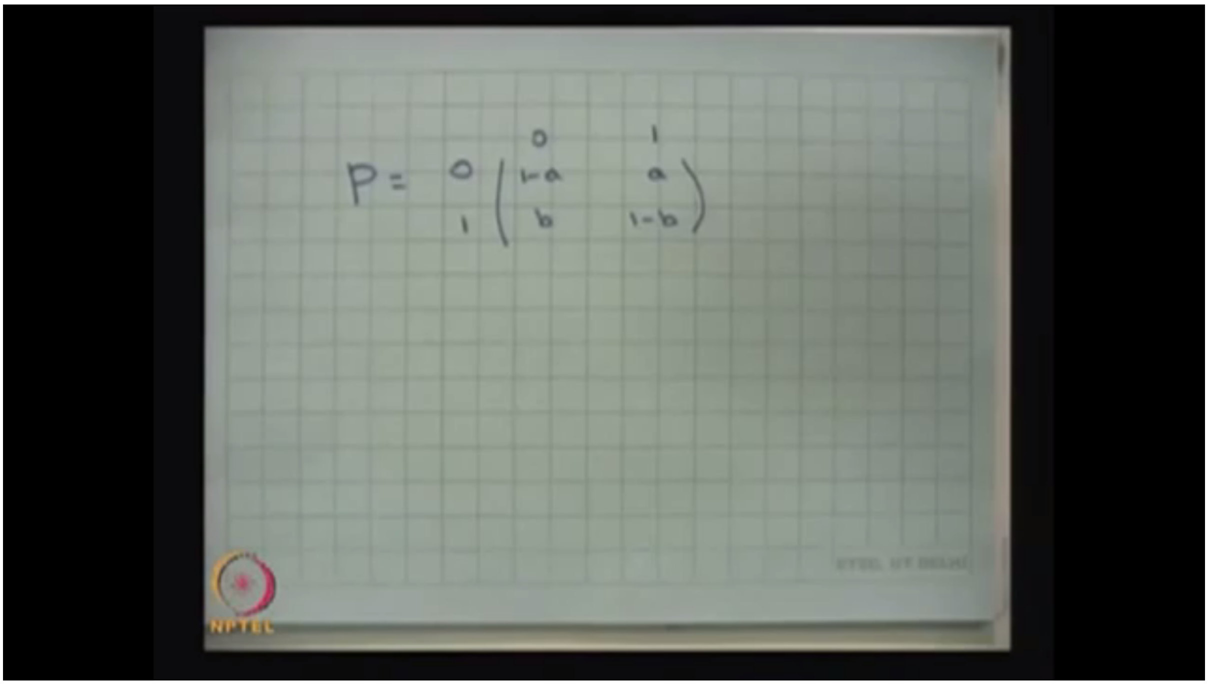

So this is the one-step transition probability matrix and from this model, you can see that it is since a and b is open interval 0 to 1, this is going to be a irreducible Markov chain and with the finite state space. Therefore, using the result we can conclude all the states are going to be a positive recurrent. That can be verified from the classification of the states also. You can verify that first one is recurrent state. That means you can find out the probability of F_{00} that is going to be 1 and similarly you can conclude, you can find out F_{11} that is also going to be 1. So you can conclude both the states are going to be a positive recurrent and you can find out μ_{00} that is going to be a finite quantity as well as μ_{11} that is also going to be a finite quantity.

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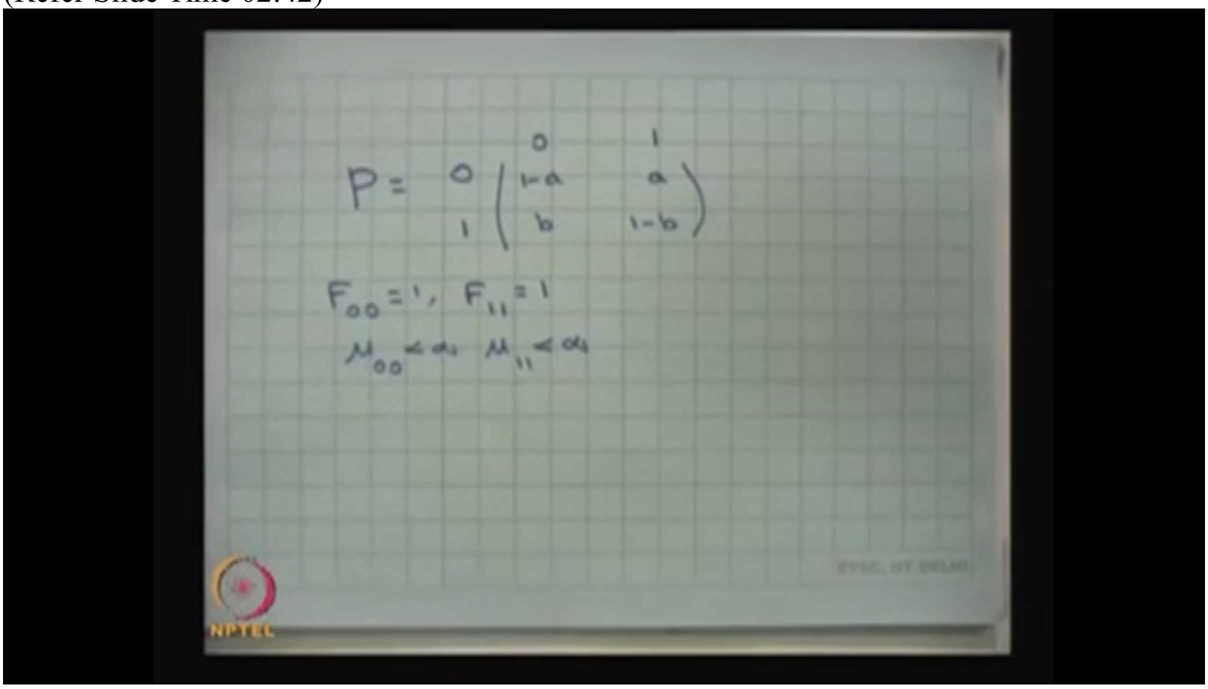

Therefore, you can conclude it is going to be a positive recurrent.

Now our interest is what is the limiting distribution? That means you find out what is a limiting distribution matrix. That is nothing but limit n tends to infinity P power n where P power n is nothing but the n-step transition probability matrix that is same as the one-step transition probability matrix power n.

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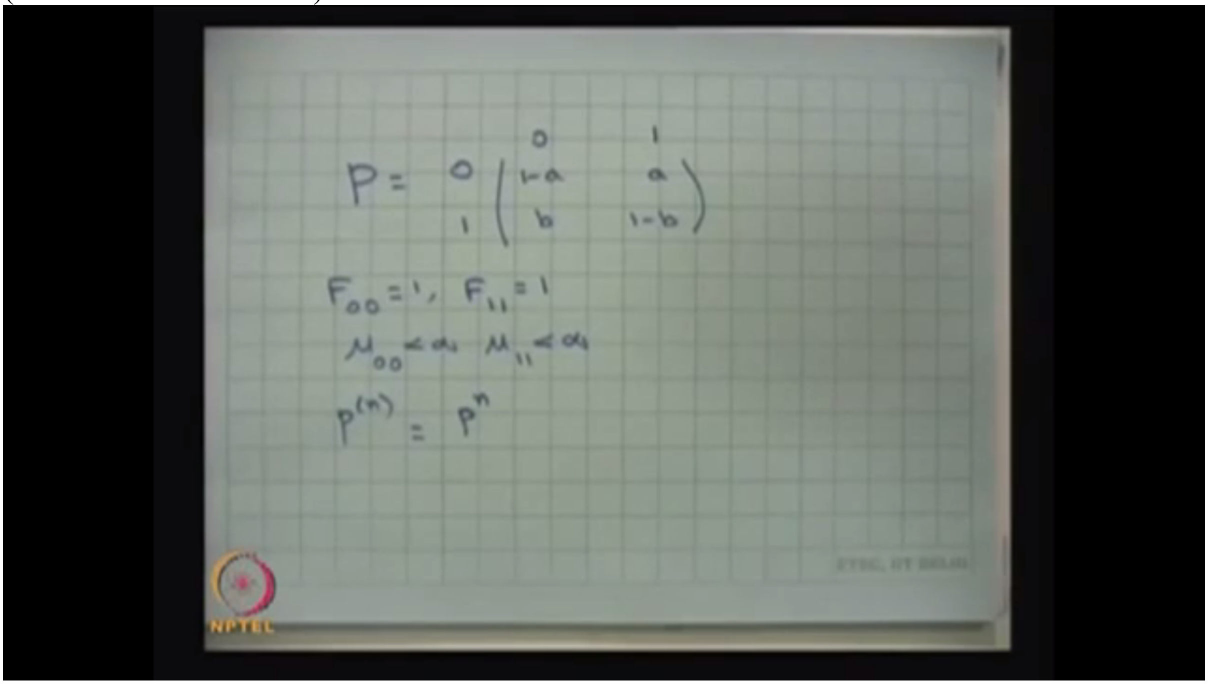

That means you have to find out what is P power n for any n. Then you have to find out what is the P power n matrix as n tends to infinity.

So you can use either eigenvalue and eigenvector method or you can use the by induction method. That means you find out P power 2, then P power 3 and so on. Then you find out what is the P power n by mathematical induction or you find out the eigenvalues or eigenvectors. Then you find out the P power n.

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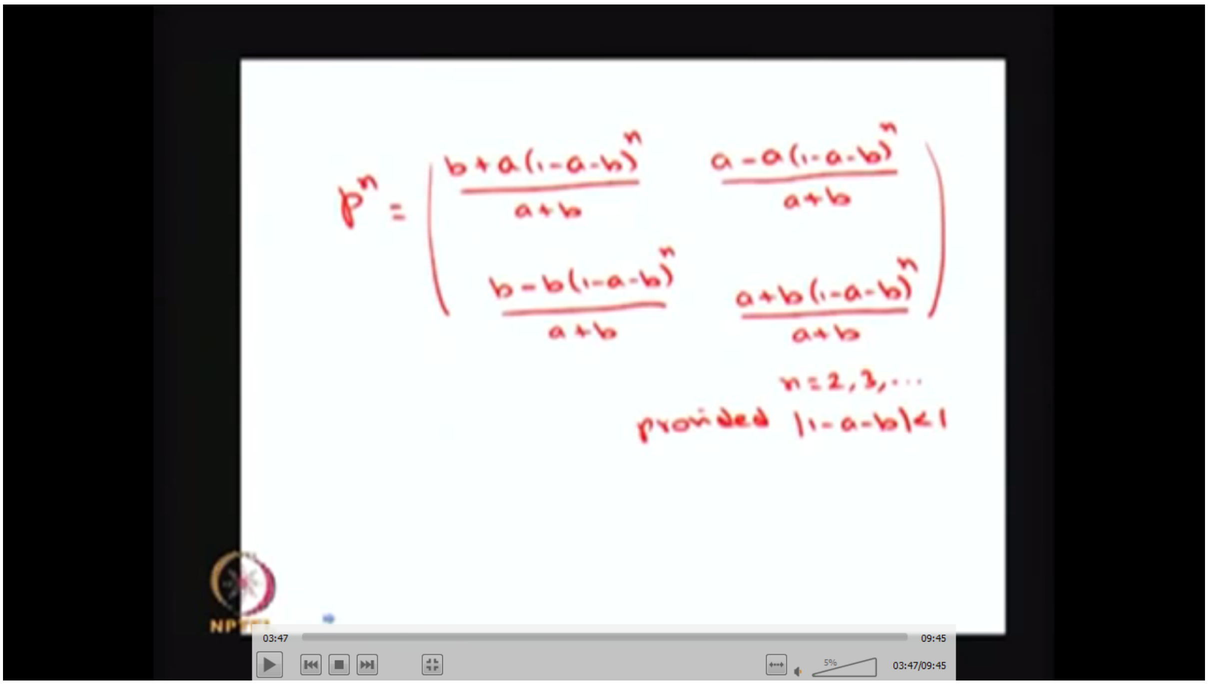

So here I am directly giving the P power n values matrix. So this consists of four elements with the function of a, b and n. This will exist provided the absolute of 1 minus a minus b is less than 1. Otherwise, this won't -- P power n won't exist. Now we are going for as n tends to infinity, what is the matrix?

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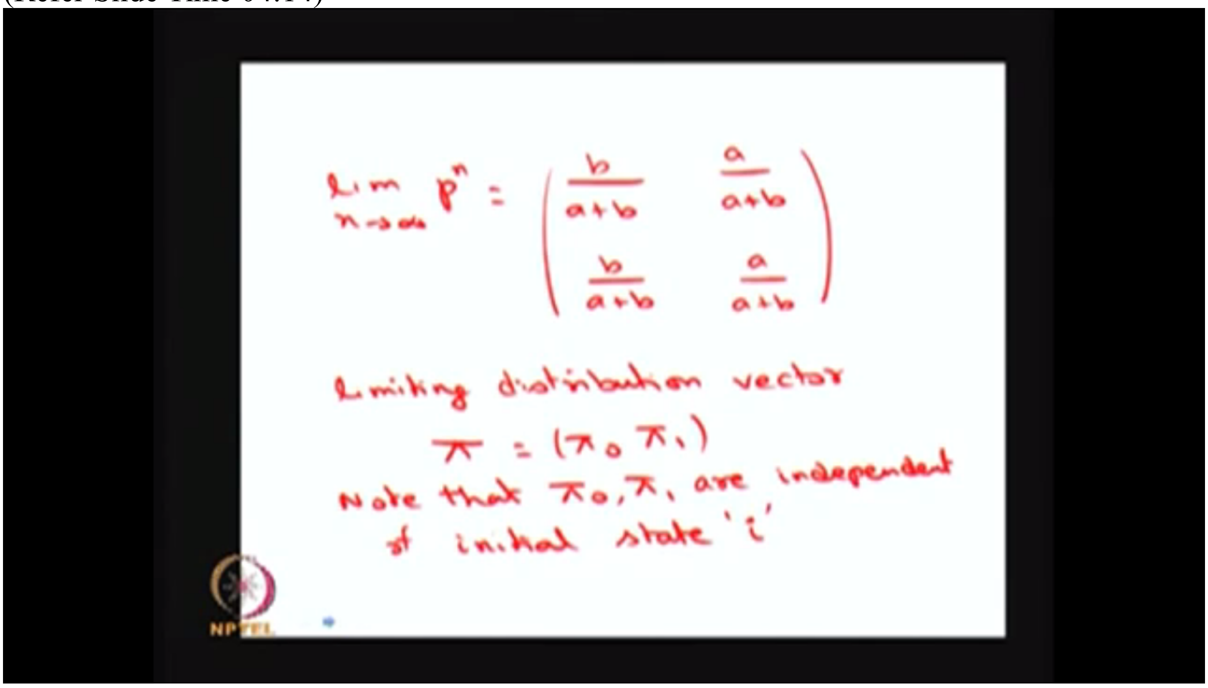

That is limit n tends to infinity, the P power n is -- that matrix is going to be, again, it is going to be a stochastic matrix because the row sum is going to 1 and all the elements are greater than or equal to 0.

Therefore, if the limiting probability matrix exists, then it is going to be unique. The limit exists means it is unique and the row sums are -- row values are -- all the rows are going to be identical. That you can visualize. So that vector is going to be π, that is π_0 and π_1. So the π_0 is nothing but -- π_0 is nothing but b divided by a plus b and π_1 is nothing but a divided by a plus b.

These are all the limiting state probability. That means in a longer run, in a longer run, the system will be in the state 0 or in the state 1 and the system will be in the state 0 in a longer run with the probability b divided by a plus b. In a longer run, the system will be in the state 1 with the probability a divided by a plus b.

Note that these probabilities are independent of initial state i. That means whether you start at time 0 in the state 0 or 1, it doesn't matter. In a longer run, the system is going to be in the state 0 or 1 with these probabilities. So this is the situation for a time-homogeneous discrete-time Markov chain with the finite state space and irreducible Markov chain. Therefore, all the states are positive recurrent and we are getting the limiting state probabilities, which are going to be independent of initial state.

So this information is going to be useful later. Based on this, I am going to (inaudible 06:09) three different probabilities distribution. The one is a limiting distribution. The next one is a stationary distribution. The third one is the steady state or equilibrium distribution.

In general, all these three results are -- all three, these three distributions are different. That is the limiting distributions, stationary distributions and steady state or equilibrium distribution. All three are different in general, but there are in some situation that means for a special case of discrete-time Markov chain, all these three results are going to be same. So for that this example is going to be important one.

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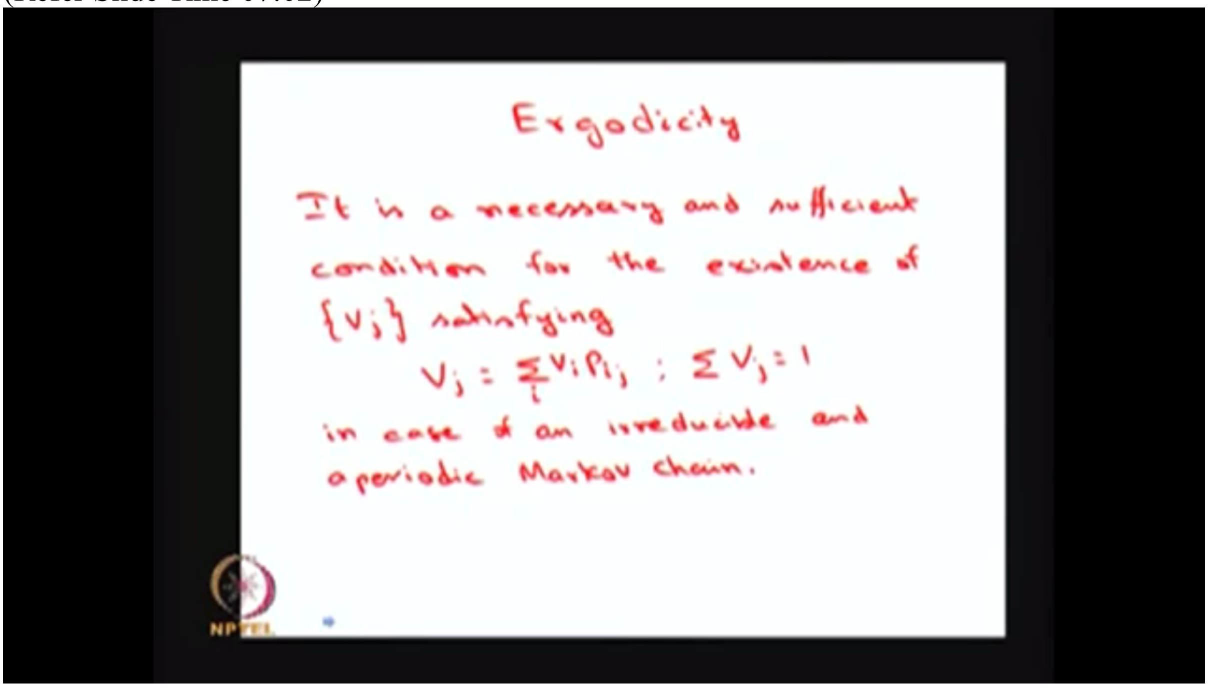

Now I am going to discuss the ergodicity. This is a very important concept in the any dynamical system, but here we are discussing the Markov process or with the -- or we are going to discuss the time-homogeneous discrete-time Markov chain, but the ergodicity is the important concept for any dynamical system.

So I can give the easy definition that is it is necessary and a sufficient condition for existence of V_j's that is nothing but some probability, state probabilities. If that is satisfying, V_j's are going to be summation V_i P_{ij} and the V_i's are going to be -- summation is going to be 1 for j in case of irreducible aperiodic Markov chain, then we are going to say the system is a ergodic system. That means whenever the system is a irreducible and aperiodic Markov chain and then that system is going to be call it as a ergodicity -- ergodic Markov chain or the absorb -- this process is called the ergodicity. That means if you have a irreducible and aperiodic Markov chain, the ergodicity properties are satisfied.

What is the use of ergodicity property in the Markov chain? Since it is a irreducible and aperiodic, these are limiting distributions. These probabilities are going to be independent of initial state. Therefore, this is used in the discrete event simulation. That means if you want to find out the what is the proportion of the time the system being in some state in a longer run, that you can compute by finding the -- that is nothing but the limiting probabilities and this limiting probability is the same as these probabilities V_j's can be computed in this way using the one-step transition probability matrix and that probability is going to be always independent of an initial distribution.

That means whatever the seed you are going to provide in the discrete event simulation, that doesn't matter and you are interested only in the longer run what is the proportion of the time the system being in some state. So that can be easily computed for a ergodic system. That means before you use the ergodic property in the any dynamical system, you have to make sure that that system is a irreducible aperiodic. Then you can use the ergodicity concept.