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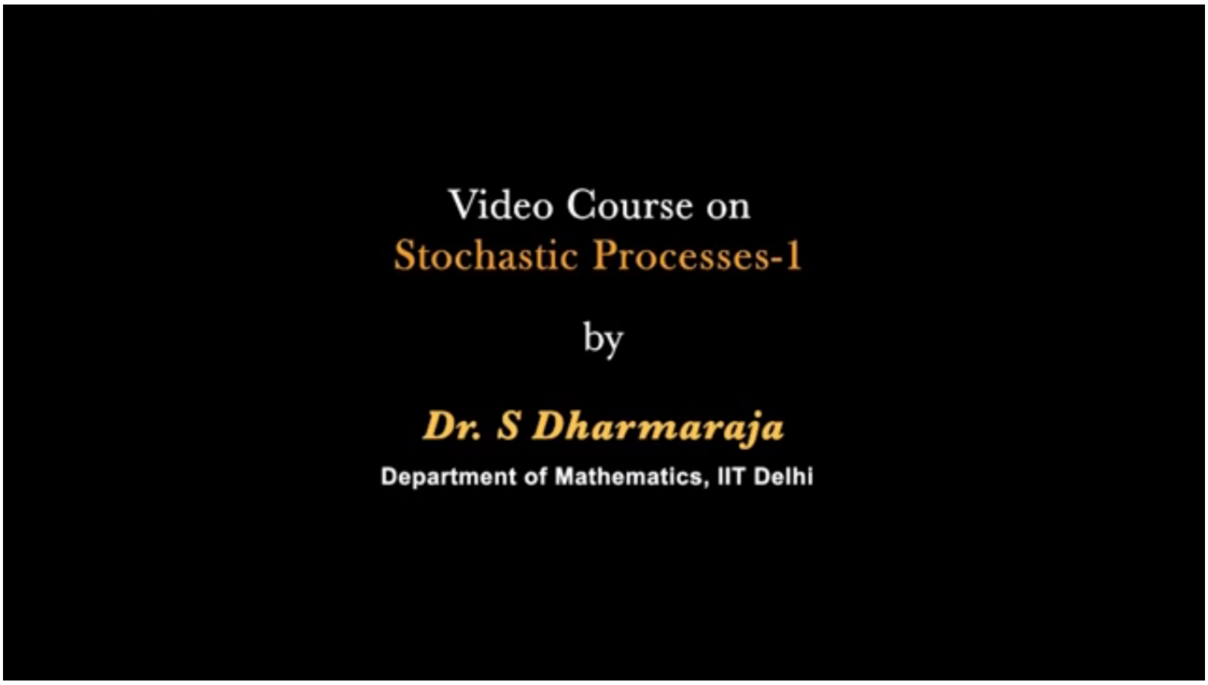

Video Course on
Stochastic Processes-1

by

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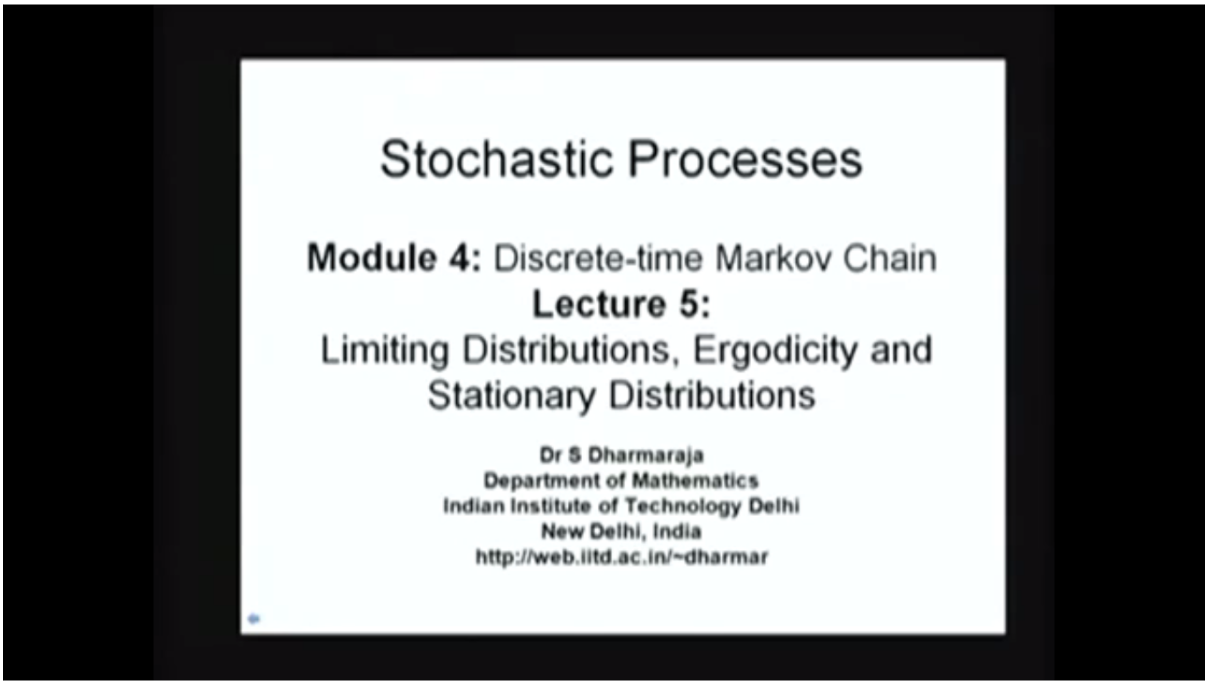

Good morning. This is Module 4, Lecture 5: Limiting Distributions, Ergodicity and Stationary Distribution.

In the last four lectures, we have discussed the discrete-time Markov chain starting with the definition, transition probability matrix.

Then in the second lecture, we have discussed the Chapman–Kolmogorov equations. Then we have discussed the one-step transition probability matrix. Followed by that, we have discussed the N-step transition probability matrix.

In the Lecture 3, we have classified the states of the discrete-time Markov chain as a recurrent, that is a positive recurrent and a null recurrent transient states, absorbing state and the periodicity.

Then we have, in the fourth lecture, we have given a simple examples.

In the fifth lecture, we are going to discuss the limiting distribution, ergodicity, stationary distributions.

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If I am not able to complete limiting distribution and the ergodicity, then I will discuss the stationary distribution in the next lecture, and followed by the limiting distribution and the ergodicity I am going to give some simple examples also.

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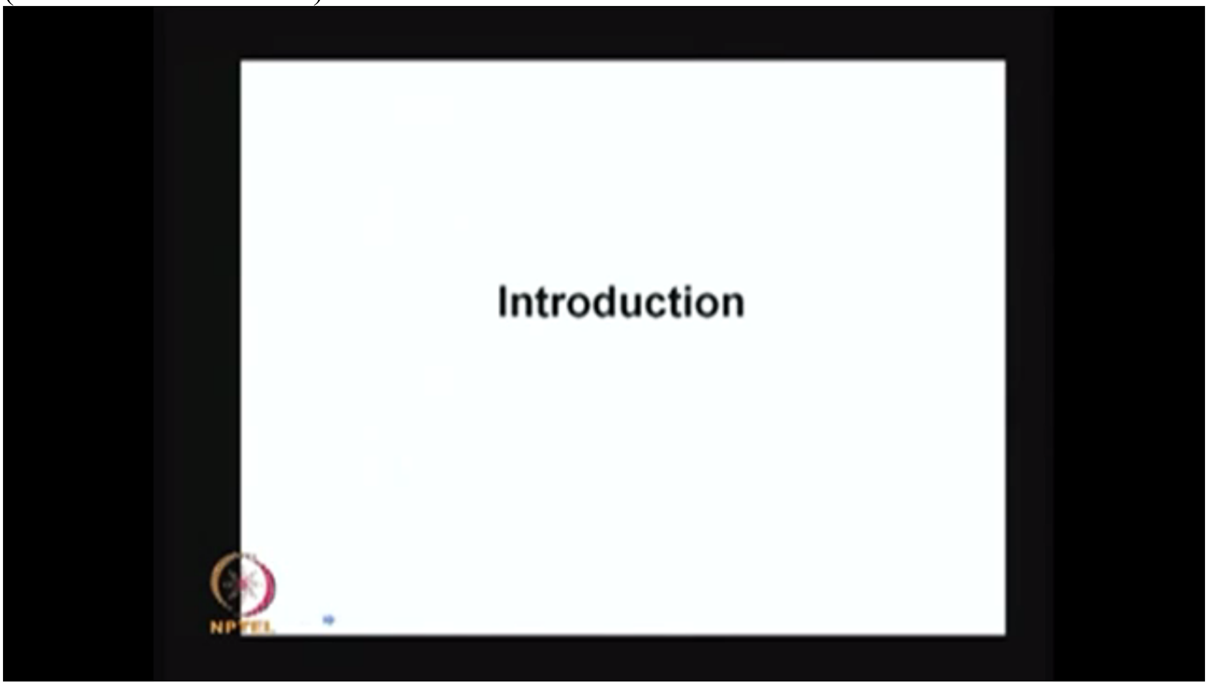

So the introduction. What is the meaning of a limiting distribution? It is very important concept in time-homogeneous discrete-time Markov chain and the limiting distribution is going to give some more information about the behaviour of the discrete-time Markov chain. And before I move into the limiting distribution, let me discuss some of the important results. Then I am going to give the limiting distribution.

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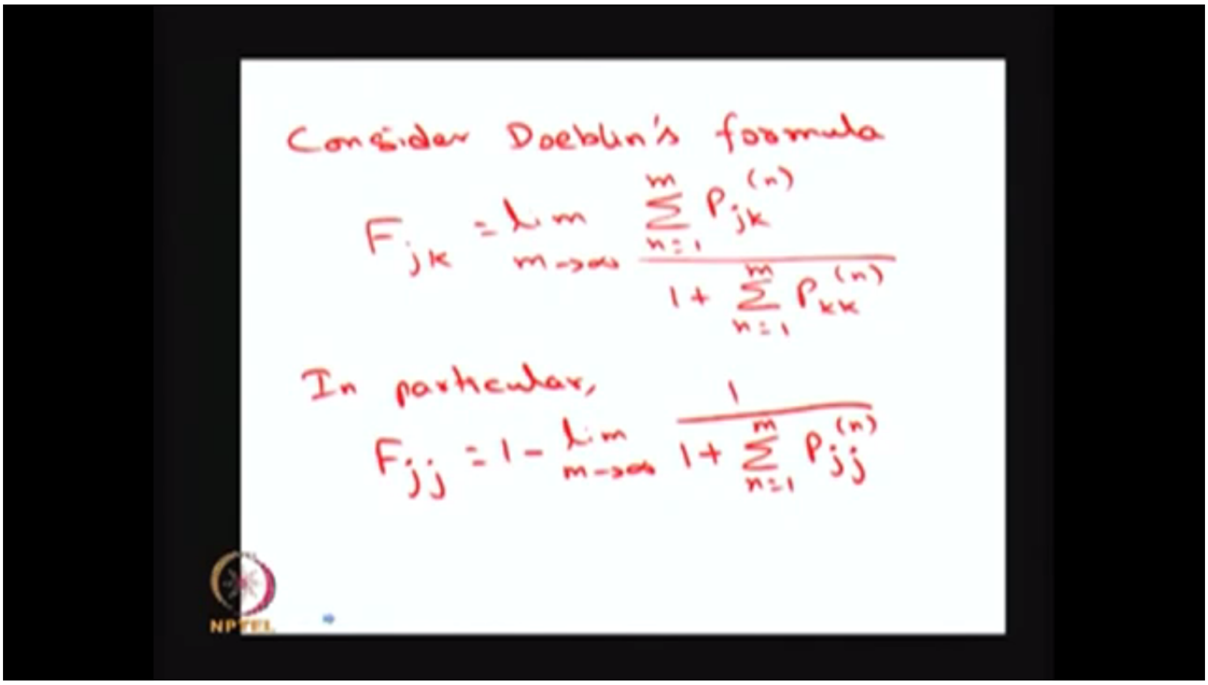

So consider the Doeblin's formula, that is f_{jk} in terms of a limit m tends to infinity of summation, we know that the P_{IJ} of n is nothing but what is the probability that the system will be in the state j given that the system was in the state i whereas the small F -- the capital F_{jk} can be written as in terms of $f_{jk}^{(n)}$ where n is running from 1 to infinity. Here the small $f_{jk}^{(n)}$ is nothing but the first visit to the state k starting from the state j in n^{th} step and all the combination of n steps that will give capital F_{jk}.

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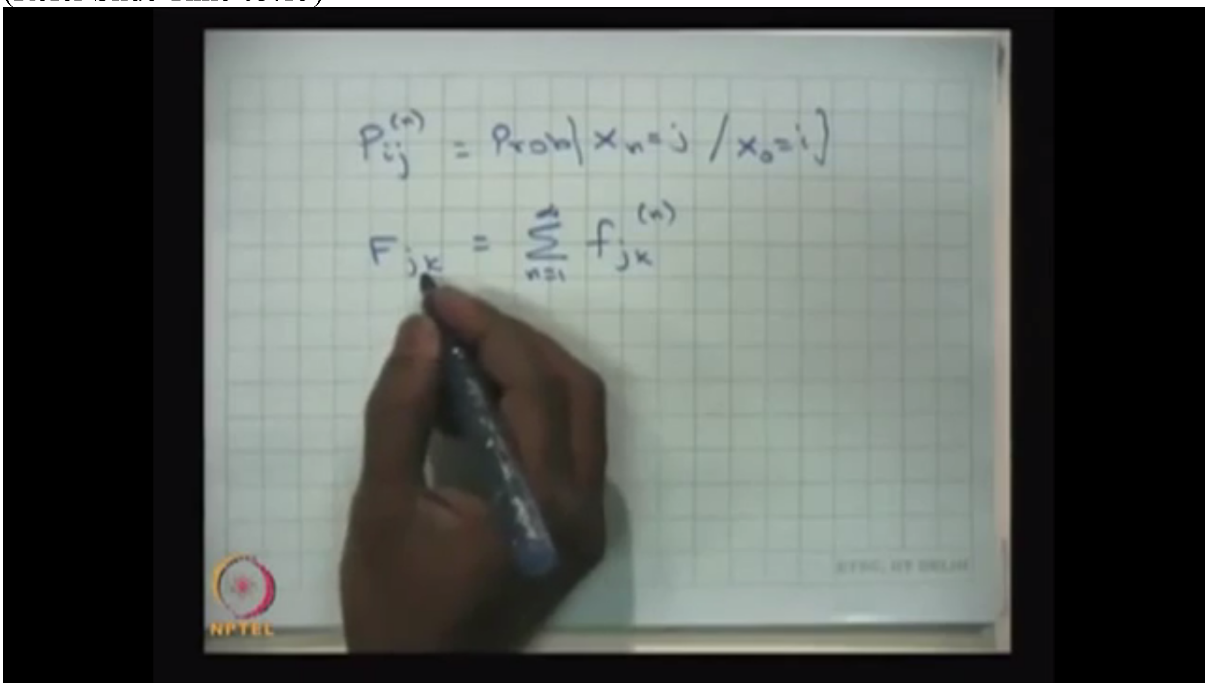

So now you see the capital F_{jk} is nothing but the limit m tends to infinity the summation divided by 1 plus the summation. In particular, we can go for k equal to j. So that is nothing but 1 minus this. Now based on the state is a recurrent, transient and so on, I can discuss the further results.

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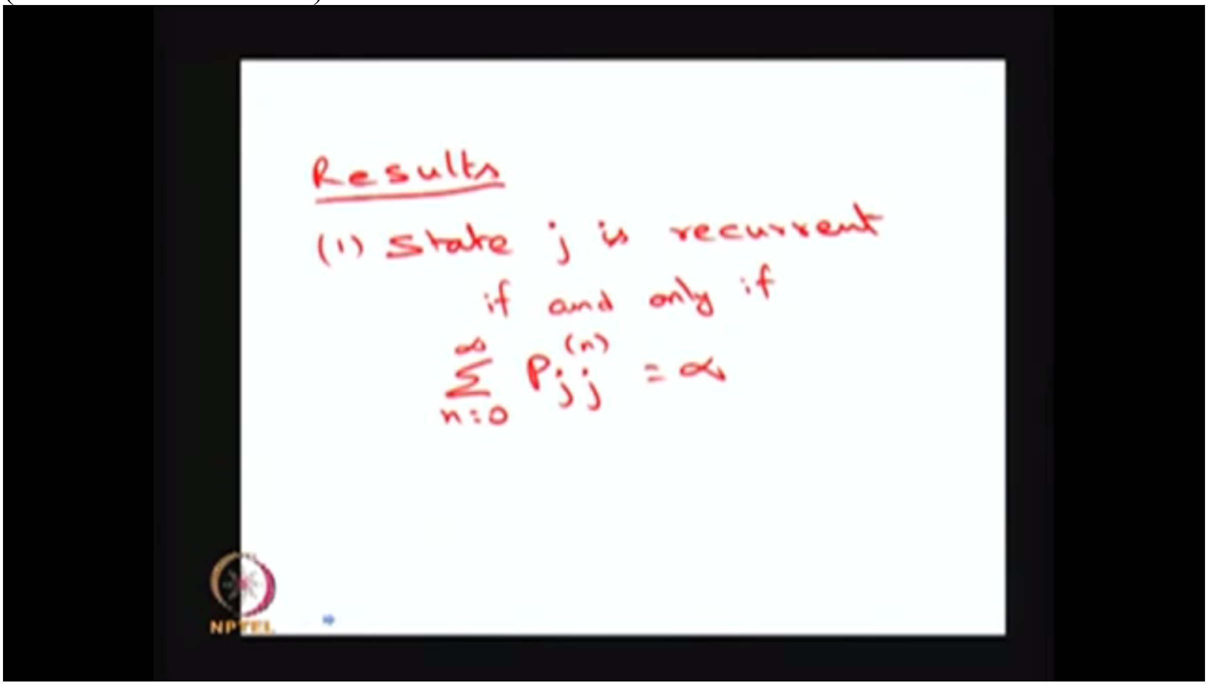

The first result, the state j is going to be a recurrent if and only if the summation of P_{jj} of n has to be infinity. The if and only if means if the recurrent -- the state is recurrent, then you can come to the conclusion this summation of the probability, not the first visit, starting from the state j to j in n steps, that summation is going to be infinity. If for any state j the summation is going to be infinity, then that state is going to be recurrent.

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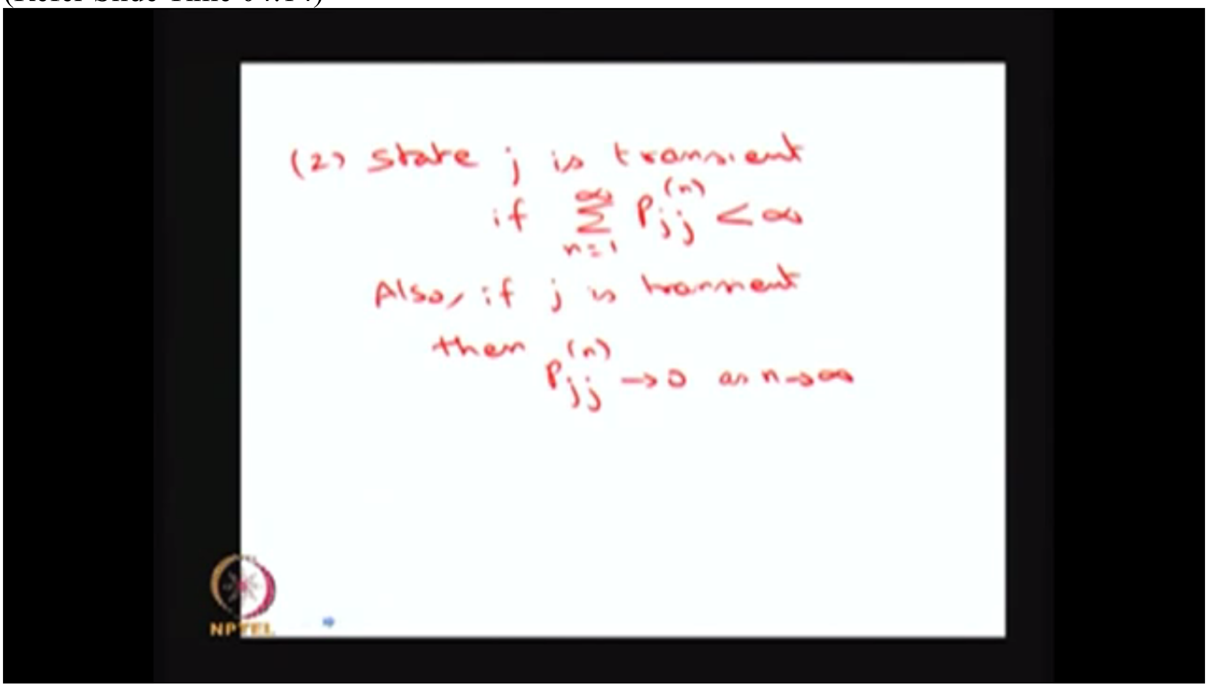

The second result, suppose the state is a transient, then you can have the P_{jj} of n tends to infinity as n tends to infinity. This we can conclude easily. If the state is a transient, then you know that the F_{jj} is going to be less than 1. The probability of the system coming back to the state is going to be less than 1.

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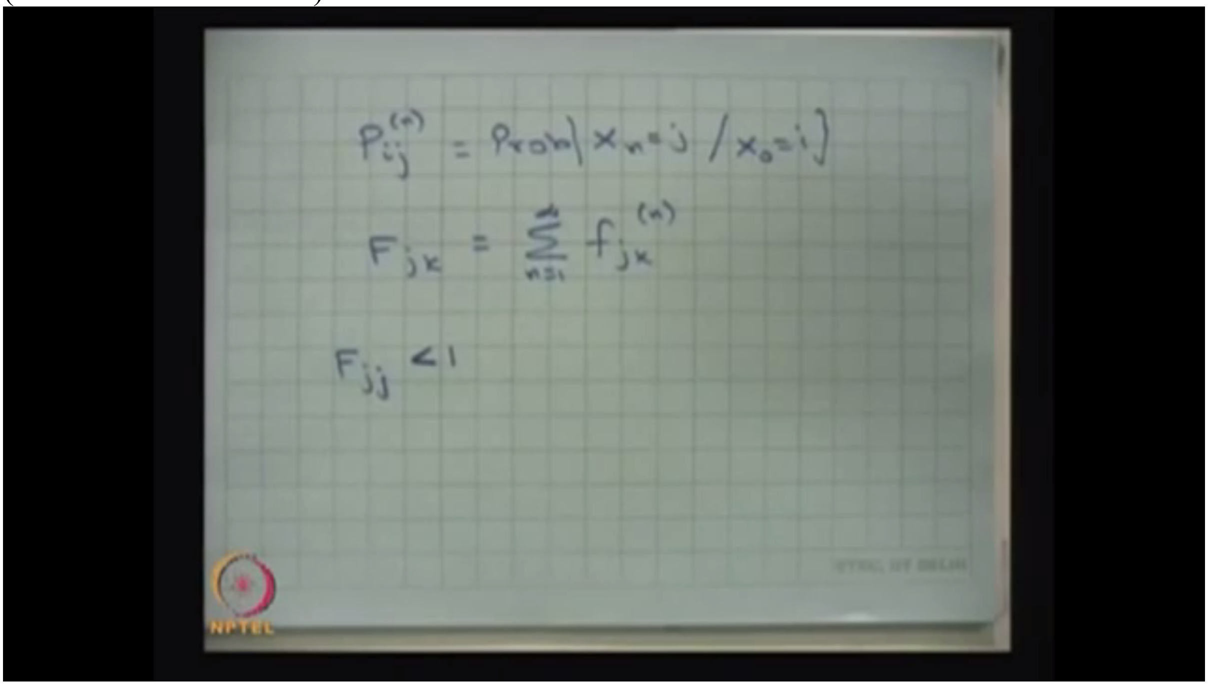

Therefore, the P_{jj} of n tends to infinity as n tends to infinity for the transient state and also if the state is a transient, then, sorry, if the summation is going to be a finite quantity, then you can conclude the state is going to be transient.

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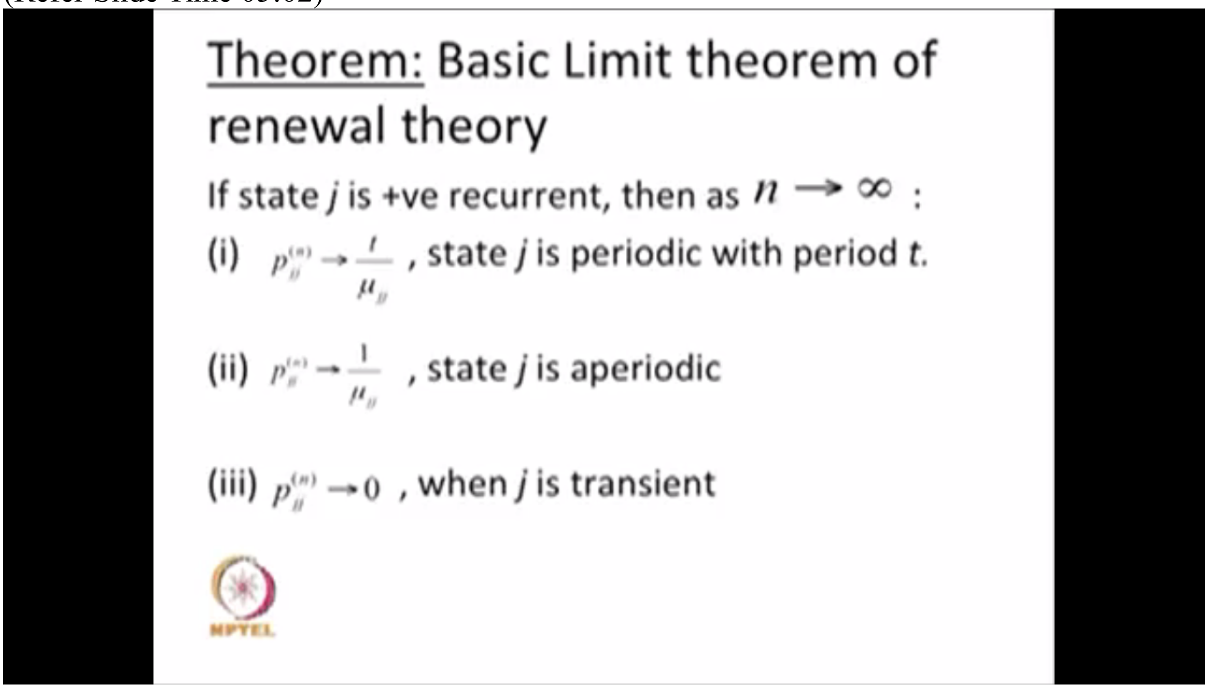

Based on these, I am going to give the next theorem that is Basic Limit theorem of renewal theory. I am not giving the proof here. I am just only stating the theorem. If the state j is a positive recurrent, that means the state is going to be a recurrent as well as it satisfies the positive recurrent property, that means the mean recurrence time is going to be a finite value for that state j, then the P_{jj} of n that will tends to t divided by μ_{jj} where μ_{jj} is nothing but the mean recurrence time for the state j and the t is nothing but the periodicity for the state j. If the periodicity is going to be 1, then as n tends to infinity, the P_{jj} of n, that is nothing but what is the probability that the system start from the state j and reaches the state j in n steps will tends to the 1 divided by the mean recurrence time for a positive recurrent state with the aperiodic.

If state j is transient, then limit P_{jj} as n tends to infinity is 0.

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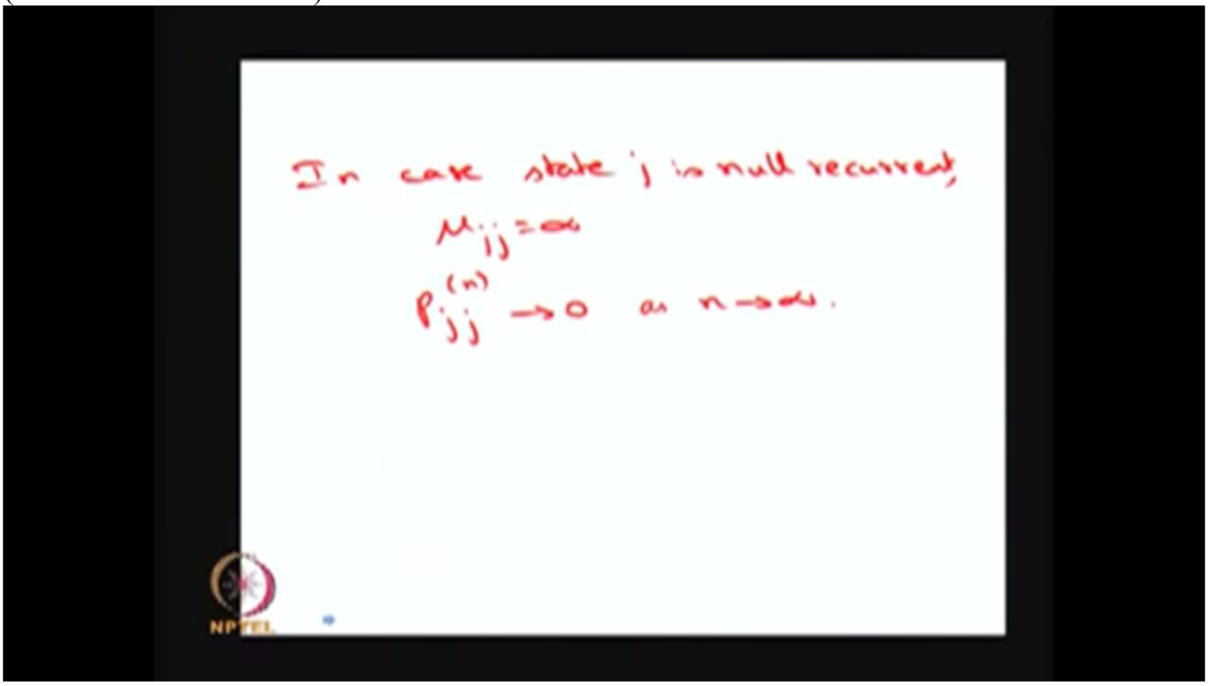

In a case of a null recurrent, if the state j is a null recurrent, then you know that for a null recurrent, the mean recurrence the time is going to be infinity. Therefore, as n tends to infinity, the P_{jj} of n will tends to 0.

Now I'm going to give some more important results for a discrete-time Markov chain. Here I am considering a time-homogeneous discrete-time Markov chain only.

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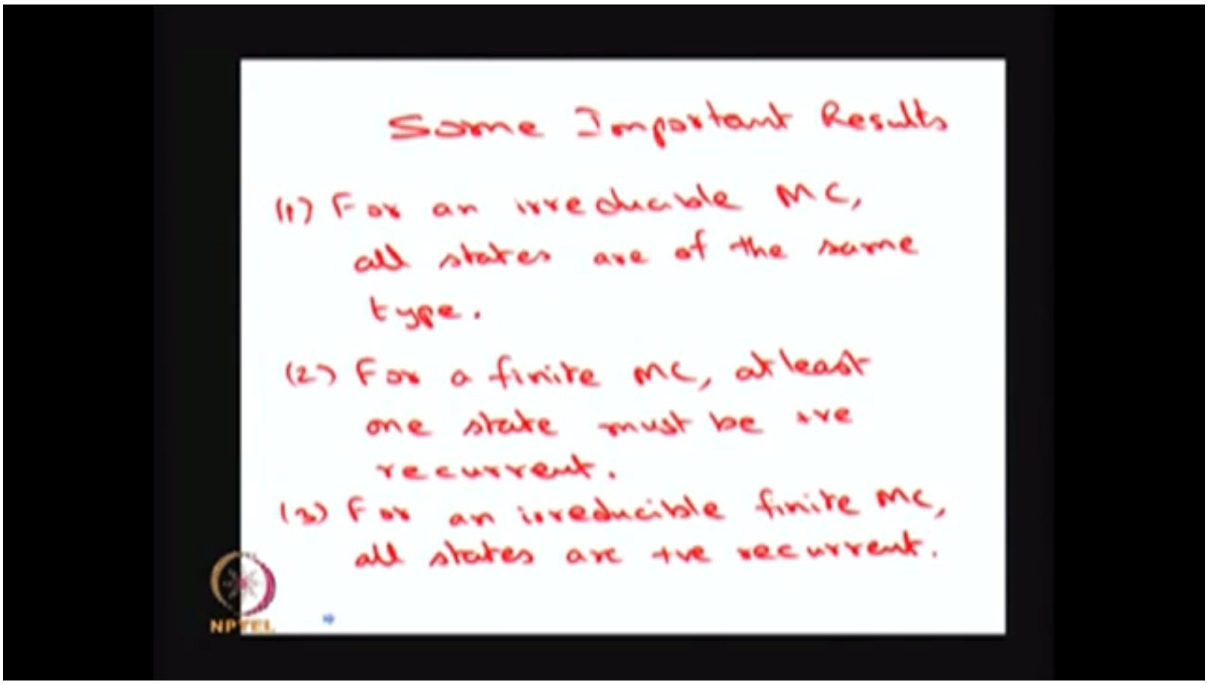

So for a irreducible Markov chain, all the states are of the same type. That means if the Markov chain is going to be irreducible, that means each state is communicating with each other state. Then only the Markov chain is going to be call it as a irreducible Markov chain. That means for a irreducible Markov chain, all the states are of the same type. That means if one state is going to be a positive recurrent, then all the states are going to be positive recurrent. If one state is going to be a null recurrent, then all the states are going to be null recurrent.

The second result for a finite Markov chain, the discrete-time Markov chain with the finite state space, at least one state must be a positive recurrent. This can be proved easily, but here I am not giving the proof. At least one state must be a positive recurrent because it is a finite Markov chain that means it has finite states. Therefore, the mean recurrence time that is nothing but on average time spending in the state starting from the state j and coming back to the state j, that mean recurrence time, that is going to be always a finite value at least for one state.

Now I am combining the result 1 and 2 gives the third result. That means the finite Markov chain has at least one positive recurrent state and the first result states that if the Markov chain is irreducible, then all the states are of the same type. Therefore, the third result is for a irreducible finite Markov chain, that means it is a time-homogeneous discrete-time Markov chain with the finite state space and all the states are communicating with all other states, let's say irreducible, then all the states are going to be a positive recurrent.

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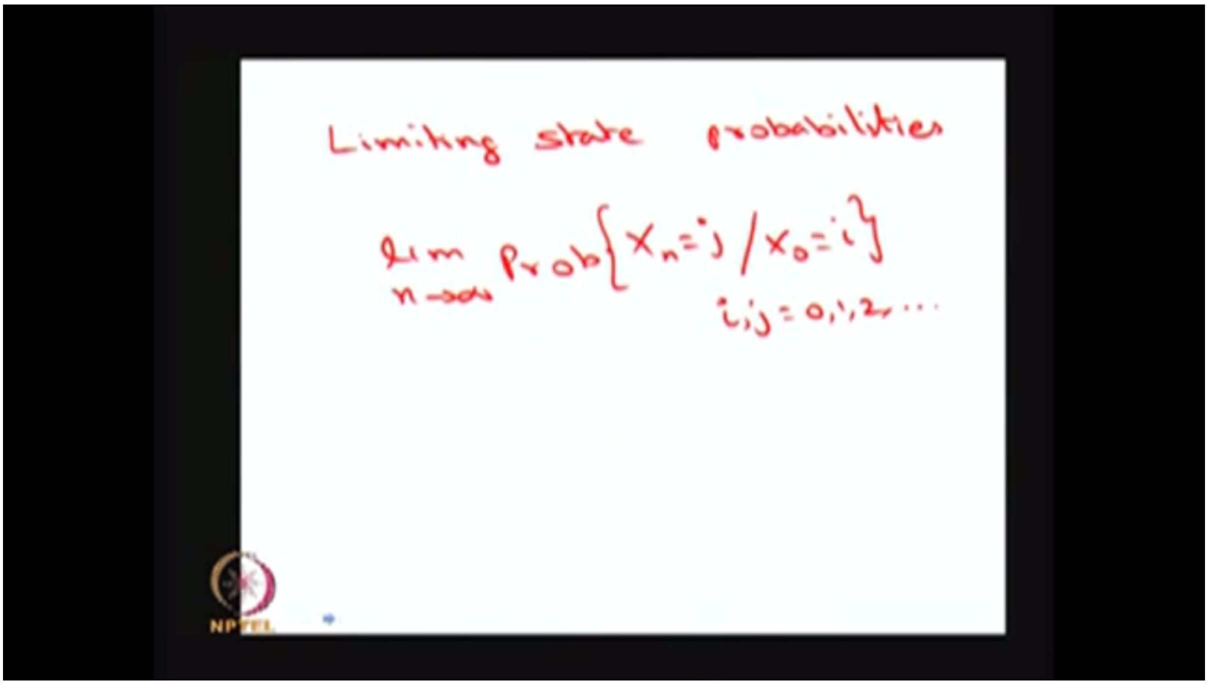

Now I'm describing the limiting distribution. The limiting distribution means what is the probability that the system starting from the state i and reaches the state j as n^{th} steps as n tends to infinity. So this is nothing but this is the definition of a limiting state probability.

We are only considering a time-homogeneous discrete-time Markov chain. So if this limit is going to exist, then it is going to be unique. So what is the limiting state probability for any time-homogeneous discrete-time Markov chain? Whether it will exist? If it exists, what is the value? So that -- that is what we are going to discuss in the further in this class, in this lecture.

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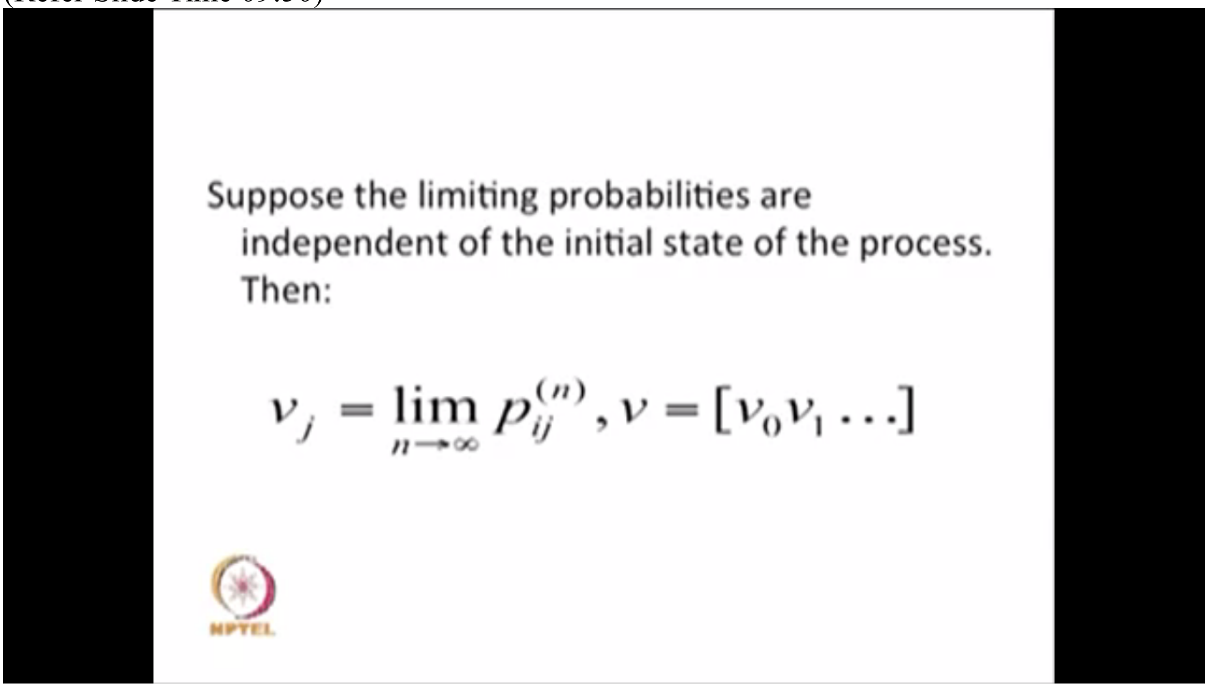

Suppose limiting probability is independent of initial state of the process, P_0 vector. Suppose, I am just making the assumption, if the limiting probability is going to exist as well as if it is independent of the initial probability distribution, we can write as V_j because that is nothing to do with i. So V_j is nothing but what is the limiting state probability of system being in the state j as n tends to infinity. That is nothing but limit n tends to infinity P_{ij} of n. So now I can write a vector V consists of V_0, V_1. So those entries are nothing but the limiting state probabilities.

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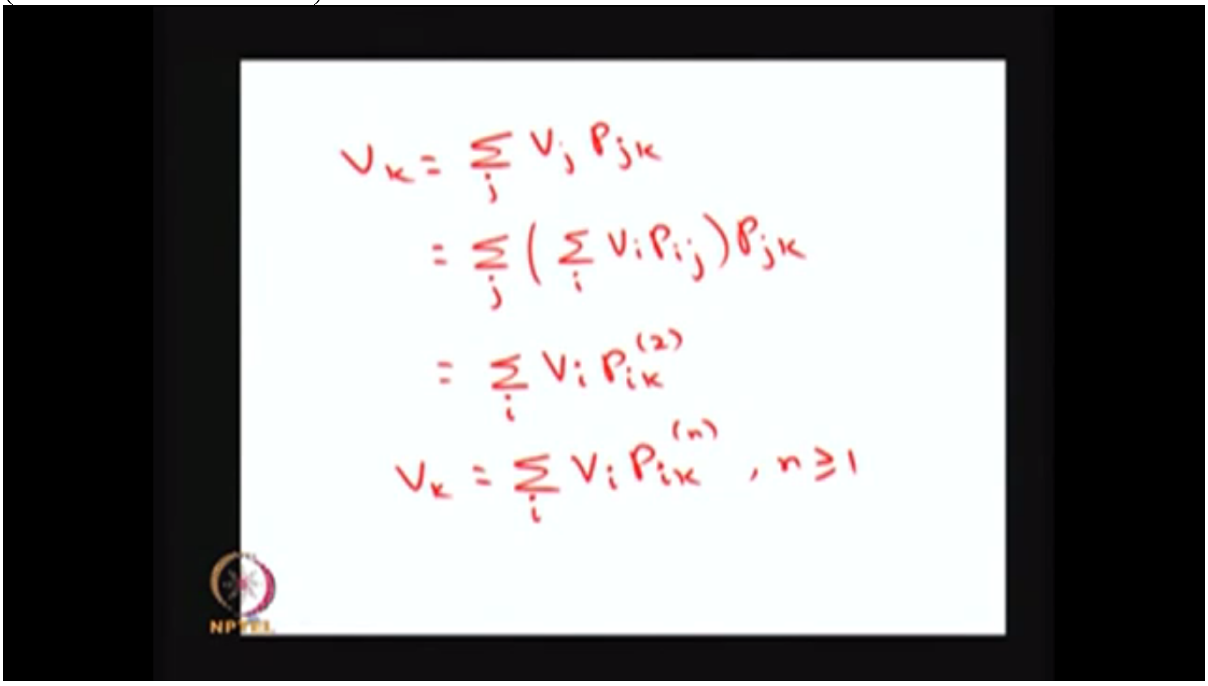

So this I can compute as a V_k is equal to summation j V_j P_{jk}. That means the P_{jk} is nothing but the one-step transition probability. So that possibility summation will give V_k. Now I can replace V_j by again the summation over i V_I P_{ij}. I can do simple calculation. It land up V_k is equal to summation i V_i P_{ik} of 2. Again, I can repeat the same thing for V_i. So I will get a V_k is equal to summation over i V_i P_{ik} of n for n is greater than or equal to 1. That means this is the entry of n step transition probability matrix having the probability that is the probability of system is moving from the state i to k in n steps for n is equal to 1, 2, 3 and so on.