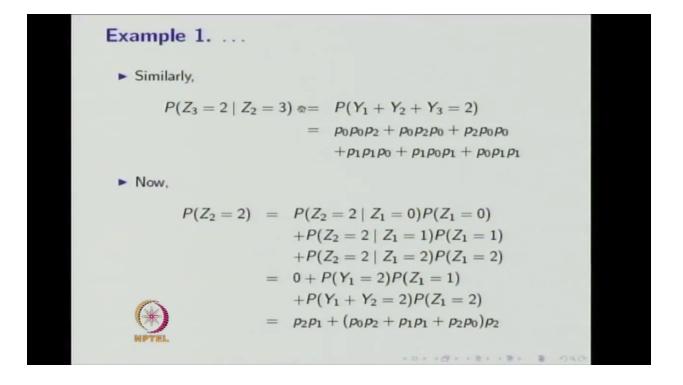


Now let us consider a simple example. Consider the Galton Watson process Zn with offspring distribution Pk. Assume that P naught is equal to 1 by 5, P1 is equal to 3 by 5 and P2 is equal to 1 by 5. So this is a probability mass function for the random variable Z1. Our interest is to find out the conditional probability P probability of Z2 is equal to 3 given Z1 was 2. And also probability that Z2 is equal to 2 given Z2 was 3. And also probability of Z2 is equal to 2.

So the conditional probability PZ2 is equal to 3 given Z1 is equal to 2 that is same as probability of Y1 plus Y2 equal to 3. That is possible either Y1 is equal to 0 or Y2 is equal to 3 or Y1 is equal to 1, Y2 is equal to 2 or Y1 is equal to 2 or and Y2 is equal to 1 or Y1 is equal to 3 and Y2 is equal to 0. Since Yi's are iid random variables you can write down this has a probability of Y1 is equal to 0 into probability of Y2 is equal to 3 and so on. Probability of Y1 is equal to 0 that is nothing better P naught. Probability of Y2 is equal to 3 that is nothing but P3.

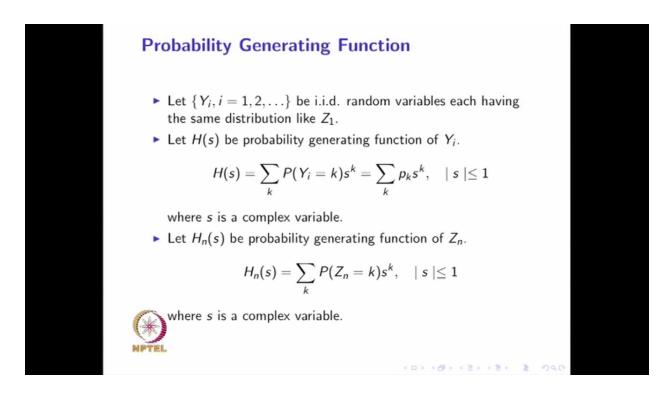
Similarly the second expression is P1 into P2. Third one is a P2 into P1. The last one is P3 P naught. Since P3 is equal to 0 the first term and the last term will be 0. So you will get a P1 P2 plus P2 P1 that is same as 6 by 25. So this is a conditional probability of P of Z2 is equal to 3 given Z1 is equal to 2.



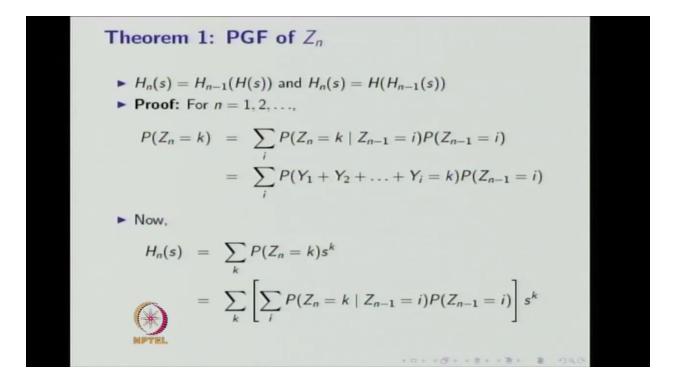
Similarly one can find P of Z3 is equal to 2 given Z2 is equal to 3. That is possible when Y1 plus Y2 plus Y3 is equal to 2. So the probability of Y1 plus Y2 plus Y3 is equal to 2 we have six possibilities. Substitute the value of P naught P1 P2 you will get the numerical value of this conditional probability.

Similarly one can find the probability of Z2 is equal to 2 also. Probability of Z is equal to 2 is same as probability of Z2 is equal to 2 given Z1 is equal to 0 multiplied by probability of Z1 is equal to 0 plus the combination with a probability of Z1 is equal to 1, probability of Z1 is equal to 2. Substitute the values then you will get the probability of Z is equal to 2.

Now we are going to discuss the probability generating function for the branching process. Let Yi's be iid random variables each having the same distribution like Z1. Let H of s be the probability generating function of Yi's that Hn of s be the probability generating function of Zn.

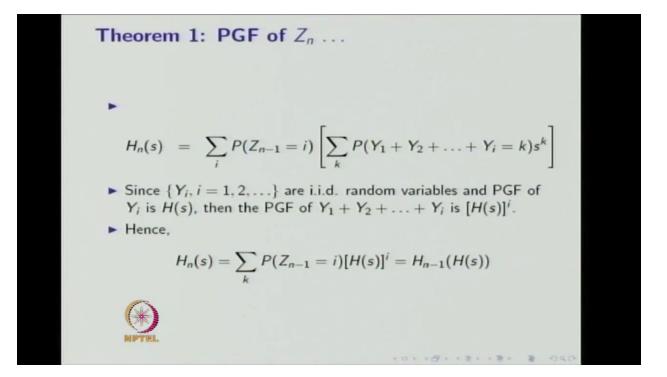


Since Z naught is equal to 1 you will get H naught of S is same as s. Now our interest is to find out the probability generating function for Z1 that is same as the probability generating function of Yi's. The probability generating function of Yi's is H of s therefore H1 of s is same as H of s. Our interest is to find out the probability generating function for Zi, Zn where n is 1, 2, 3 and so on. This we are going to give it as a theorem. Hn of s this is nothing but the probability generating function for s and Hn of s and Hn of s also can be written in the form of H of Hn minus minus 1 of s. Let us see the proof for this.

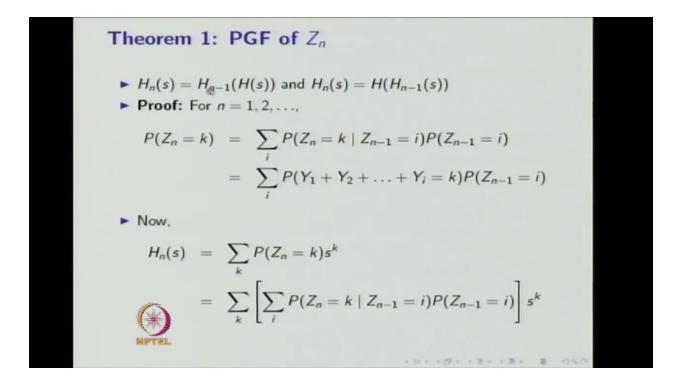


You know that the probability of Zn is equal to K that is nothing but summation over i's probability of Zn is equal to K given Zn minus 1 is equal to i multiplied by probability of ZN minus 1 is equal to i. You know that this conditional probability is nothing but probability of Y1 plus Y2 and so 1 plus Yi is equal to k multiplied by probability of Zn minus 1 is equal to i.

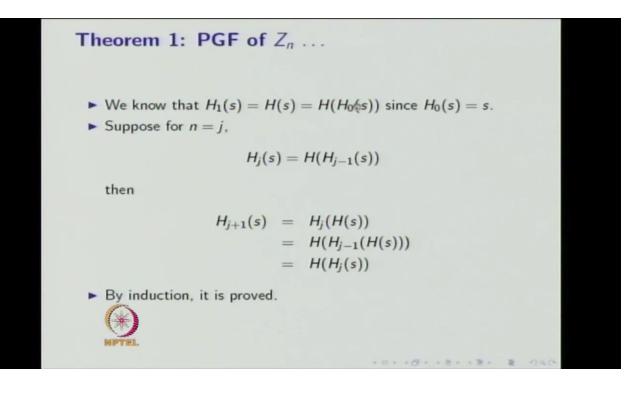
Now we'll go for finding out the probability generating function for the random variable Zn. Substitute probability of Zn is equal to K from above in this equation therefore you will have a summation of K substitute the probability of Zn is equal to K that is nothing but the summation over i probability of a conditional probability multiplied probability of Zn minus 1 is equal to i multiplied by s power k.



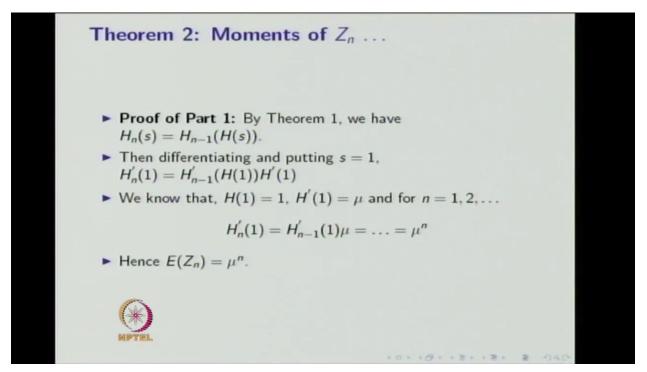
Substitute the conditional probability is nothing but the probability of Y1 plus Y2 plus so on Yi is equal to k into a spot. Since Yi's are iid random variables and the probability generating function of Yi's is are nothing but Hi sorry H of s the probability generating function of a sum of i random variables Yi's that is nothing but H of s whole power i because of Yi's are iid random variables and probability generating function of Yi is equal to H of s therefore the probability generating function of y1 plus y2 and so one plus Yi is H of s whole power i. Therefore substitute this is nothing but the probability generating function of Y1 plus Y2 plus and so on plus Yi therefore H n of s is nothing but the summation over K. Hn of s is nothing but a summation over i probability of Zn minus 1 is equal to I times H of s whole power i. Replace this by H of s whole power I therefore summation I probability of Zn minus 1 is equal to H of S whole i. So that is nothing but the probability generating function for the random variable Zn minus 1 with the replacement s by H of s. Therefore Hn of s is nothing but Hn minus 1 of H of s.



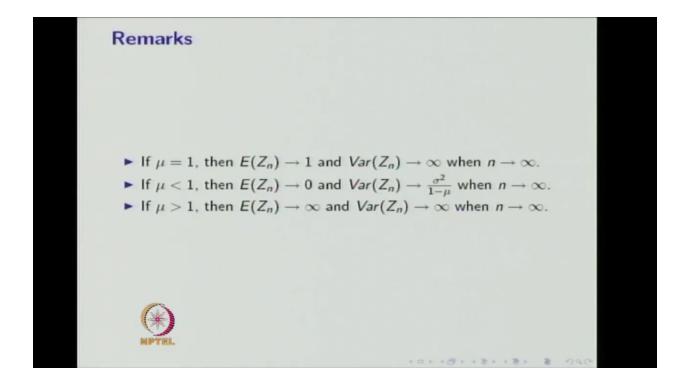
Now we are going to prove the second part. We know that H1 of s is H of s that is same as H of H naught of s because H naught of s is nothing but s. Suppose this is a true for n is equal to j that means H of j of s is H of s H of j minus 1 of s then we can find out what is H of j plus one of s. That is nothing but Hj of H of s that is same as H of Hj minus one times H of s that is same as H of Hj of s. By induction it is a proved.



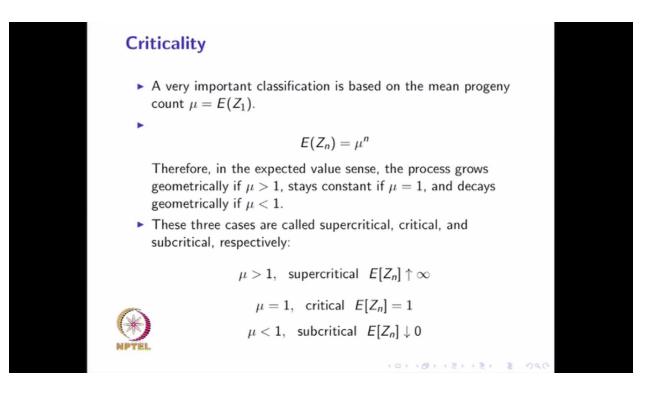
Now we are finding the moments of Zn. Let that mu equal to expectation of Z1 and Sigma square is nothing but the variance of Z1 then expectation of Zn that is mu power n and the variance of Zn will be for mu equal to 1 it is n times Sigma square for mu is not equal to 1 it will be mu power n minus 1 times mu n minus 1 times Sigma square divided by mu minus 1. We will prove the part one that is expectation of Zn is equal to mu power n. By theorem one you know that Hn of s is same as Hn minus 1 times sorry, by theorem one, we know that Hn of s is Hn minus 1 of H of s. Then differentiating and putting s equal to one you will get Hn dash of 1 that is same as Hn minus 1 dash of H of 1 into H dash of 1. We know that H of 1 is 1 H dash of 1 is mu therefore for n is equal to 1, 2 and so one you will get a Hn dash of 1 is same as Hn minus 1 dash of 1 into mu that is same as Hn minus 2 dash of 1 into mu square and so on; therefore you will get mu power n because you know that H dash of 1 is mu. By recursively you will get a Hn dash of 1 is same as mu power n.



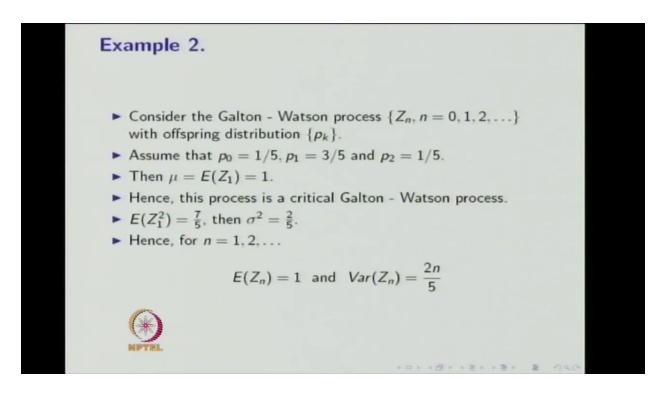
So till now we have discussed the probability generating function of Z power n and also we have discussed the mean and variance of Z of n. As a remark if mu is equal to 1 the expectation of Zn tends to 1 whereas a variance of Zn tends to infinity as n tends to infinity. You can see it from this theorem as mu tends to 1 expectation of Zn will tends to 1 whereas the variance it tends to infinity because that is same as the n times Sigma square therefore as mu tends to 1 the variance of Zn tends to infinity. Whereas if mu is less than 1 the expectation of Zn tends to 0 and the variance of Zn will be sigma square divided by 1 minus mu n tends to infinity. Similarly if mu is greater than one then the expectation of Zn will tends to infinity and the variance of Zn tends to infinity as n tends to infinity. This also you can see it from the theorem.



Now we are going to discuss the criticality. A very important classification is based on mean progeny count mu is equal to expectation of Z1. A very important classification is based on the mean progeny count mu is equal to expectation of Z1. You know that expectation of Zn is equal to mu power n just now we have proved it in the theorem one. Therefore in the expected value since the process grows geometrically if mu is greater than 1, stays constant if mu is equal to 1 and decays geometrically if mu is less than 1 from the expectation of Zn is equal to 1 then the process stays constant whereas if mu is less than 1 the process decays geometrically. Thus three cases are called a super critical, critical and a subcritical respectively. That means if mu is greater than 1 the process is called supercritical. In this case the expectation of Zn tends to infinity. If mu is less than 1 the process is called a critical and expectation of Zn is equal to 1 as n tends to infinity. When mu is less than 1 the process is called subcritical and expectation of Zn is equal to 1 as n tends to 0 as n tends to infinity.



Now we are going to consider the second example. Consider the Galton Watson processor Zn with offspring distribution Pk. We choose the same problem example 1 if the assumption P naught is equal to 1 by 5, P1 is equal to 3 by 5 and P2 is equal to 1 by 5.



Now we can find out the mean of Z1 that is nothing but 1 because Pnaught is equal to 1 by 5, P1 is equal to 3 by 5, P2 is equal to 1 by 5 we will get mean of Z1 will be 1. Hence this process is called a critical Galton Watson process because mu is equal to 1. You can find out the variance of Z1 also. Also you find out expectation of Z1 square that will be 7 by 5 hence a variance equal to expectation of Z1 square that will be 2 by 5.