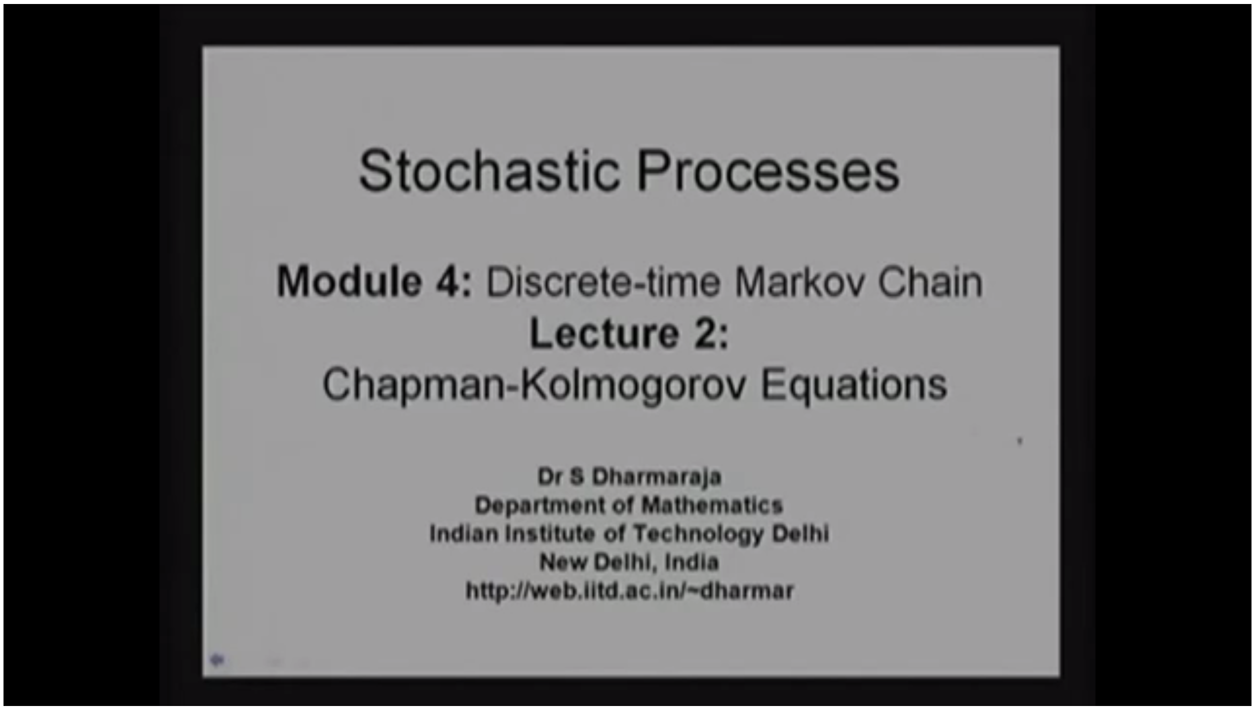
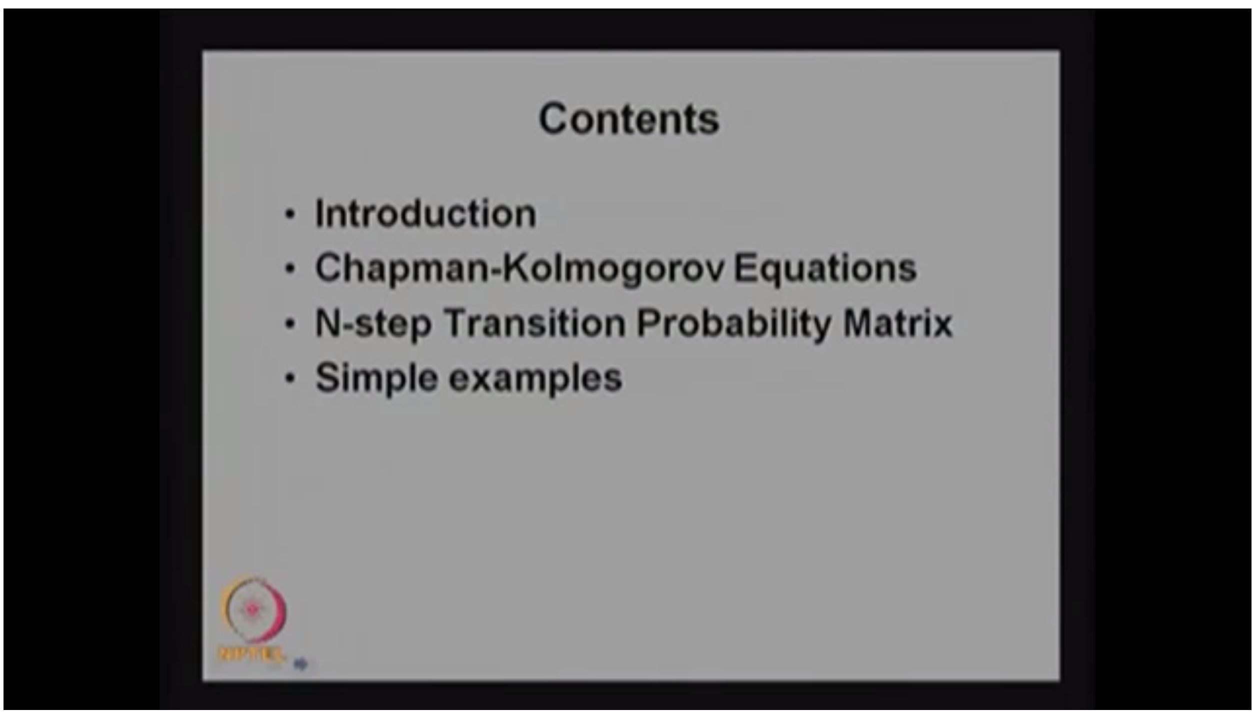
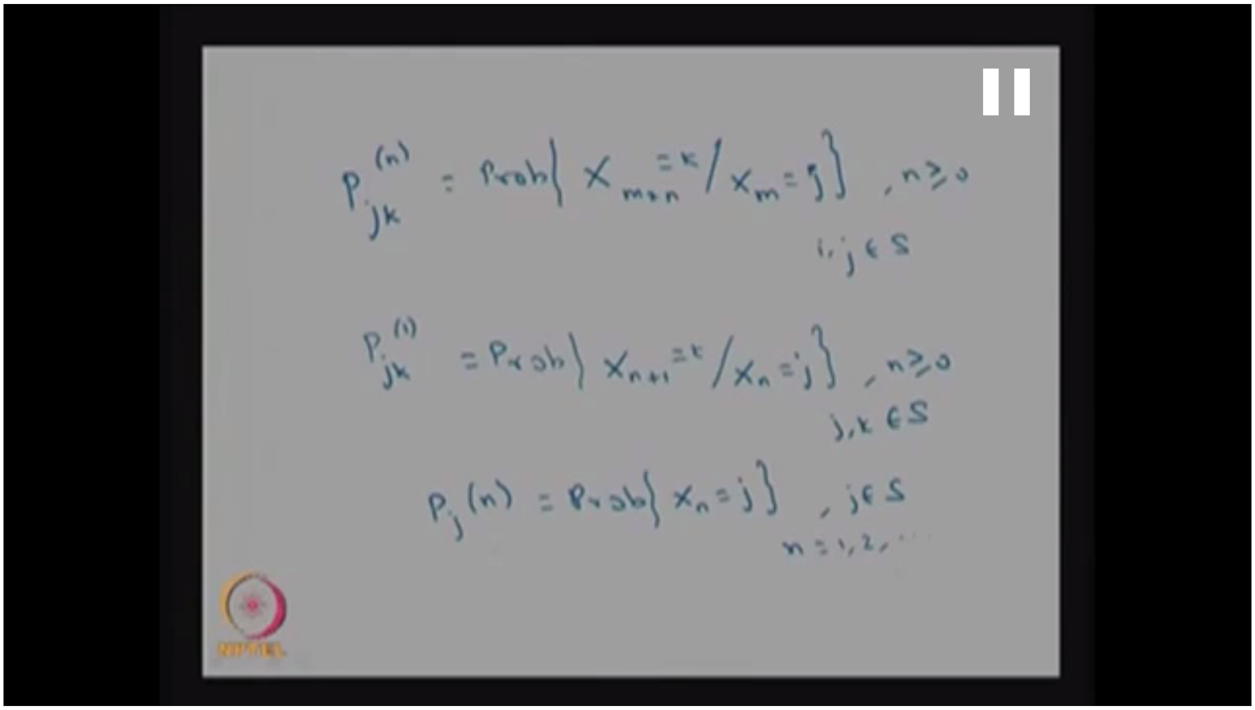



Good morning. This is a Stochastic Process, Module 4: Discrete-time Markov Chain. This is the Lecture 2.

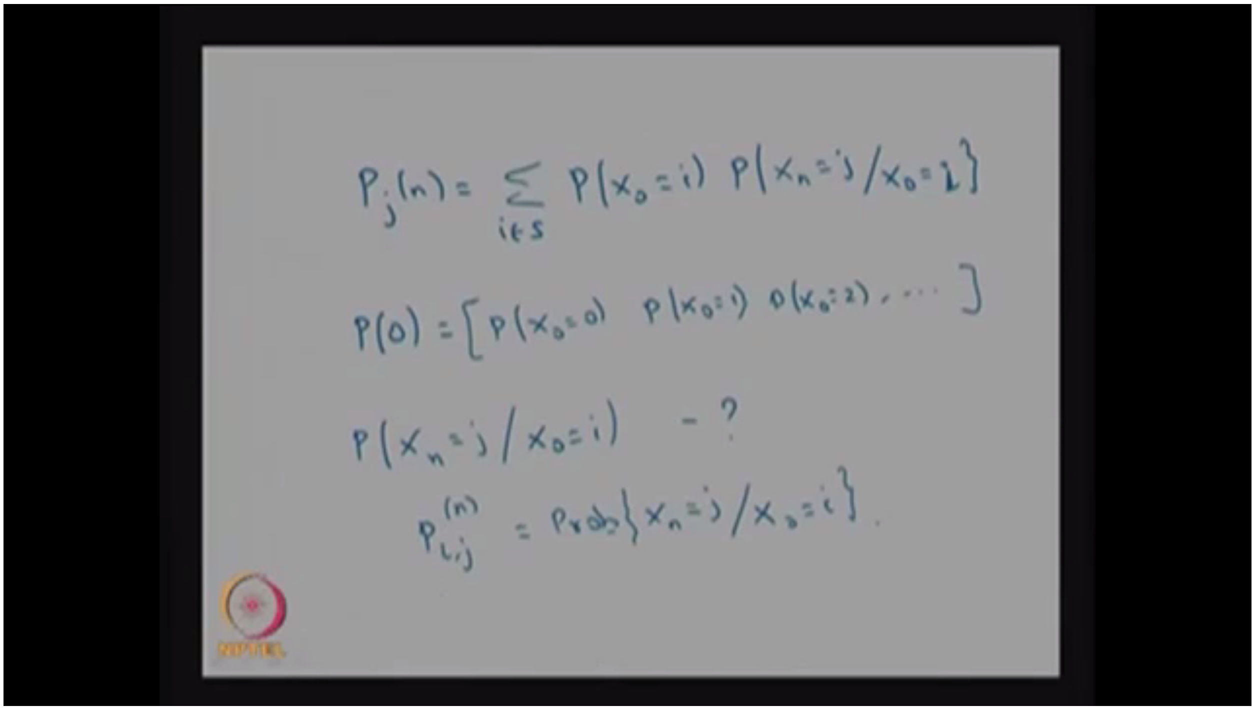



And in this lecture, we are going to discuss about the Chapman-Kolmogorov equations. Then we are going to discuss the N-step Transition Probability Matrix, and we are going to discuss a few more examples in this lecture.




In the last class, we have discussed the transition probability of J to K in N steps as the probability that the X, m+n takes a value k given that the Xm was J, fir n is great not equal to 0 and J belonging to S. Since the underlying DTMC is a time homogeneous, this is the N-step transition probability of system is moving from the state J to K in N steps. So this we denoted as a conditional probability of P J, K in N step transition probability, where I, J is belonging to S, where S is the state space and the N can take the value great not equal to 0. Also, we have discussed in the last class what is the one step transition probability of P J, K. We can write it within the (1) or we can remove the 91) in the superscript also. That is nothing but what is the probability that the system will be in the state K in the n+1 at the step given that it was in the state J in the N step, here also J, K belonging to S. So this is the one step transition probability. So our interest is to find out what is the distribution of Xn. Whenever the sequence of random variable Xn is a time homogeneous a DTMC, our interest is to find out the distribution of Xn.

So it has the probability mass function, the Pj of n that is nothing but what is the probability that the system will be in the state J at the N step, so the J is belonging to S and the n can be 1 or 2 and so on, because you know the distribution of n=0, that means you know the initial probability vector of X0. So our interest is to find out what is the distribution of Xn for m=1, 2, 3 and so on.

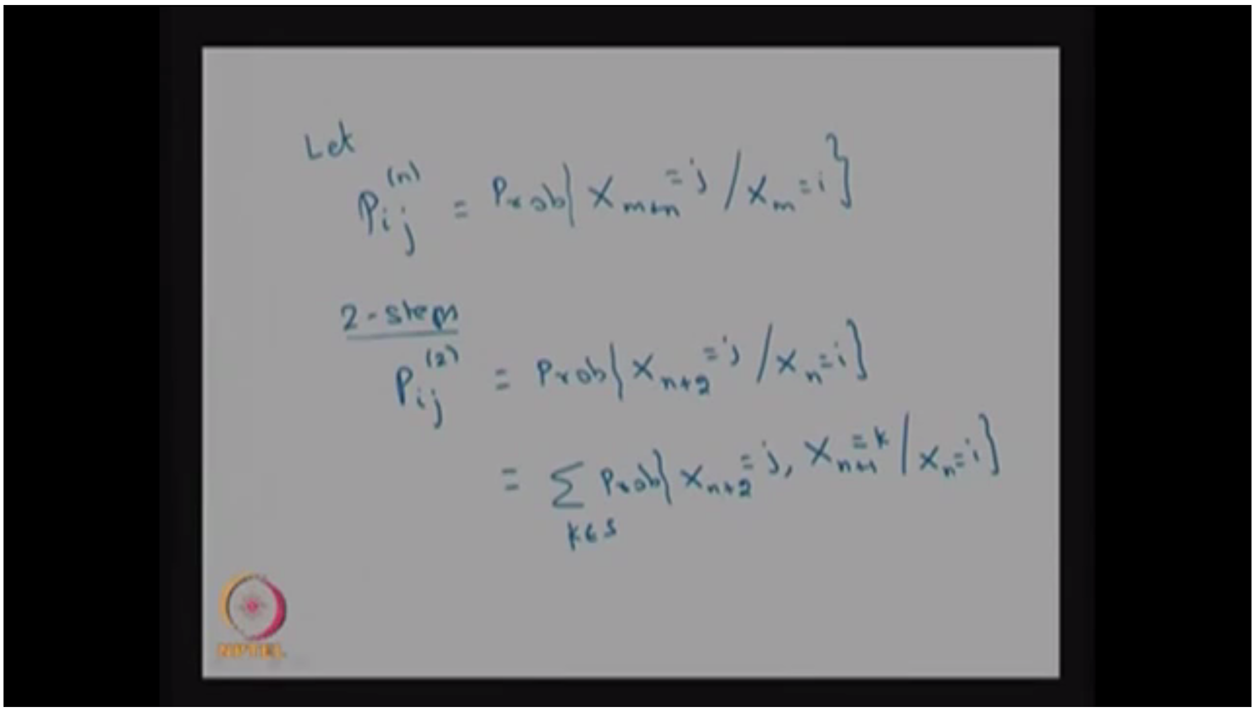



So how we are going to find out this distribution? So this distribution can be written using the Pj of n is nothing but the summation over I belonging to S, such that the system was in the state I at 0th step and multiplied by what is the probability that it will be in the state J given that it was in the state I at 0th step. So this is nothing but what is a probability that the system will be in the state J at the end of the step, that is same as what are all the possible ways the system would have been moved from the state I from the 0th step to the state J at the Nth step. So this is a product of one marginal distribution and one conditional distribution for all possible values of I that gives the distribution of Xn in the Nth step.

So for that you need to compute this distribution of Xn, you need N step transition probability as well as the initial distribution vector or initial probability vector or the distribution of X naught. So the distribution of X naught can be given as the vector P of 0. This vector P of 0, it consists of the element what is the probability that X naught takes the value 0, what is the probability that X naught takes a value 1, what is the probability that X naught takes the value 2, and so on. So this is the initial probability vector. Why we have taken the state 0, 1, 2, and so on unless otherwise as mentioned in the set of the state space that is going to be the possible values of 0, 1, 2, and so on, unless otherwise it is assumed you can take always this values. So this is the initial probability vector or initial distribution vector.

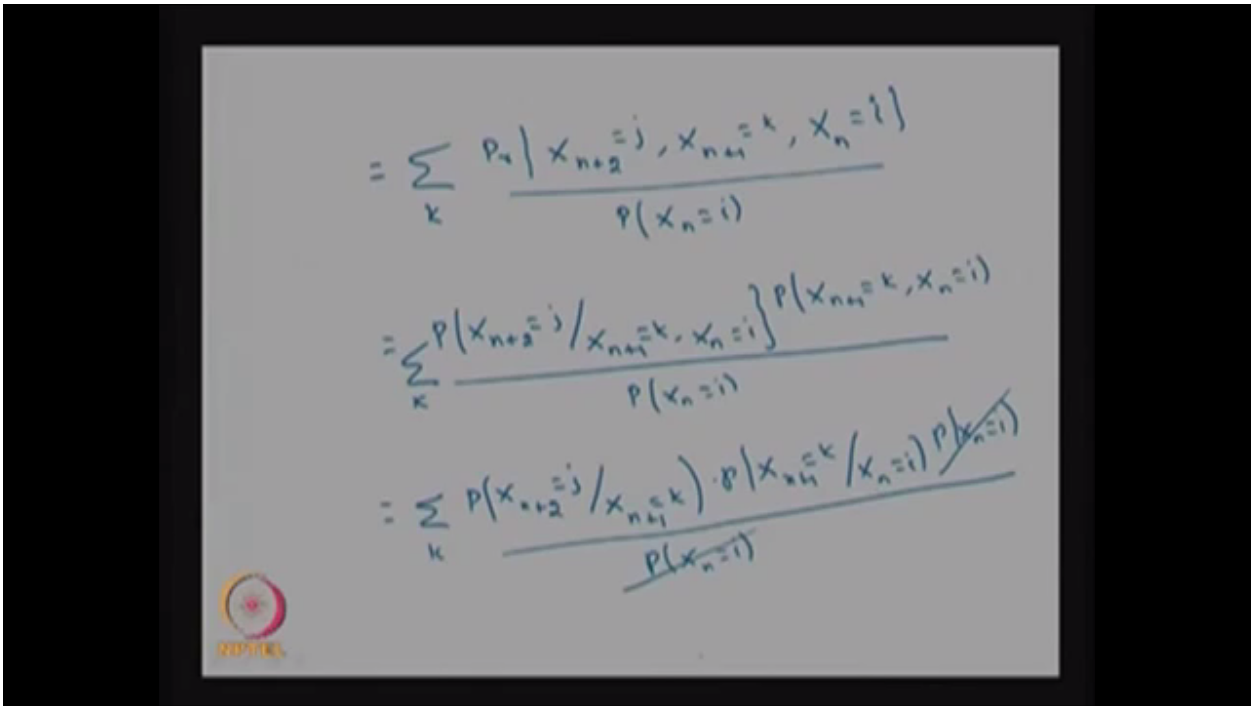
So what we need, what is the N step transition probability of the system will be in the state J given that it was in the state I at the 0th step. This is what do you want to find out, what is the conditional probability mass function of N step transition probability vector, so that we can write it in the form of Pi, j of superscript n that is nothing but the probability of the system will be in the state J given that the system was in the state I at the 0th step. That is we need to compute the N step transition probabilities that is a Pi, j of n.

So this can be computed by using the method called a Chapman-Kolmogorov equations. So this Chapman-Kolmogorov equation provide a method for computing this N step transition probabilities.



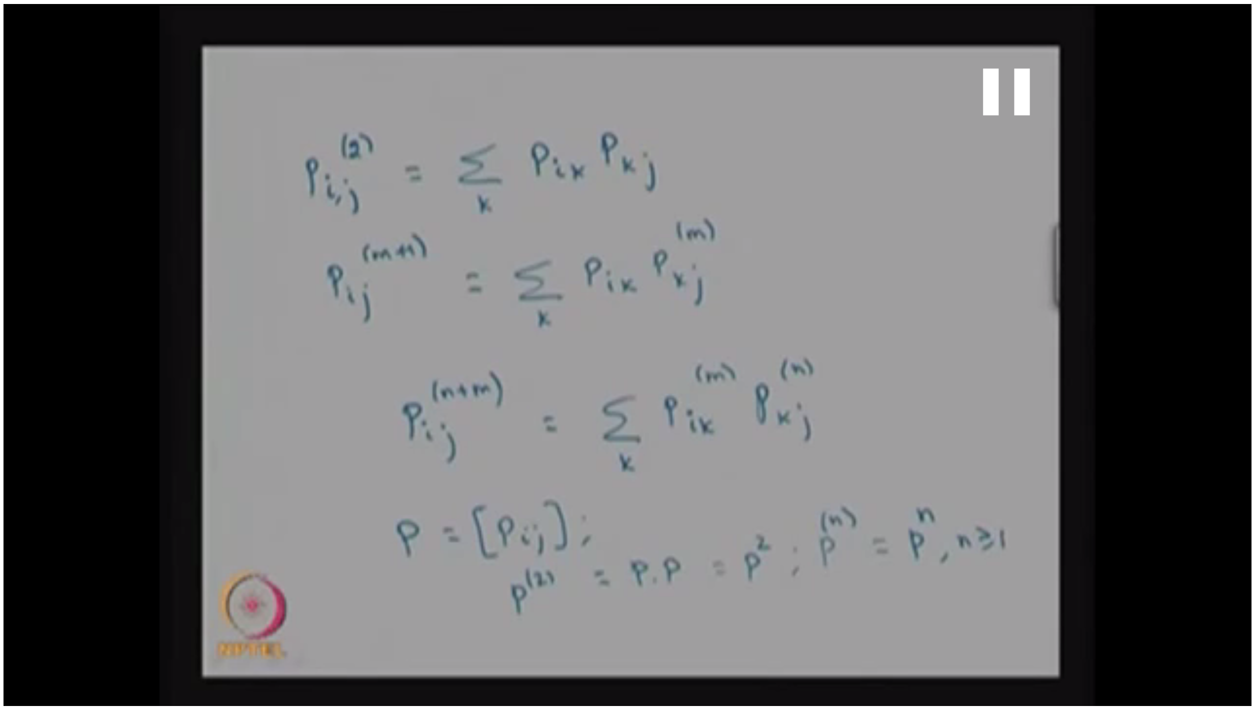

So how we are going to derive this Chapman-Kolmogorov equation that I am going to do it now. So we are going to derive the Chapman-Kolmogorov equations for the time homogeneous discrete time Markov chain. So let the Pi j of superscript n that is nothing but what is the probability that the X m+n takes a value J given that Xm was I. Since the discrete time Markov chain is the time homogeneous, so this is the transition probability of system moving from the state I to J from the M at step 2, m+n at the step. Therefore, this transition is the N step transition probability matrix for the time homogeneous discrete time Markov chain.

Let us start with the 2-step. The 2-step is nothing but what is the probability that system is moving from the state I to J in two steps. So n+2 takes a value J given that the Xn was I. This is for all n it is true, because the DTMC is this time emerges. So this probability you can write it as this 2-step transition probability of system moving from I to J, the state I to the state J in two steps, that you can write it as what are all the possible ways the system is moving from the state I to J by including one more state in the first step the state is K given that the system was in the state I in the Nth step for all possible values of K belonging to S. I can write this a conditional 2-step conditional probability mass function from the Nth step to n+2 second step, that is same as I can include one more possible state of K in the n+1 of the step.




Now I can expand these as that is same as for possible values of K and what is the probability that the system was in the state J in the n plus second step and the system was in the state K in the n+1 naught step. The system was in the state I in the Nth step, divided by what is the probability that in the Nth step it is in the state I. The numerator joint distribution of these three state – these three random variables that I can write it as in the form of conditional, what is the conditional probability that the X n+2 takes a value J given that X n+1 takes the value K and the Xn takes a value I product of X n+1 takes the value K, X n takes a value I, divided by what is the probability that X n takes a value I and here the summation is over the key. So basically, I am writing the numerator joint distribution of these three random variables as the product of our conditional distribution with the marginal distribution of those two random variable.

Since the X I's are the time homogeneous Markov chain, this conditional distribution by using the Markov property is same as the conditional distribution of X n+2 takes a value J given that only the latest value is important. The latest value is needed, not the previous history; therefore, because of the memoryless property X n takes a value I is removed. Therefore, this conditional distribution is the conditional distribution to only X n+1 with the X n+2. Similarly, I can apply the joint distribution of these two random variables X n+1 and Xn, I can again write it as the probability of the X 1+1 takes a value K given that Xn takes the value I and the probability of Xn takes a value I, whole divided by probability of X n and takes the value I. So this and this get cancelled, so this is nothing but the conditional probability, this is nothing but the one-step transition probability of system moving from K to J and the second term is a one-step transition probability of system is moving from I to K.

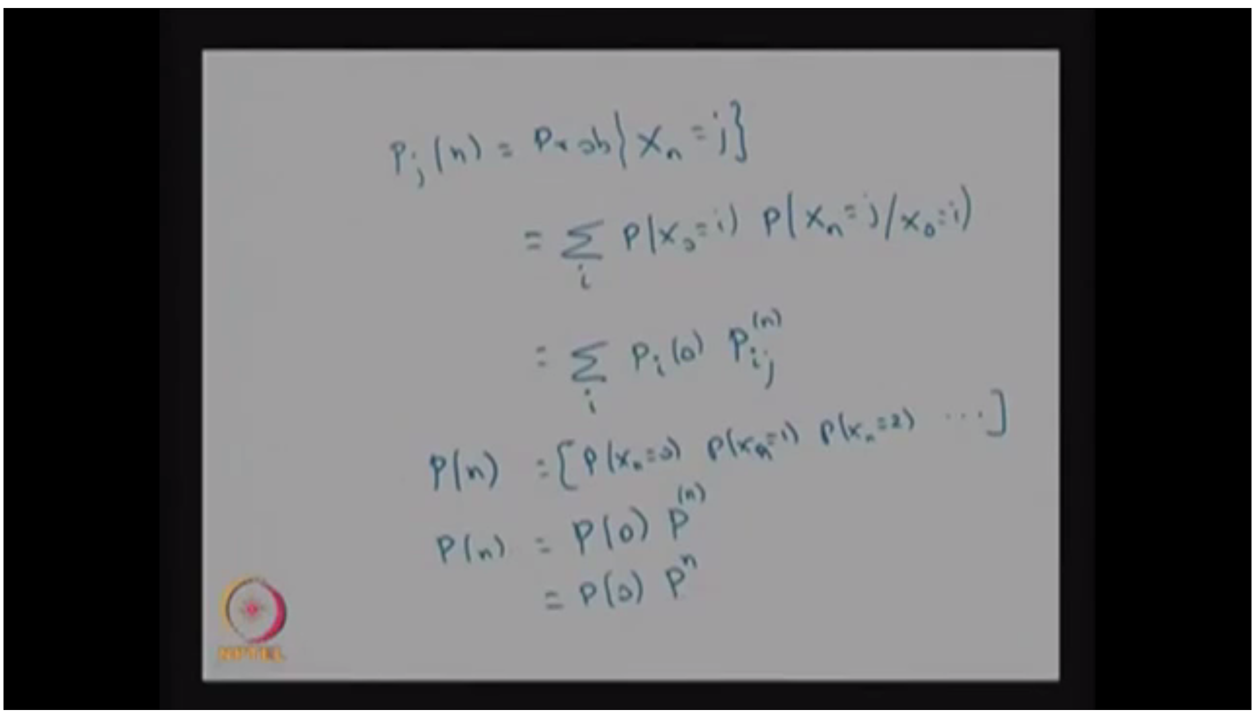



Therefore, the left-hand side we have what is the two-step transition probability of I to J is same as all possible values of K what is the one-step transition probability of system is moving from I to K and the one step transition probability of K to J. So this product will give the two-step transition probability of system is moving from the state I to J. That is same as what is the possible values of K, the system is moving from the state I to K and K to J. So this is for the two-step. Similarly, by using the induction method, one can prove I to J of m+1 step that is same as what is the possible values of K, the system is moving from one-step from I to K and m steps from K to J. This is a two-step so this one step from I to K and one Step from K to j, by induction I can prove the m+1 step will be I to K and K to J in n step. Similarly, I can make it in the other way around also. It is I to K in n steps and to K to J in one step also. That combination also land up. The m+1 step the system is moving from I to J.

In general, we can make the conclusion, the system is moving from I to J in in n+m steps, that is same as the possible values of K of probability of system is moving from I to K in m steps and the n step the system is moving from K to J. That will give -- for all possible values of k, that will give the possibility of system is moving from I to J in n+m steps. So this equation is known as Chapman Kolmogorov equation for the time homogeneous discrete time Markov chain.

So whenever you have a stochastic process as time homogeneous discrete time Markov chain, then that satisfies this equation and this equation is known as the Chapman Kolmogorov equations. In the matrix form, you can write the capital P is the matrix which consists of the element of one-step transition probability method, one-step transition probabilities. In that case, if you make m=1 and n=1, then the matrix of P of superscript 2, that is the in matrix form of a two-step transition probability. That is nothing but if you put n=1 and m=1, you will get P x P and that is going to be P square. So the right hand side the P of superscript (2) means, it is a two-step transition probability matrix, and the right hand side the P square that is the square of the P matrix where P is the one-step transition probability matrix.

So in this form, in general, you can make the N step transition probability matrix is nothing but P of n for n is greater than or equal to 1. For n=1, it is obvious; for n=2 onwards, the P power n. That is same as the N step transition probability matrix.




Hence, so now we got the n step transition probability is nothing but the P power n, where P is the one-step transition probability matrix. Therefore, in matrix form, I can give the P of n, the P of n is nothing but in the matrix form of the distribution of Xn or this is nothing but the vector which consists of the Nth step where the system will be. So this is nothing but what is the probability that in the Nth step, the system will be in the state 0 or in the Nth step the system will be in the state 1 and in the Nth step, the system will be in the 2 and so on. This is the vector. So the P of n, you can find out in the matrix form by using the above equation. It is going to be P of 0 that is also vector, the initial probability vector, multiplied by P power P of (n), that is N step transition probability matrix, but the N step transition probability matrix is nothing but the P power n. Therefore, this is same as the P of 0 into P power n.

In the last slide, we got a P of superscript within (n), that is the N step transition probability matrix is same as the one-step transition probability with the power n. Therefore, this is going to be the distribution of e Xn in the vector form, that is same as the P 0 multiplied by P power n, where the P is nothing but the one-step transition probability matrix. That means if you want to find out the distribution of Xn for any n, you need only the initial probability vector and one-step transition probability matrix. Because the discrete time Markov chain is a time homogeneous, we need only the one-step transition probability matrix and the initial probability vector that gives to find out the distribution of Xn for any n. So with the help of one-step transition probability matrix and the initial probability vector, you can find the distribution of Xn for any n.