Example of Classification of state(Contd.)

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Example two. Here I am going to discuss a reducible Markov chain. Here also we have only two states. The probability of system is moving from state 0 to 0 in the next step, it is the probability is 1 and the system is coming from the state 1 to 0 in one step that probability is 1.

So this is a state transition diagram of a time homogeneous discrete-time Markov chain. So I am going to write what is the one step transition probability matrix for this state transition diagram or for this discrete-time Markov chain.

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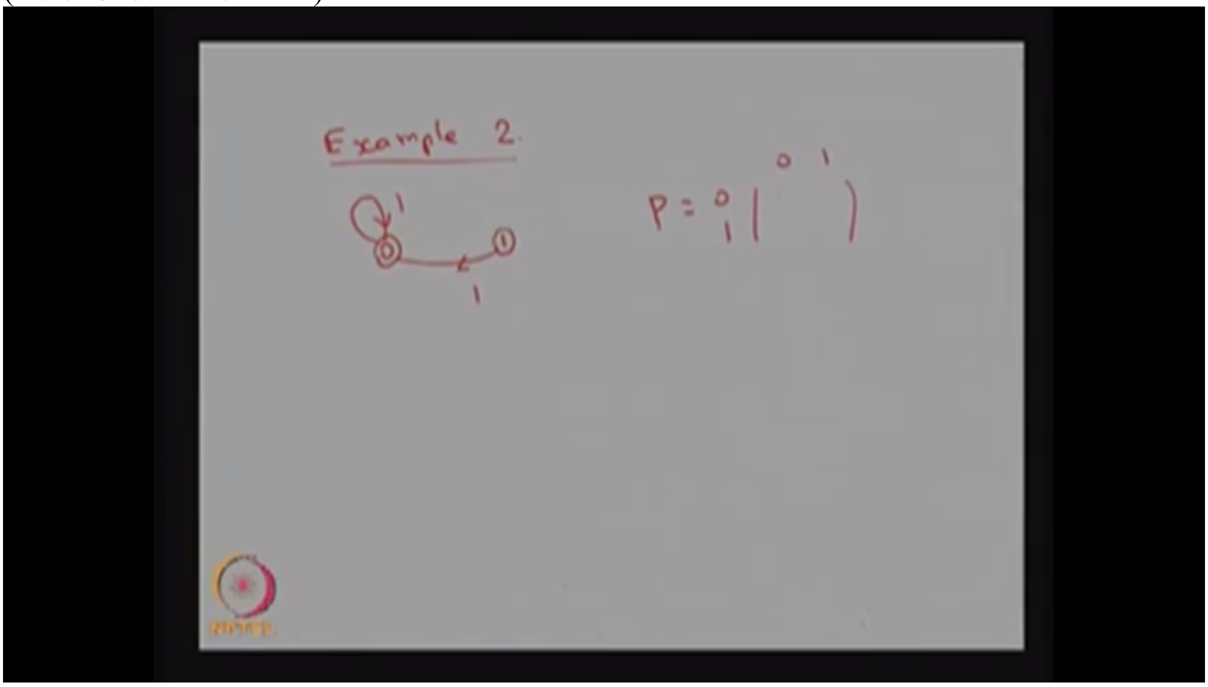

So 0 to 0, one step, that probability is 1. 0 to 1 is 0. 1 to 0 is -- 1 to 0 is 1. 1 to 1 is 0.

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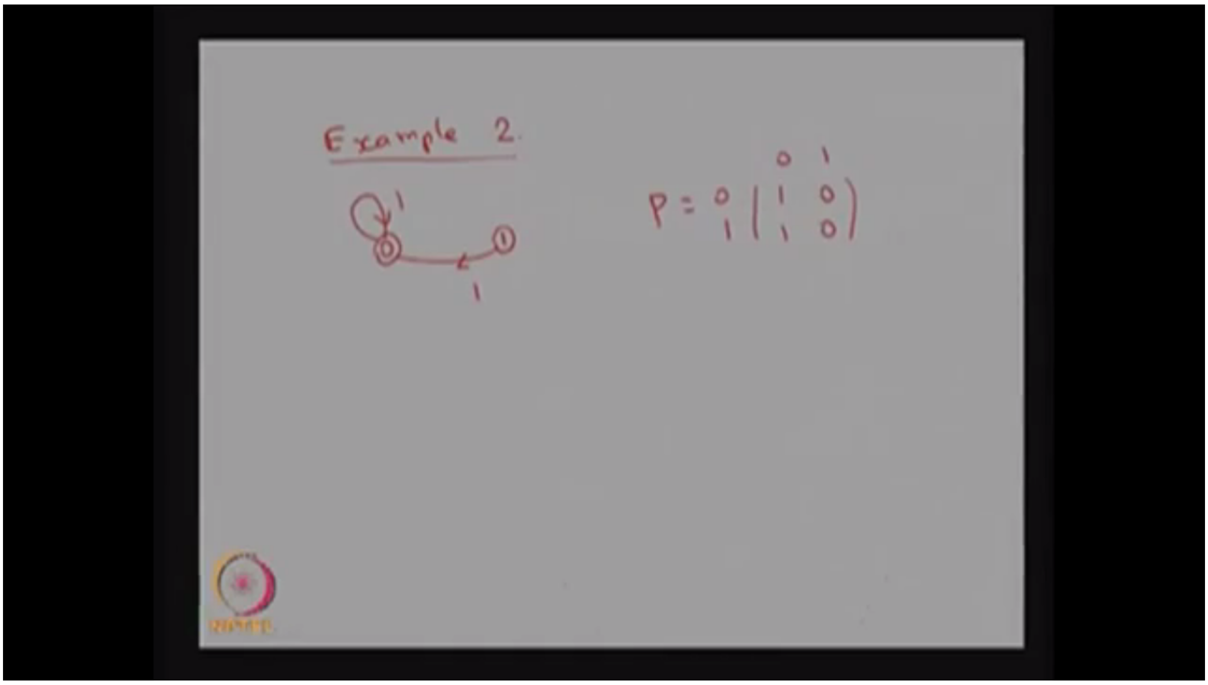

You can verify whether this is going to be a stochastic matrix because each elements are lies between 0 to 1 and the row sum is 1. Therefore, this is a stochastic matrix. So both are equal. The state transition diagram and the one step transition probability matrix is one and the same.

Now we will try to find out what is the classification of the states. Go for the state 0. The P_{00} of 1 that is 1. That is one step transition of system is moving from the state 0 to 0. That is going to be 1. This implies the state 0 is a absorbing state.

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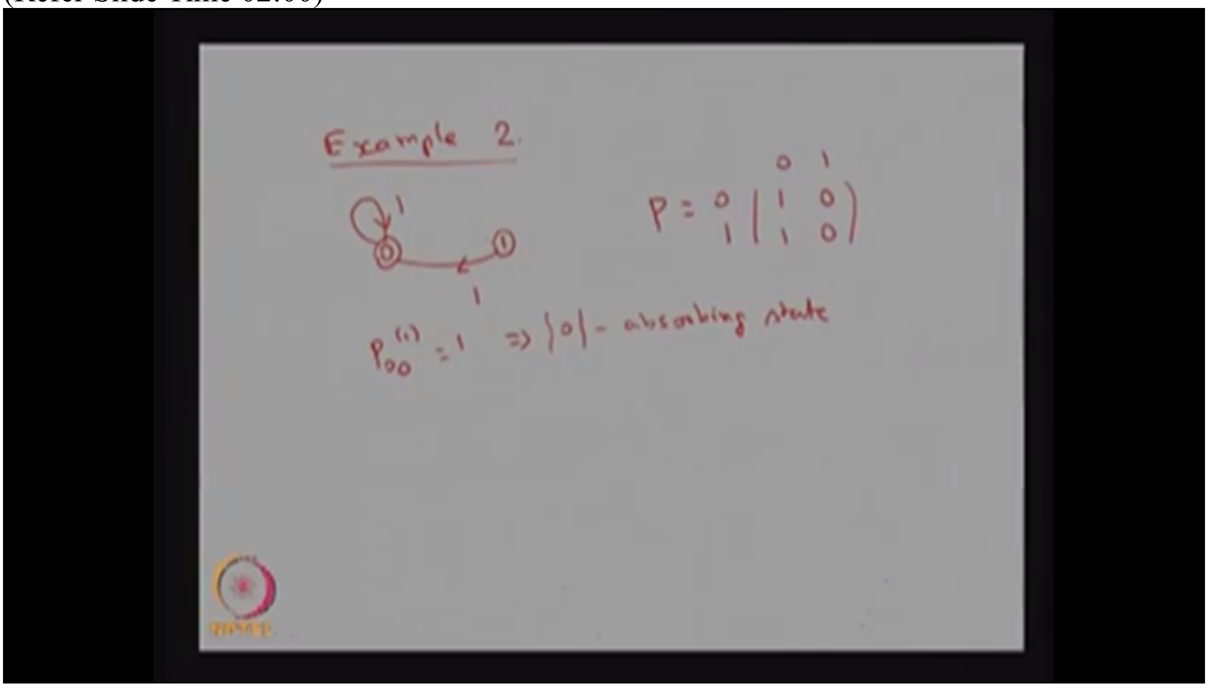

Now we will try to find out what is the classification of the state 1. So if you find out f_{11} of 1, what is the probability that the system will come to the state 1 given that it was in the state 1 and the first time visit to the state 1 exactly in the first step. So that is going to be not possible because with the probability 1, it moved to the state 0. Therefore, this is going to be 0.

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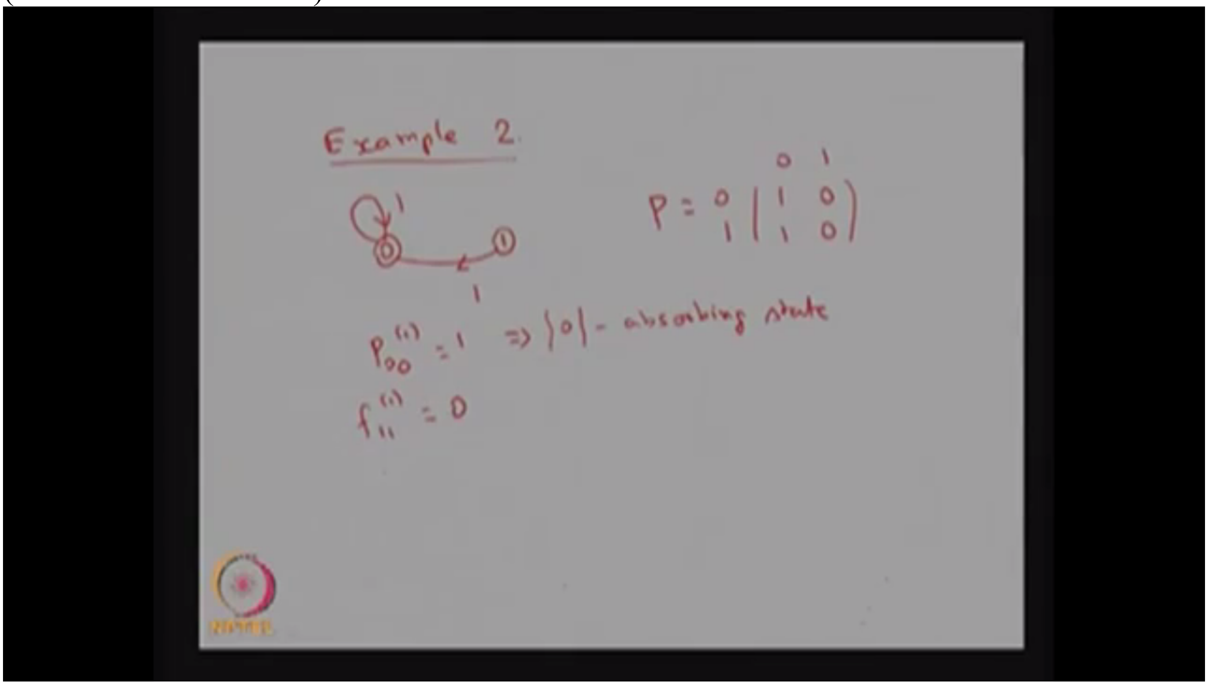

And if you find out f_{11} of all the subsequent steps also that is also going to be 0 because if the system starts from the state 1, in the next step itself it goes to the state 0 with the probability 1 and it is not coming back. Therefore, now you try to find out what is the capital F_{11}. That is nothing but the summation of all the f_i's, summation of all the f_i's and that is going to be 0.

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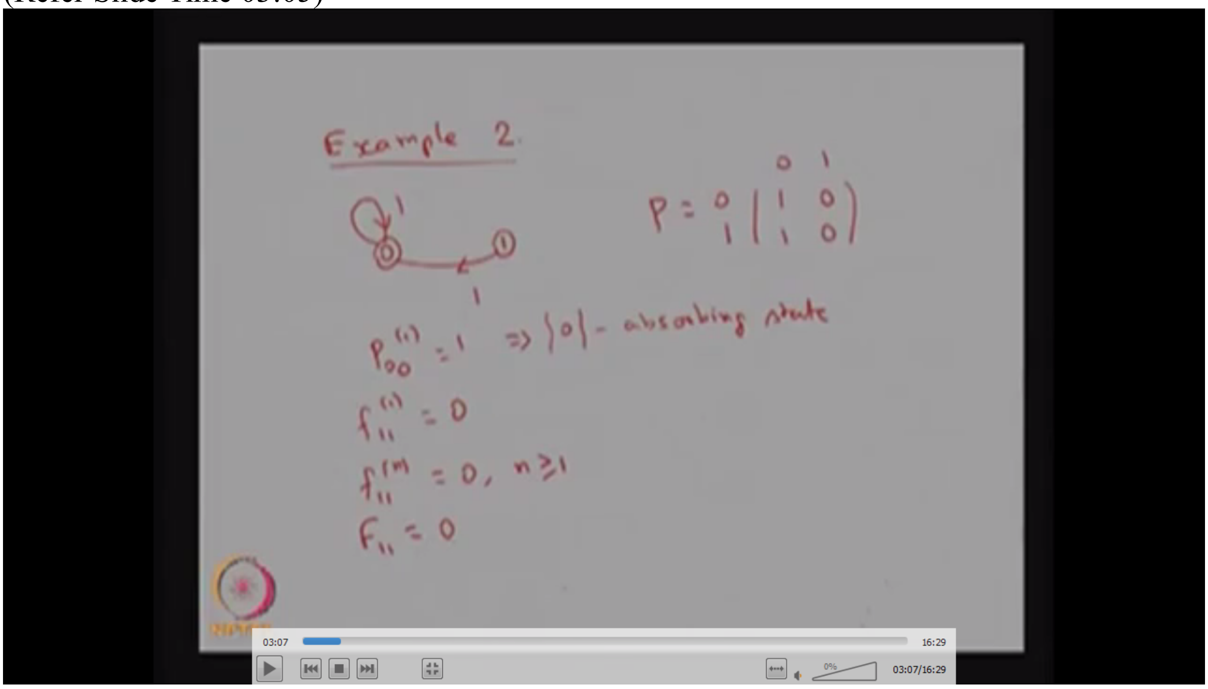

If you recall the way we classify the state is going to be a recurrent or transient, we said F_{ii} is going to be 1 or F_{ii} is going to be less than 1. So that less than 1 includes F_{ii} is equal to 0. So basically our interest is to classify whether with the proper distribution the system is coming back to the same state with the probability 1 that is F_{ii} is equal to 1 and all other things we say that's a transient state. It includes F_{ii} equal to 0.

So here with the probability 0, the system is not coming back to the state 1 if the system starts from the state 1. This is always a conditional probability and this conditional probability F_{11} is equal to 0 implies the state 1 is going to be a transient state. So whenever any -- for any state i, F_{ii} is equal to 1, that concludes the state is going to be a recurrent state and whenever the F_{ii} is lies between including 0, excluding 1, that is less than 1, then that state is going to be call it as a transient state.

Since the close -- since you have only two states, that's the state space is 0 and 1, and you will land up having one absorbing state and one transient state. Therefore, the state space is partition into one closed communicating class, which has only one element and the transient state is 1. Therefore, I can say the state space S is a partition into closed communicating class C_1, which consists of only one element and the collection of all the transient states that is only one element.

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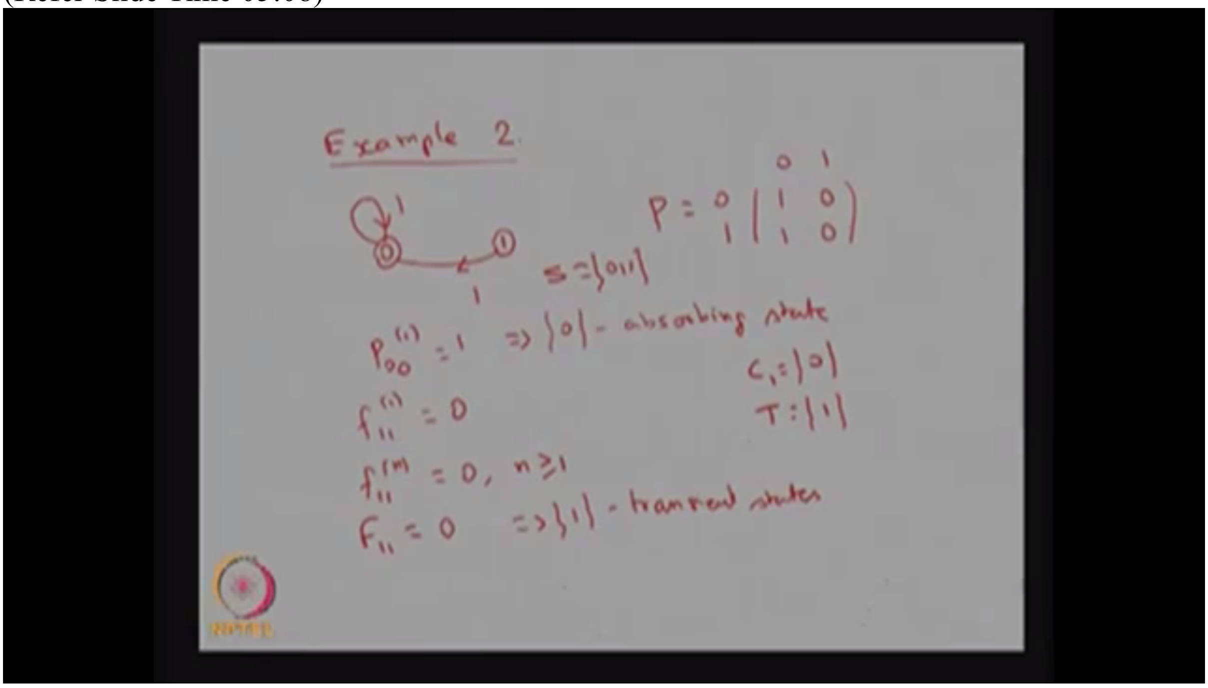

So this is a notation for capital T collecting all the transient states in the -- state space in the DTMC and C_1 is the first closed communicating class and which has only one element. If any closed communicating class has only one element, then it is going to be call it as absorbing state. Therefore, 0 is the absorbing state and 1 is a transient state. Since you have a C_1 union T becomes a state space S, therefore, this Markov chain is not a reducible -- irreducible Markov chain. Therefore, this is called a reducible Markov chain.

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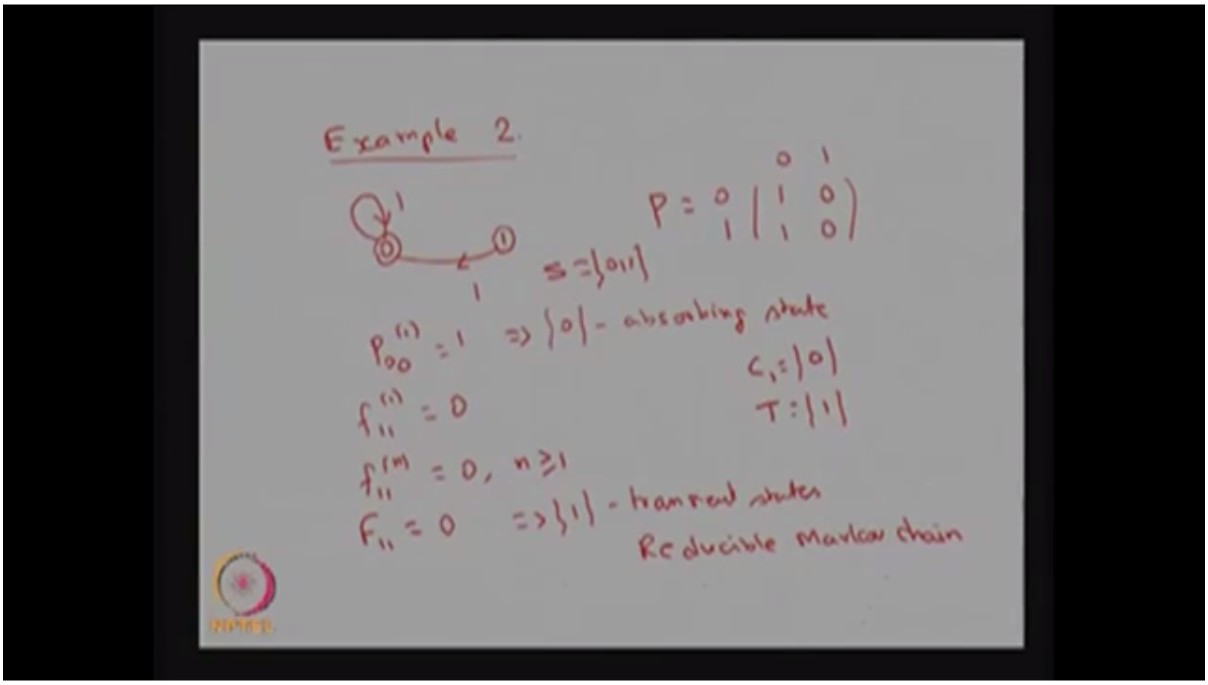

This is Markov -- whereas the previous example is a irreducible Markov chain. There we have two elements and both the elements from only one closed communicating class whereas here you have one closed communicating class with one element and the transient state is 1. Therefore, it is going to be a reducible Markov chain. There can be more than one transient state.

Now I'm moving into the third example so that I am explaining some more concepts through the examples. Example three.

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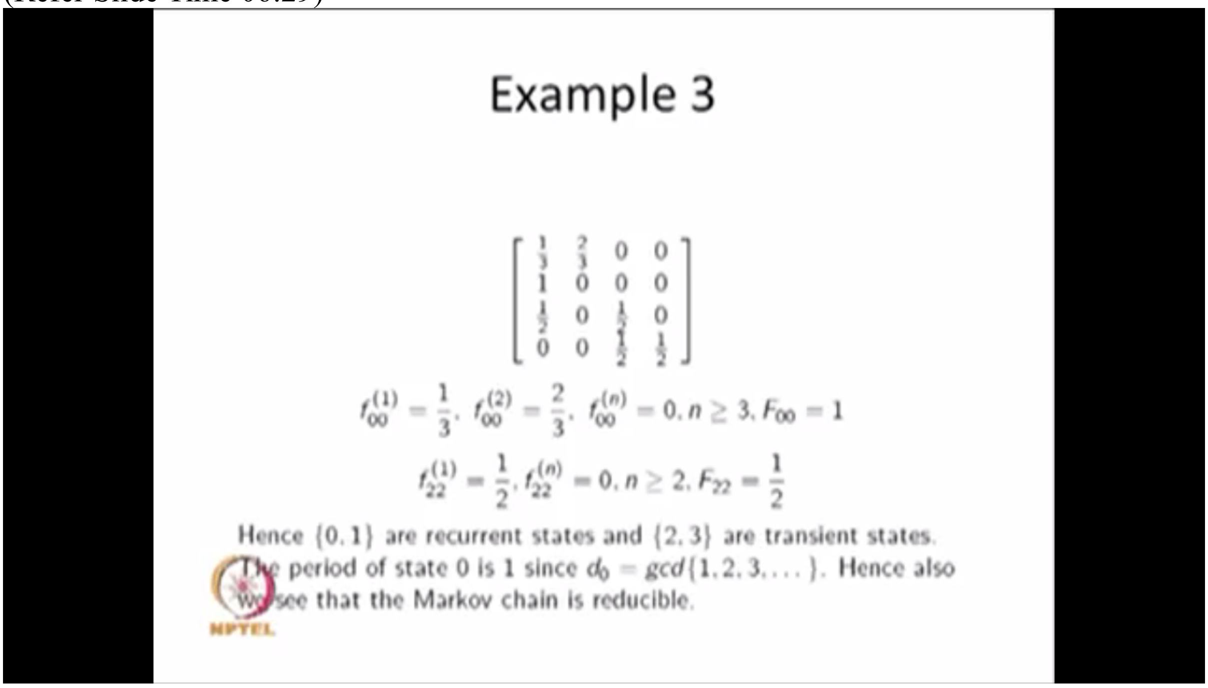

Here I go for four state -- four states. It consist of states 0, 1, 2 and 3. It is easy to explain through the state transition diagram than the one-step transition probability matrix. So I am just drawing the state transition diagram for this DTMC. So 0 to 0, one-step that probability is 1/3 and 0 to 1 is a 2/3. Therefore, row sum is taken care. That probability -- summation of probability is 1.

Now I am going for the state 1. State 1 to 0, that probability is 1. Therefore, that row is taken care.

Now I'm moving to the state 2. State 2, the self loop has the probability 1/2 and going from the state 2 to 0, that probability is 1/2. Therefore, this row is also taken care.

Now I'm moving into the state 3. State 3, it has the self loop with the probability 1/2 and it has the moving from the state 3 to 2, that probability is 1/2.

My interest is to classify the states for this Markov chain. Markov chain has four states 0, 1, 2 and 3. That's the state space capital S. Now we will start with the state 0. So you find out what is the f_{00} of 1? In one step as a first visit, the system has to come back to the same state to 0. So that probability is 1/3.

You find out f_{00} of 2. f_{00} of 2, exactly two steps. As a first visit, you have to come back to the state 0. That means you can go to the state 1 by starting from the state 0 and come back to the state 0 in the next step. Therefore, it is a 2/3 into 1. Therefore, it is going to be 2/3.

Then you go for what is the possibility I will take three steps, exactly three steps coming to the state 0. As a first visit, it is not possible whereas the p_{00} of 3 is possible. f_{00} of 3 is not possible because in three steps you cannot make a first visit. Therefore, that is going to be 0. Not only f_{00} of 3 and for all other things also it's going to be 0. f_{00} of n equal to 0 for n is greater than or equal to 3.

So now I can find out what is capital F_{00}. If I find out F, capital F_{00}, I have to add all the values. So it's 1/3 plus 2/3 plus all the further terms are 0. Therefore, it is going to be 1. Since F_{00} is equal to 1, you can conclude the state space 0 is -- the state 0 is going to be the recurrent state.

The similar exercise you can do it for the state 1. The same way you can conclude F_{11} is also going to be 1. The other way, since the state 1 is communicating with the state 0, since the state 1 is communicating with the state 0, therefore, this is also going to be of the same type. Therefore, the state 1 is also going to be the recurrent state.

Now we can go to the state 2. So the state {0, 1} that is going to be the current state.

Now I will move it to the state 2. So whereas the state 2, if you find out f_0, f_{22} of 1 in one step coming back to the same state, that is going to be 1/2. f_{22} of two steps, exactly two steps, that is not possible, that is going to be 0 and so on. Not only 2 and all the further steps also going to be 0 because with the probability ½, it takes only one step come back and all the further steps it takes with the probability 1/2, it is not coming back at all. Therefore, this is going to be for greater than or equal to 2, it is going to be 0. Therefore, if you compute F_{22}, capital F_{22}, then that is going to be 1/2 plus 0 and so on. Therefore, you will land up 1/2, which is less than 1. Therefore, you can conclude the state 2 is going to be transient state.

Not only the state 2, if you do the similar exercise for the state 3, the same thing you may land up F_{33} is also going to be less than 1 whatever be the number. You can conclude the state 3, that is also going to be the transient state.

You can find out the periodicity for the recurrent state only, not for the transient state. Therefore, now we can try to find out what is the periodicity for the state is 0 and 1. Before that we will try to find out what is the type of a recurrent state, whether it is going to be a positive recurrent or null recurrent. If you find out μ_{00}, that is nothing but 1 times 1/3, 2 times 2/3, 3 times 0, 4 times 0 and so on. So if you sum it up everything, you may land up 1 times 1/3 plus 2 times 2/3. That is going to be 1/3 plus 2 times or 2/3 so that is going to be 3, 1 plus 4, that is 5, so which is a finite quantity. You can conclude the state 0 is going to be a positive recurrent.

Similarly, if you calculate f μ_{11} also, you may land up with the finite quantity. So you can conclude both the states are going to be positive recurrent states. Here the -- the state space is classified into two positive recurrent state and two transient state. Therefore, this Markov chain is going to be a reducible Markov chain. In short for MC, it's a reducible Markov chain because the whole state space capital S is a partition into one closed communicating class which consists of the state 0 and 1 and the transient states 2 and 3. Therefore, this is going to be a reducible Markov chain.

You can find out the periodicity of the -- these two recurrent state also. So if you find out d_0, that is going to be greatest common divisor 1/2, what are all the steps in which the system will be come back if the system starts from the state 0? So either it can come back with the one step or either it can come back with the two steps or it can make a one loop here, then one loop then here. Therefore, it can come back from the three steps and four steps and so on. It need not be the first visit. Therefore, the gcd of one step or two steps and three steps and so on, therefore, this is going to be 1. That means it is aperiodic state.

Therefore, whatever we have done it for the state 0, you can do it for the state 1 also. So that is also going to be 1. The period is going to be 1. Therefore, both the states 0 and 1 are the positive recurrent and aperiodic states and other two are going to be the transient states. Since this state 0 and 1 are going to be positive recurrent as well as aperiodic, these two states are ergodic states also.

Later we are going to explain ergodicity, the property. For that property, you need to understand what is ergodic state. So whenever the Markov chain has few states going to be a positive recurrent and aperiodic, then those states are going to be call it as a ergodic states. So later I am going to give the definition of ergodicity and so on.