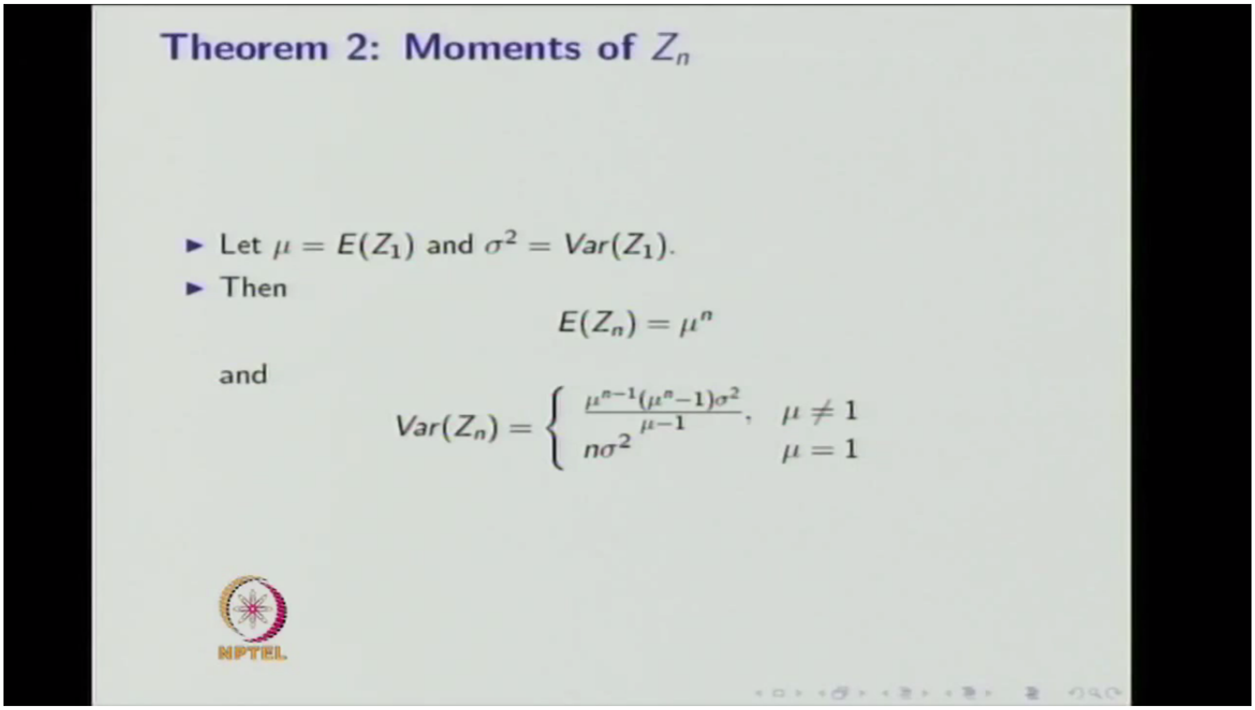
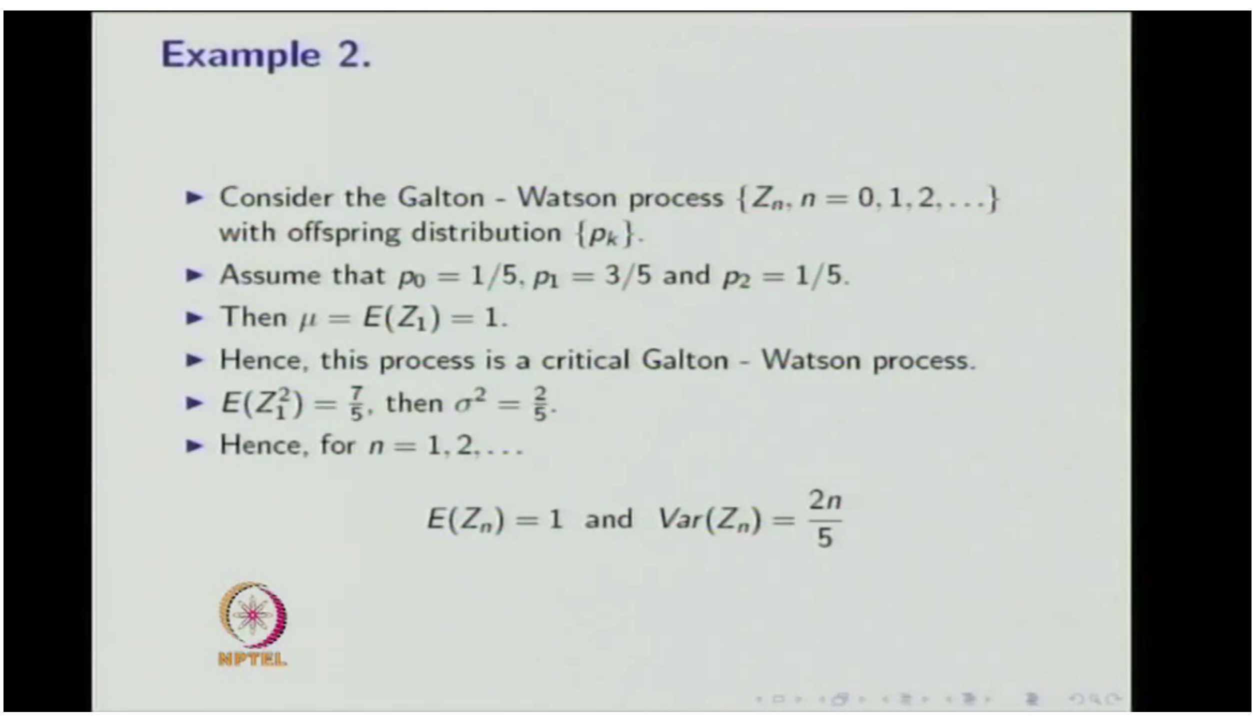
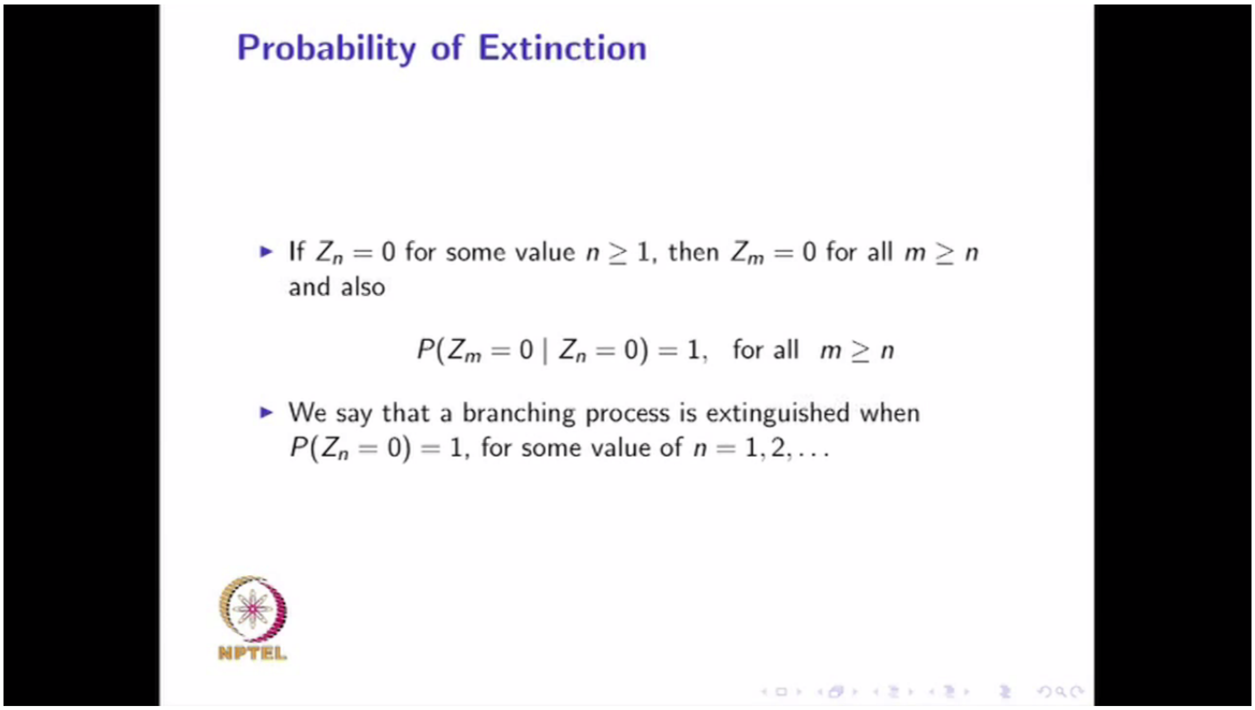


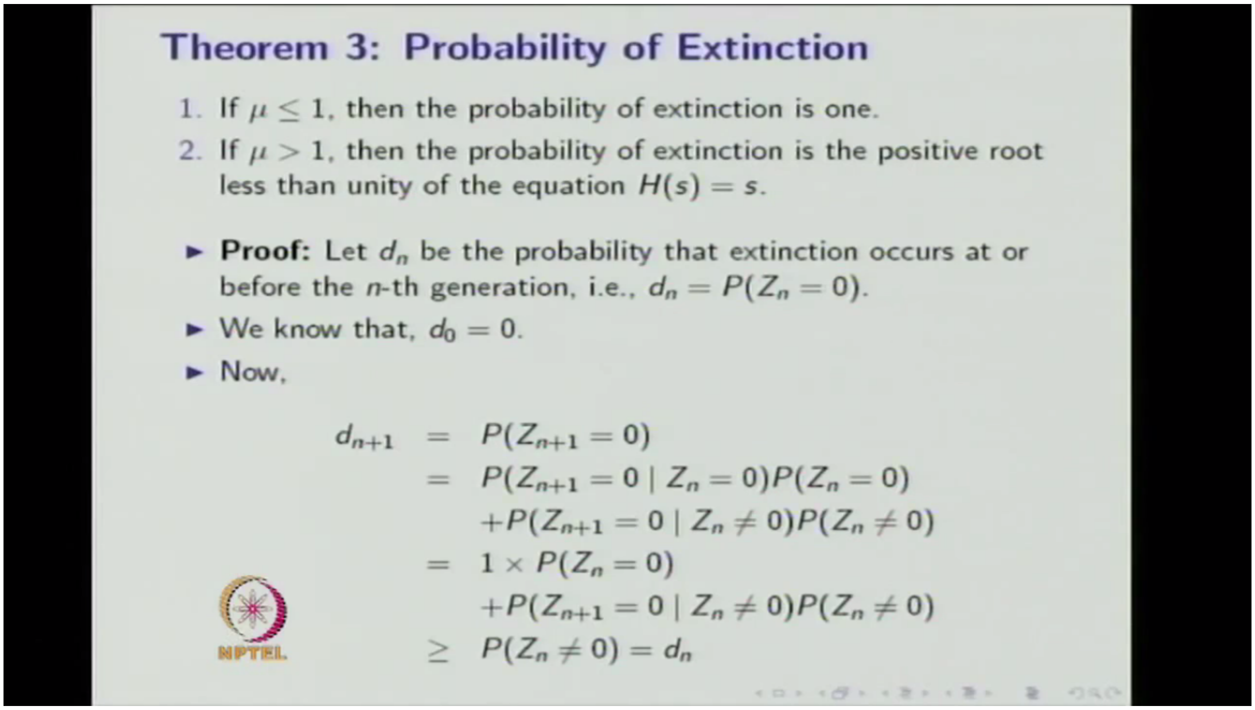
Since you know mu and sigma square using the theorem 1 you can find out the moments of Zn first and the second order moment, expectation of Zn and the variance of Zn. Since mu is equal to 1 the expectation of Zn will be 1, variance of Zn will be n times Sigma square. In this problem the Sigma square is a 2 by 5 therefore mean of Zn is 1 variance of Zn is 2n by 5.



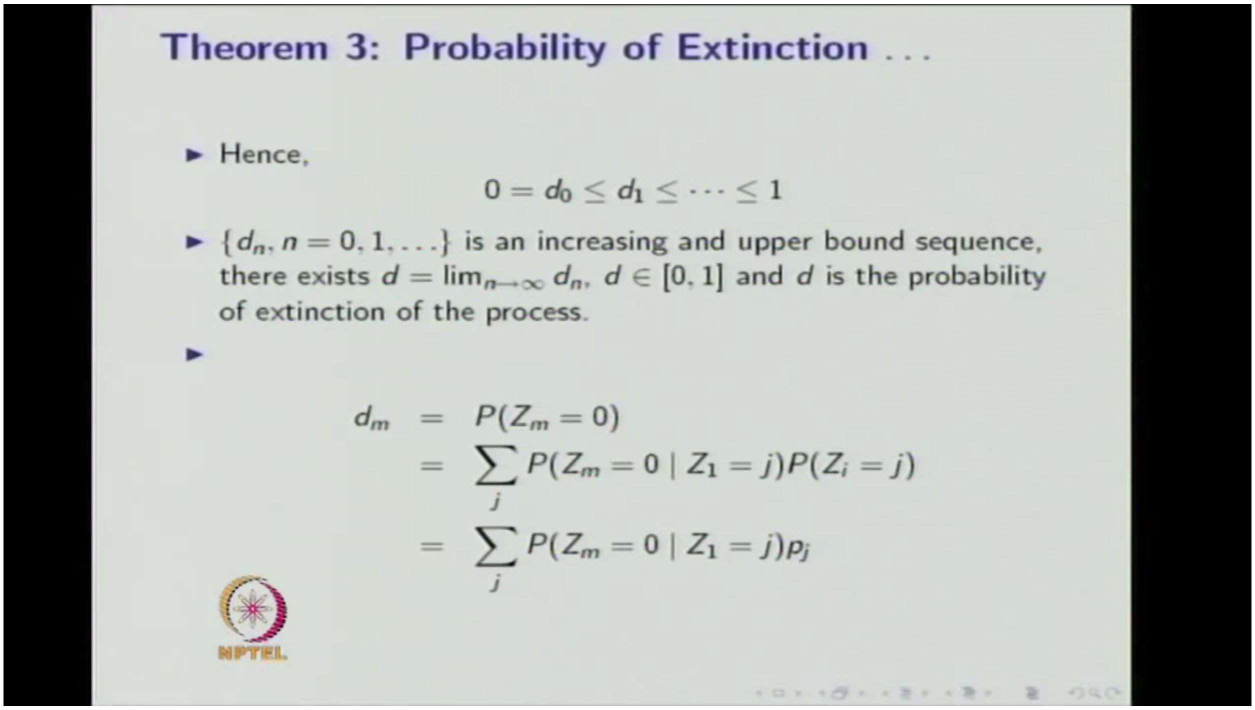
Now we are going to discuss the very important concept called the probability of extinction. If Zn is equal to 0 for some value for n is greater or equal to 1 then Zm will be 0 for all m greater or equal to n and also the conditional probability of Zm is equal to 0 given Zn is equal to 0 that'll be 1 for all m greater or equal to 1. We say that the branching process is extinguished when probability of Zn is equal to 0 will be 1 for some value of n.



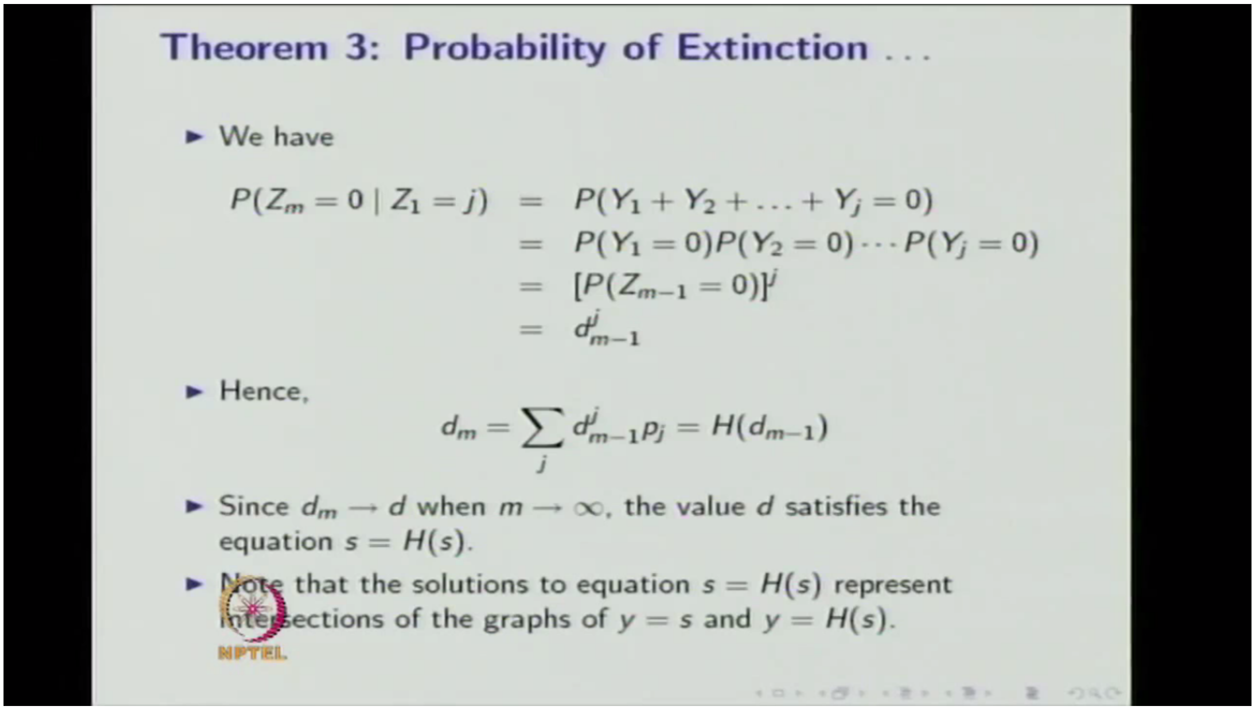
Now we discussed the probability of extinction in theorem 3. If mu is less than or equal to 1 then the probability of extinction is 1. If mu is greater than 1 then the probability of extinction is the positive root less than unity of the equation H of s equal to s where H of s is the probability generating function of Z1. Here we consider Galton-Watson branching process Zn with the offspring distribution Pk. The Pk forms a probability mass function for the random variable Z1. If mu is less than or equal to 1 then the probability of extinction is 1. If mu is greater than 1 then the probability of extinction is the positive root less than unity of the equation H of s equal to s; that means we have to solve the equation H of s equal to s from that you can get the probability of extinction.



Here mu is mean of the random variable Z1. Let us see the proof. Let dn be the probability that extinction occurs at or before the nth generation. Hence, dn is nothing but the probability of Zn is equal to 0. We know that d naught will be 0 because we made the assumption Z naught is equal to 1. Now we will find out dn plus 1. Dn plus 1 is nothing but by the definition dn plus 1 is nothing but probability of Zn plus 1 equal to 0. We can write a probability of Zn plus 1 equal to 0 using conditional probabilities. That is same as probability of Zn plus 1 is equal to 0 given that Zn was 0 multiplied by probability of Zn is equal to 0 plus probability of Zn plus 1 is equal to 0 given Zn is not equal to 0 multiplied by probability of Zn is not equal to 0. We know that probability of Zn plus 1 is equal to 0 given Zn is equal to 0 that is equal to 1 therefore this will be 1 times the probability of Zn is equal to 0 plus probability of Zn plus 1 is equal to 0 given Zn is not equal to 0 multiplied by probability of Zn not equal to 0.

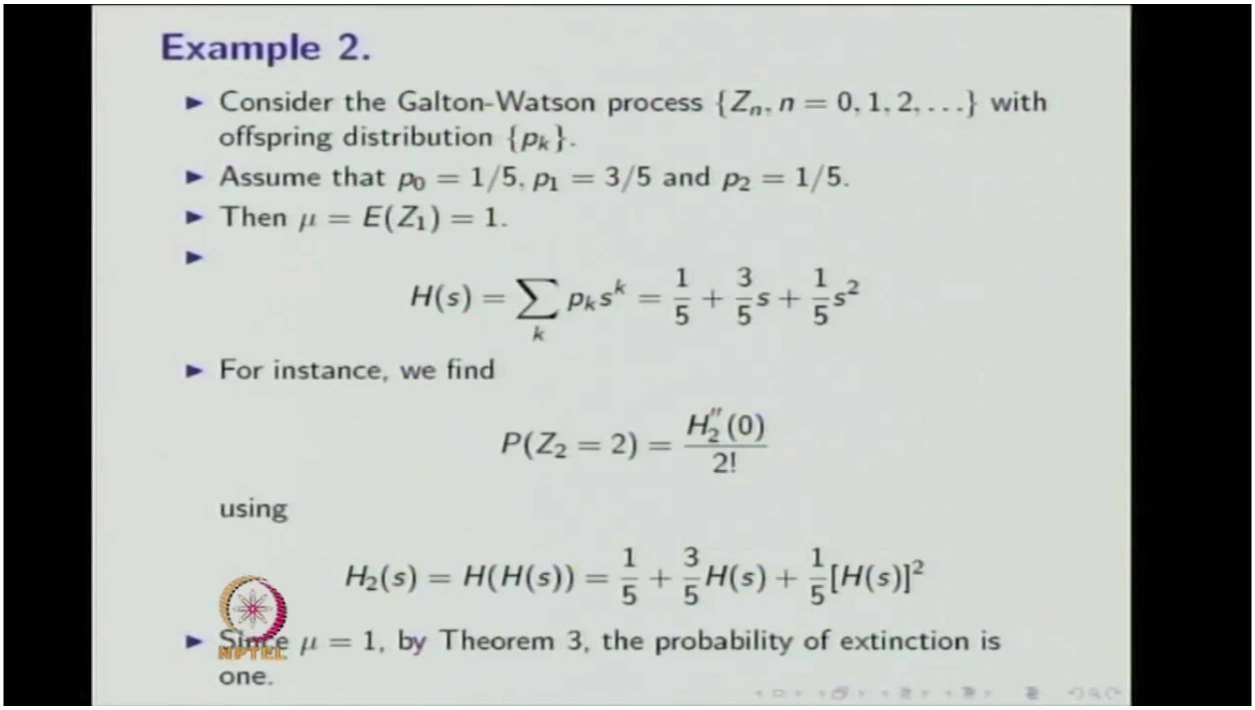


Obviously this will be greater or equal to probability of Zn is not equal to 0 because you are adding some probability plus probability multiplied by probability of Zn is not equal to 0 therefore this quantity will be greater than or equal to probability of Zn not equal to 0. This quantity will be this is same as 1 times probability of Zn is equal to 0 plus probability of Zn plus 1 equal to 0 given probability of Zn is not equal to 0 multiplied by probability of Zn is not equal to 0 obviously this quantity will be greater than or equal to 0 the second term therefore the whole result will be greater than or equal to probability of Zn equal to 0. You know that probability of Zn is equal to 0 is nothing but dn therefore dn plus 1 will be greater than or equal to dn. This is true for n is equal to 1, 2 and so on. Hence, the d naught will be 0, d naught is less than or equal to d1. d1 will be less than or equal to d2 and so on and since the da's are the probability of extinction therefore that will be less than or equal to 1. Hence, dn is a sequence and upper bound sequence dn is a increasing and upper bound sequence there exists d that is nothing but the limit n tends to infinity of tn and the d is belonging to the closed interval 0 to 1 and d is the probability of extinction of the process.



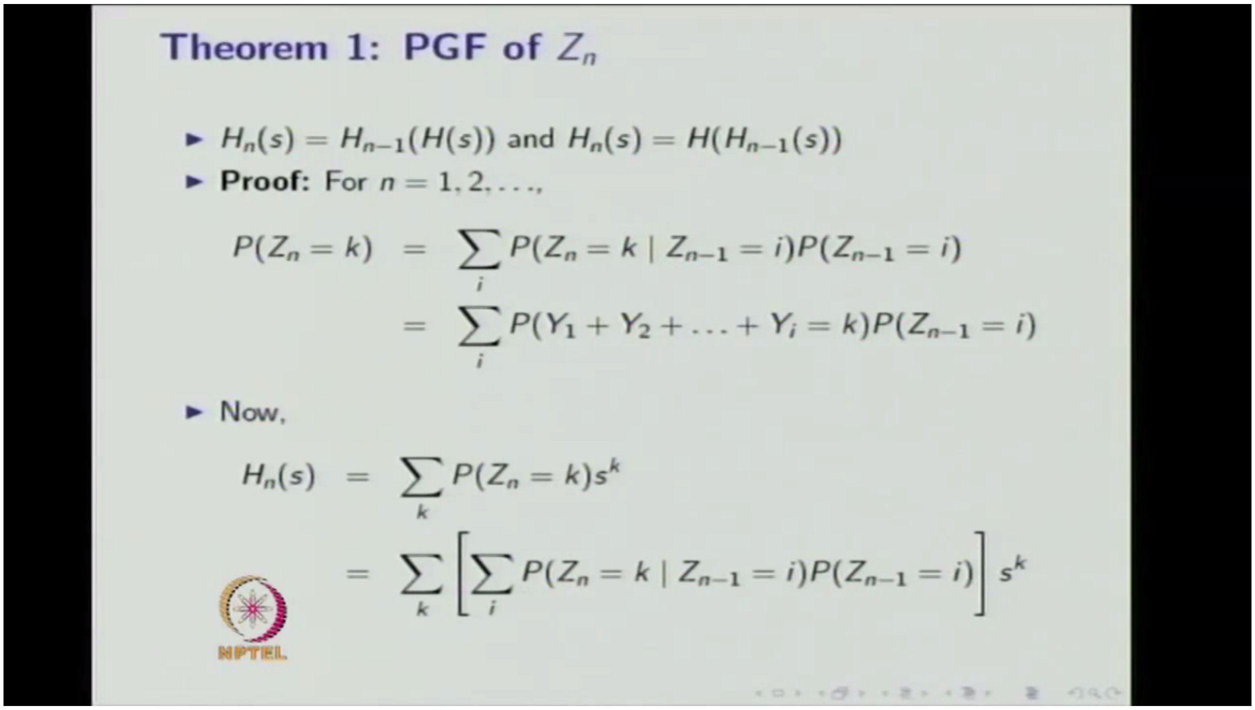
Now we will find out what is the d. You know that dm is nothing but probability of Zm equal to 0 that is nothing but summation over j probability of Zm is equal to 0 given Z1 is equal to j multiplied by probability of Z1 is equal to j. You know that probability of Z1 is equal to j is nothing but Pj therefore the dm is nothing but summation over j probability of Zm is equal to 0 given Z1 is equal to j multiplied by Pj. We know that probability of Zm is equal to 0 given Z1 is equal to j is nothing but probability of Y1 plus Y2 and so on Yj is equal to 0. Since Yi's are iid random variables that is nothing but probability of Y1 is equal to 0, Y2 is equal to 0 and so on probability of Yj is equal to 0. That is same as the probability of Zm minus 1 equal to 0 whole power j that is nothing but d suffix m minus 1 power j. Hence, you can substitute this result in this equation. This conditional probability is nothing but $ suffix m minus 1 power j therefore dm will be summation over j d of m minus 1 power j Pj and that is nothing but the probability generating function of dm minus 1. So hence we get a dm is equal to H of d m minus 1. Since dm tends to d as a m tends to infinity the value d satisfies the equation s is equal to H of s because a dm is equal to H of d m minus 1 therefore the value d satisfies the equation s is equal to H of s.

Note that the solutions to the equation s is equal to H of s is of a sense intersections of the graphs of Yis equal to s and y is equal to H of s. So the d will be probability of extinction and we are not going to discuss further more about how to solve s is equal to H of s and finding the probability of extinction.



We will consider the third example. Zn be the sequence of random variable which is a Galton-Watson process with the offspring distribution Pk. Similar to the example 1 and 2 P naught is equal to 1 by 5, P1 is equal to 3 by 5 P2 is equal to 1 by 5. Already we got the mean of a Z1 that is equal to 1 therefore this is a critical Galton-Watson process.

Now we can find out the probability generating function of Z1. Using these one can find the probability of Z is equal to or we can find the distribution of Zn also. For instant we find a probability of Z is equal to 2 for that you need the probability generating function of Z. That means you should know what is the probability generating function of H2 of s. Obvious with the help of H1 of s and H of s one can find H2 of s. That's what we have proved it in the theorem 2. Hn of s will be Hn minus 1 of H of s.



So here we put n is equal to 2 find the probability generating function of Z using the probability generating function of Z2 we are finding the probability of Z2 is equal to 2. So probability of Z is equal to 2 will be H2 double dash of 0 divided by 2 factorial. So first you find out H2 of s that is nothing but a H1 of H of s that is same as H of s. So replace s by H of s in the H of s expression that is 1 by 5 plus 3 by 5 times s plus 1 by 5 s square. So replace s by H of s so you will get H2 of s. Once you know the H2 of s you differentiate twice then substitute s is equal to 0 then divide by 2 you will get the probability of Z2 is equal to 2. Since mu is equal to 1 this is a critical Watson process by using the theorem 3 we conclude the probability of extinction is 1.

Theorem 3 says if mu is less than or equal to 1 then the probability of extinction is 1. So since in this problem the mu is equal to 1 hence by theorem 3 the probability of extinction is 1.

So in this lecture we have covered the definition and the examples of branching process. In particular we have discussed Galton-Watson discrete branching process. We have discussed the probability generating function of Zn. We have discussed the mean and variance of Zn. Also we have seen three examples through that we found the conditional probability, probability generating function, mean and variance of Zn. Finally, we have discussed the probability of extinction for the Galton-Watson process.

In the next lecture we are going to cover another important branching process that is a Markov branching process which is a continuous type branching process. In this lecture we have discussed a discrete type branching process that is a Galton-Watson process. In the next lecture we are going to cover continuous type branching process that is a Markov branching process and some more important branching processes. Here is the references.

References

- G. Sankaranarayanan, "Branching Processes and Its Estimation Theory", Wiley Eastern Ltd, New Delhi, 1989.
- K.B. Athreya and P.E. Ney, "Branching Processes", Dover Publications, New York, 2004.
- J Medhi, "Stochastic Processes", 3rd edition, New Age International Publishers, 2009.
- T.E. Harris, "Theory of Branching Processes", Dover Publications, New York, 1989.