

### Example 1.

- ▶ Consider the Galton - Watson process  $\{Z_n, n = 0, 1, 2, \dots\}$  with offspring distribution  $\{p_k\}$ .
- ▶ Assume that  $p_0 = 1/5, p_1 = 3/5$  and  $p_2 = 1/5$ .
- ▶ Calculate  $P(Z_2 = 3 | Z_1 = 2), P(Z_3 = 2 | Z_2 = 3)$  and  $P(Z_2 = 2)$ .
- ▶  $P(Z_2 = 3 | Z_1 = 2)$

$$\begin{aligned} &= P(Y_1 + Y_2 = 3) \\ &= P(Y_1 = 0, Y_2 = 3) + P(Y_1 = 1, Y_2 = 2) \\ &\quad + P(Y_1 = 2, Y_2 = 1) + P(Y_1 = 3, Y_2 = 0) \\ &= P(Y_1 = 0)P(Y_2 = 3) + P(Y_1 = 1)P(Y_2 = 2) \\ &\quad + P(Y_1 = 2)P(Y_2 = 1) + P(Y_1 = 3)P(Y_2 = 0) \\ &= p_0p_3 + p_1p_2 + p_2p_1 + p_3p_0 \\ &= 0 + \frac{3}{25} + \frac{3}{25} + 0 = \frac{6}{25} \end{aligned}$$



Now let us consider a simple example. Consider the Galton Watson process  $Z_n$  with offspring distribution  $P_k$ . Assume that  $P$  naught is equal to 1 by 5,  $P_1$  is equal to 3 by 5 and  $P_2$  is equal to 1 by 5. So this is a probability mass function for the random variable  $Z_1$ . Our interest is to find out the conditional probability  $P$  probability of  $Z_2$  is equal to 3 given  $Z_1$  was 2. And also probability that  $Z_2$  is equal to 2 given  $Z_2$  was 3. And also probability of  $Z_2$  is equal to 2.

So the conditional probability  $P_{Z_2 \text{ is equal to } 3 \text{ given } Z_1 \text{ is equal to } 2}$  that is same as probability of  $Y_1$  plus  $Y_2$  equal to 3. That is possible either  $Y_1$  is equal to 0 or  $Y_2$  is equal to 3 or  $Y_1$  is equal to 1,  $Y_2$  is equal to 2 or  $Y_1$  is equal to 2 or and  $Y_2$  is equal to 1 or  $Y_1$  is equal to 3 and  $Y_2$  is equal to 0. Since  $Y_i$ 's are iid random variables you can write down this has a probability of  $Y_1$  is equal to 0 into probability of  $Y_2$  is equal to 3 and so on. Probability of  $Y_1$  is equal to 0 that is nothing better  $P$  naught. Probability of  $Y_2$  is equal to 3 that is nothing but  $P_3$ .

Similarly the second expression is  $P_1$  into  $P_2$ . Third one is a  $P_2$  into  $P_1$ . The last one is  $P_3$   $P$  naught. Since  $P_3$  is equal to 0 the first term and the last term will be 0. So you will get a  $P_1 P_2$  plus  $P_2 P_1$  that is same as 6 by 25. So this is a conditional probability of  $P$  of  $Z_2$  is equal to 3 given  $Z_1$  is equal to 2.

### Example 1. . . .

► Similarly,

$$\begin{aligned}P(Z_3 = 2 \mid Z_2 = 3) &= P(Y_1 + Y_2 + Y_3 = 2) \\ &= p_0 p_0 p_2 + p_0 p_2 p_0 + p_2 p_0 p_0 \\ &\quad + p_1 p_1 p_0 + p_1 p_0 p_1 + p_0 p_1 p_1\end{aligned}$$

► Now,

$$\begin{aligned}P(Z_2 = 2) &= P(Z_2 = 2 \mid Z_1 = 0)P(Z_1 = 0) \\ &\quad + P(Z_2 = 2 \mid Z_1 = 1)P(Z_1 = 1) \\ &\quad + P(Z_2 = 2 \mid Z_1 = 2)P(Z_1 = 2) \\ &= 0 + P(Y_1 = 2)P(Z_1 = 1) \\ &\quad + P(Y_1 + Y_2 = 2)P(Z_1 = 2) \\ &= p_2 p_1 + (p_0 p_2 + p_1 p_1 + p_2 p_0) p_2\end{aligned}$$



Similarly one can find  $P$  of  $Z_3$  is equal to 2 given  $Z_2$  is equal to 3. That is possible when  $Y_1$  plus  $Y_2$  plus  $Y_3$  is equal to 2. So the probability of  $Y_1$  plus  $Y_2$  plus  $Y_3$  is equal to 2 we have six possibilities. Substitute the value of  $P$  naught  $P_1$   $P_2$  you will get the numerical value of this conditional probability.

Similarly one can find the probability of  $Z_2$  is equal to 2 also. Probability of  $Z$  is equal to 2 is same as probability of  $Z_2$  is equal to 2 given  $Z_1$  is equal to 0 multiplied by probability of  $Z_1$  is equal to 0 plus the combination with a probability of  $Z_1$  is equal to 1, probability of  $Z_1$  is equal to 2. Substitute the values then you will get the probability of  $Z$  is equal to 2.

Now we are going to discuss the probability generating function for the branching process. Let  $Y_i$ 's be iid random variables each having the same distribution like  $Z_1$ . Let  $H$  of  $s$  be the probability generating function of  $Y_i$ 's that  $H_n$  of  $s$  be the probability generating function of  $Z_n$ .

## Probability Generating Function

- ▶ Let  $\{Y_i, i = 1, 2, \dots\}$  be i.i.d. random variables each having the same distribution like  $Z_1$ .
- ▶ Let  $H(s)$  be probability generating function of  $Y_i$ .

$$H(s) = \sum_k P(Y_i = k)s^k = \sum_k p_k s^k, \quad |s| \leq 1$$

where  $s$  is a complex variable.

- ▶ Let  $H_n(s)$  be probability generating function of  $Z_n$ .

$$H_n(s) = \sum_k P(Z_n = k)s^k, \quad |s| \leq 1$$



where  $s$  is a complex variable.

Since  $Z$  naught is equal to 1 you will get  $H$  naught of  $S$  is same as  $s$ . Now our interest is to find out the probability generating function for  $Z_1$  that is same as the probability generating function of  $Y_i$ 's. The probability generating function of  $Y_i$ 's is  $H$  of  $s$  therefore  $H_1$  of  $s$  is same as  $H$  of  $s$ . Our interest is to find out the probability generating function for  $Z_i, Z_n$  where  $n$  is 1, 2, 3 and so on. This we are going to give it as a theorem.  $H_n$  of  $s$  this is nothing but the probability generating function for the random variable  $Z_n$  is same as  $H$  of  $n$  minus 1 of  $H$  of  $s$  and  $H_n$  of  $s$  also can be written in the form of  $H$  of  $H_n$  minus minus 1 of  $s$ . Let us see the proof for this.

## Theorem 1: PGF of $Z_n$

►  $H_n(s) = H_{n-1}(H(s))$  and  $H_n(s) = H(H_{n-1}(s))$

► **Proof:** For  $n = 1, 2, \dots$ ,

$$\begin{aligned} P(Z_n = k) &= \sum_i P(Z_n = k \mid Z_{n-1} = i)P(Z_{n-1} = i) \\ &= \sum_i P(Y_1 + Y_2 + \dots + Y_i = k)P(Z_{n-1} = i) \end{aligned}$$

► Now,

$$\begin{aligned} H_n(s) &= \sum_k P(Z_n = k)s^k \\ &= \sum_k \left[ \sum_i P(Z_n = k \mid Z_{n-1} = i)P(Z_{n-1} = i) \right] s^k \end{aligned}$$



You know that the probability of  $Z_n$  is equal to  $K$  that is nothing but summation over  $i$ 's probability of  $Z_n$  is equal to  $K$  given  $Z_{n-1}$  is equal to  $i$  multiplied by probability of  $Z_{n-1}$  is equal to  $i$ . You know that this conditional probability is nothing but probability of  $Y_1$  plus  $Y_2$  and so on plus  $Y_i$  is equal to  $k$  multiplied by probability of  $Z_{n-1}$  is equal to  $i$ .

Now we'll go for finding out the probability generating function for the random variable  $Z_n$ . Substitute probability of  $Z_n$  is equal to  $K$  from above in this equation therefore you will have a summation of  $K$  substitute the probability of  $Z_n$  is equal to  $K$  that is nothing but the summation over  $i$  probability of a conditional probability multiplied probability of  $Z_{n-1}$  is equal to  $i$  multiplied by  $s$  power  $k$ .

## Theorem 1: PGF of $Z_n \dots$



$$H_n(s) = \sum_i P(Z_{n-1} = i) \left[ \sum_k P(Y_1 + Y_2 + \dots + Y_i = k) s^k \right]$$

▶ Since  $\{Y_i, i = 1, 2, \dots\}$  are i.i.d. random variables and PGF of  $Y_i$  is  $H(s)$ , then the PGF of  $Y_1 + Y_2 + \dots + Y_i$  is  $[H(s)]^i$ .

▶ Hence,

$$H_n(s) = \sum_k P(Z_{n-1} = i) [H(s)]^i = H_{n-1}(H(s))$$



Substitute the conditional probability is nothing but the probability of  $Y_1$  plus  $Y_2$  plus so on  $Y_i$  is equal to  $k$  into a spot. Since  $Y_i$ 's are iid random variables and the probability generating function of  $Y_i$ 's is are nothing but  $H$  of  $s$  the probability generating function of a sum of  $i$  random variables  $Y_i$ 's that is nothing but  $H$  of  $s$  whole power  $i$  because of  $Y_i$ 's are iid random variables and probability generating function of  $Y_i$  is equal to  $H$  of  $s$  therefore the probability generating function of  $y_1$  plus  $y_2$  and so one plus  $Y_i$  is  $H$  of  $s$  whole power  $i$ . Therefore substitute this is nothing but the probability generating function of  $Y_1$  plus  $Y_2$  plus and so on plus  $Y_i$  therefore  $H_n$  of  $s$  is nothing but the summation over  $K$ .  $H_n$  of  $s$  is nothing but a summation over  $i$  probability of  $Z_n$  minus 1 is equal to  $I$  times  $H$  of  $s$  whole power  $i$ . Replace this by  $H$  of  $s$  whole power  $I$  therefore summation  $I$  probability of  $Z_n$  minus 1 is equal to  $H$  of  $S$  whole  $i$ . So that is nothing but the probability generating function for the random variable  $Z_n$  minus 1 with the replacement  $s$  by  $H$  of  $s$ . Therefore  $H_n$  of  $s$  is nothing but  $H_n$  minus 1 of  $H$  of  $s$ . So the first part is proved.  $H_n$  of  $s$  is same as  $H_n$  minus 1 of  $H$  of  $s$ .

## Theorem 1: PGF of $Z_n$

- ▶  $H_n(s) = H_{n-1}(H(s))$  and  $H_n(s) = H(H_{n-1}(s))$
- ▶ **Proof:** For  $n = 1, 2, \dots$ ,

$$\begin{aligned} P(Z_n = k) &= \sum_i P(Z_n = k \mid Z_{n-1} = i)P(Z_{n-1} = i) \\ &= \sum_i P(Y_1 + Y_2 + \dots + Y_i = k)P(Z_{n-1} = i) \end{aligned}$$

- ▶ Now,

$$\begin{aligned} H_n(s) &= \sum_k P(Z_n = k)s^k \\ &= \sum_k \left[ \sum_i P(Z_n = k \mid Z_{n-1} = i)P(Z_{n-1} = i) \right] s^k \end{aligned}$$



Now we are going to prove the second part. We know that  $H_1$  of  $s$  is  $H$  of  $s$  that is same as  $H$  of  $H$  naught of  $s$  because  $H$  naught of  $s$  is nothing but  $s$ . Suppose this is a true for  $n$  is equal to  $j$  that means  $H$  of  $j$  of  $s$  is  $H$  of  $s$   $H$  of  $j$  minus 1 of  $s$  then we can find out what is  $H$  of  $j$  plus one of  $s$ . That is nothing but  $H_j$  of  $H$  of  $s$  that is same as  $H$  of  $H_j$  minus one times  $H$  of  $s$  that is same as  $H$  of  $H_j$  of  $s$ . By induction it is a proved.

## Theorem 1: PGF of $Z_n \dots$

- ▶ We know that  $H_1(s) = H(s) = H(H_0(s))$  since  $H_0(s) = s$ .
- ▶ Suppose for  $n = j$ ,

$$H_j(s) = H(H_{j-1}(s))$$

then

$$\begin{aligned} H_{j+1}(s) &= H_j(H(s)) \\ &= H(H_{j-1}(H(s))) \\ &= H(H_j(s)) \end{aligned}$$

- ▶ By induction, it is proved.




Now we are finding the moments of  $Z_n$ . Let  $\mu$  equal to expectation of  $Z_1$  and  $\sigma^2$  is nothing but the variance of  $Z_1$  then expectation of  $Z_n$  that is  $\mu^n$  and the variance of  $Z_n$  will be for  $\mu$  equal to 1 it is  $n$  times  $\sigma^2$  for  $\mu$  is not equal to 1 it will be  $\mu^n$  times  $\sigma^2$  divided by  $1 - \mu^n$ . We will prove the part one that is expectation of  $Z_n$  is equal to  $\mu^n$ . By theorem one you know that  $H_n(s)$  is same as  $H_{n-1}(H(s))$ , by theorem one, we know that  $H_n(s)$  is  $H_{n-1}(H(s))$ . Then differentiating and putting  $s$  equal to one you will get  $H_n'(1)$  that is same as  $H_{n-1}'(H(1))H'(1)$ . We know that  $H(1) = 1$   $H'(1) = \mu$  therefore for  $n = 1, 2$  and so on you will get a  $H_n'(1)$  is same as  $H_{n-1}'(1)\mu$  that is same as  $H_{n-2}'(1)\mu^2$  and so on; therefore you will get  $\mu^n$  because you know that  $H'(1) = \mu$ . By recursively you will get a  $H_n'(1) = \mu^n$  therefore expectation of  $Z_n$  is nothing but a  $H_n'(1)$  that is same as  $\mu^n$ .

**Theorem 2: Moments of  $Z_n \dots$**

- ▶ **Proof of Part 1:** By Theorem 1, we have  $H_n(s) = H_{n-1}(H(s))$ .
- ▶ Then differentiating and putting  $s = 1$ ,  $H_n'(1) = H_{n-1}'(H(1))H'(1)$
- ▶ We know that,  $H(1) = 1$ ,  $H'(1) = \mu$  and for  $n = 1, 2, \dots$ 

$$H_n'(1) = H_{n-1}'(1)\mu = \dots = \mu^n$$
- ▶ Hence  $E(Z_n) = \mu^n$ .



So till now we have discussed the probability generating function of  $Z$  power  $n$  and also we have discussed the mean and variance of  $Z$  of  $n$ . As a remark if  $\mu$  is equal to 1 the expectation of  $Z_n$  tends to 1 whereas a variance of  $Z_n$  tends to infinity as  $n$  tends to infinity. You can see it from this theorem as  $\mu$  tends to 1 expectation of  $Z_n$  will tend to 1 whereas the variance it tends to infinity because that is same as the  $n$  times  $\sigma^2$  therefore as  $\mu$  tends to 1 the variance of  $Z_n$  tends to infinity as  $n$  tends to infinity. Whereas if  $\mu$  is less than 1 the expectation of  $Z_n$  tends to 0 and the variance of  $Z_n$  will be  $\sigma^2$  divided by  $1 - \mu^n$  tends to infinity. Similarly if  $\mu$  is greater than one then the expectation of  $Z_n$  will tend to infinity and the variance of  $Z_n$  tends to infinity as  $n$  tends to infinity. This also you can see it from the theorem.

## Remarks

- ▶ If  $\mu = 1$ , then  $E(Z_n) \rightarrow 1$  and  $\text{Var}(Z_n) \rightarrow \infty$  when  $n \rightarrow \infty$ .
- ▶ If  $\mu < 1$ , then  $E(Z_n) \rightarrow 0$  and  $\text{Var}(Z_n) \rightarrow \frac{\sigma^2}{1-\mu}$  when  $n \rightarrow \infty$ .
- ▶ If  $\mu > 1$ , then  $E(Z_n) \rightarrow \infty$  and  $\text{Var}(Z_n) \rightarrow \infty$  when  $n \rightarrow \infty$ .



Now we are going to discuss the criticality. A very important classification is based on mean progeny count  $\mu$  is equal to expectation of  $Z_1$ . You know that expectation of  $Z_n$  is equal to  $\mu^n$  just now we have proved it in the theorem one. Therefore in the expected value since the process grows geometrically if  $\mu$  is greater than 1, stays constant if  $\mu$  is equal to 1 and decays geometrically if  $\mu$  is less than 1 from the expectation of  $Z_n$  is equal to  $\mu^n$  we can conclude if  $\mu$  is greater than 1 the process grows geometrically. If  $\mu$  is equal to 1 then the process stays constant whereas if  $\mu$  is less than 1 the process decays geometrically. Thus three cases are called a super critical, critical and a subcritical respectively. That means if  $\mu$  is greater than 1 then the process is called supercritical. In this case the expectation of  $Z_n$  tends to infinity. If  $\mu$  is equal to 1 the process is called a critical and expectation of  $Z_n$  is equal to 1 as  $n$  tends to infinity. When  $\mu$  is less than 1 the process is called subcritical and expectation of  $Z_n$  will tends to 0 as  $n$  tends to infinity.



## Criticality

- ▶ A very important classification is based on the mean progeny count  $\mu = E(Z_1)$ .



$$E(Z_n) = \mu^n$$

Therefore, in the expected value sense, the process grows geometrically if  $\mu > 1$ , stays constant if  $\mu = 1$ , and decays geometrically if  $\mu < 1$ .

- ▶ These three cases are called supercritical, critical, and subcritical, respectively:

$$\mu > 1, \text{ supercritical } E[Z_n] \uparrow \infty$$

$$\mu = 1, \text{ critical } E[Z_n] = 1$$

$$\mu < 1, \text{ subcritical } E[Z_n] \downarrow 0$$



Now we are going to consider the second example. Consider the Galton Watson processor  $Z_n$  with offspring distribution  $P_k$ . We choose the same problem example 1 if the assumption  $P_0$  is equal to 1 by 5,  $P_1$  is equal to 3 by 5 and  $P_2$  is equal to 1 by 5.

## Example 2.

- ▶ Consider the Galton - Watson process  $\{Z_n, n = 0, 1, 2, \dots\}$  with offspring distribution  $\{p_k\}$ .
- ▶ Assume that  $p_0 = 1/5$ ,  $p_1 = 3/5$  and  $p_2 = 1/5$ .
- ▶ Then  $\mu = E(Z_1) = 1$ .
- ▶ Hence, this process is a critical Galton - Watson process.
- ▶  $E(Z_1^2) = 7/5$ , then  $\sigma^2 = 2/5$ .
- ▶ Hence, for  $n = 1, 2, \dots$

$$E(Z_n) = 1 \text{ and } \text{Var}(Z_n) = \frac{2n}{5}$$



Now we can find out the mean of  $Z_1$  that is nothing but 1 because  $P_{naught}$  is equal to  $\frac{1}{5}$ ,  $P_1$  is equal to  $\frac{3}{5}$ ,  $P_2$  is equal to  $\frac{1}{5}$  we will get mean of  $Z_1$  will be 1. Hence this process is called a critical Galton Watson process because  $\mu$  is equal to 1. You can find out the variance of  $Z_1$  also. Also you find out expectation of  $Z_1$  square that will be  $\frac{7}{5}$  hence a variance equal to expectation of  $Z_1$  square minus expectation of  $Z_1$  whole square that will be  $\frac{2}{5}$ .