INDIAN INSTITUTE OF TECHNOLOGY DELHI

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NPTEL ONLINE CERTIFICATION COURSE

Stochastic Processes

Module 8: Renewal Processes

Lecture-04

Non Markovian Queues(contd.)

With

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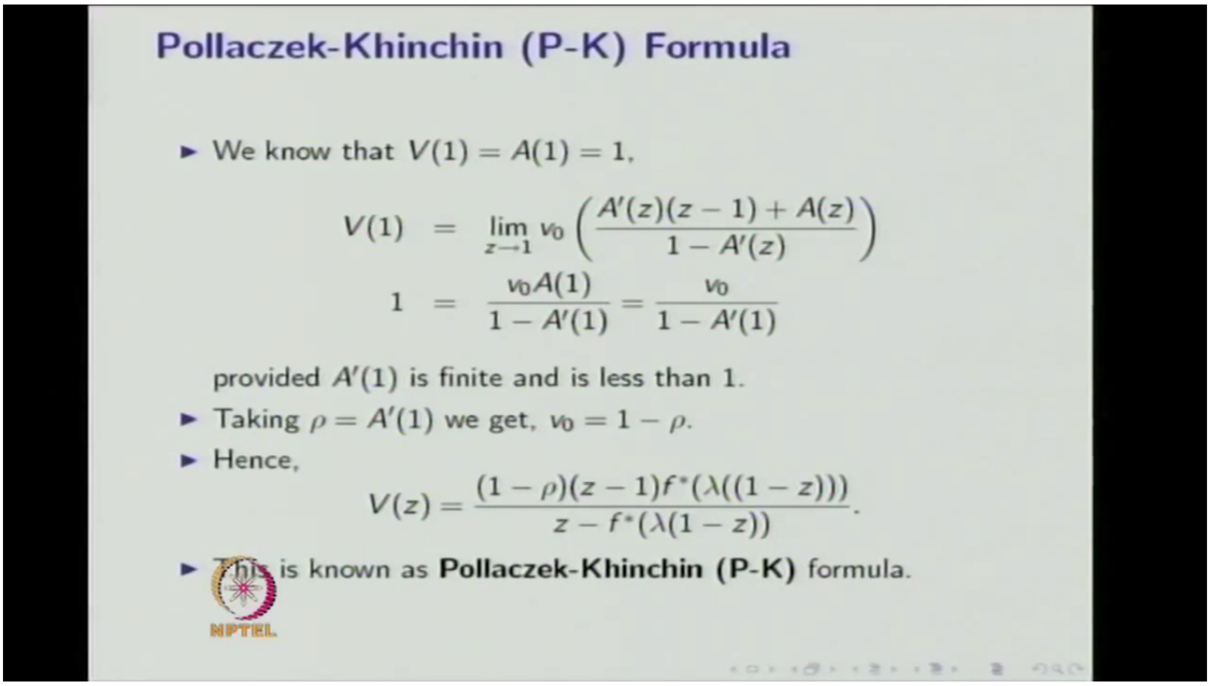
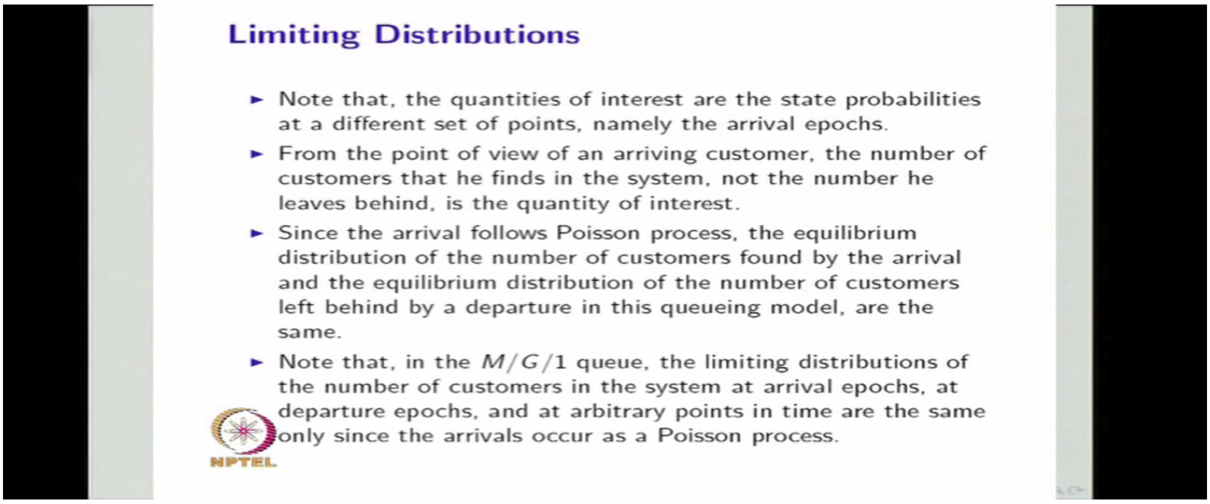
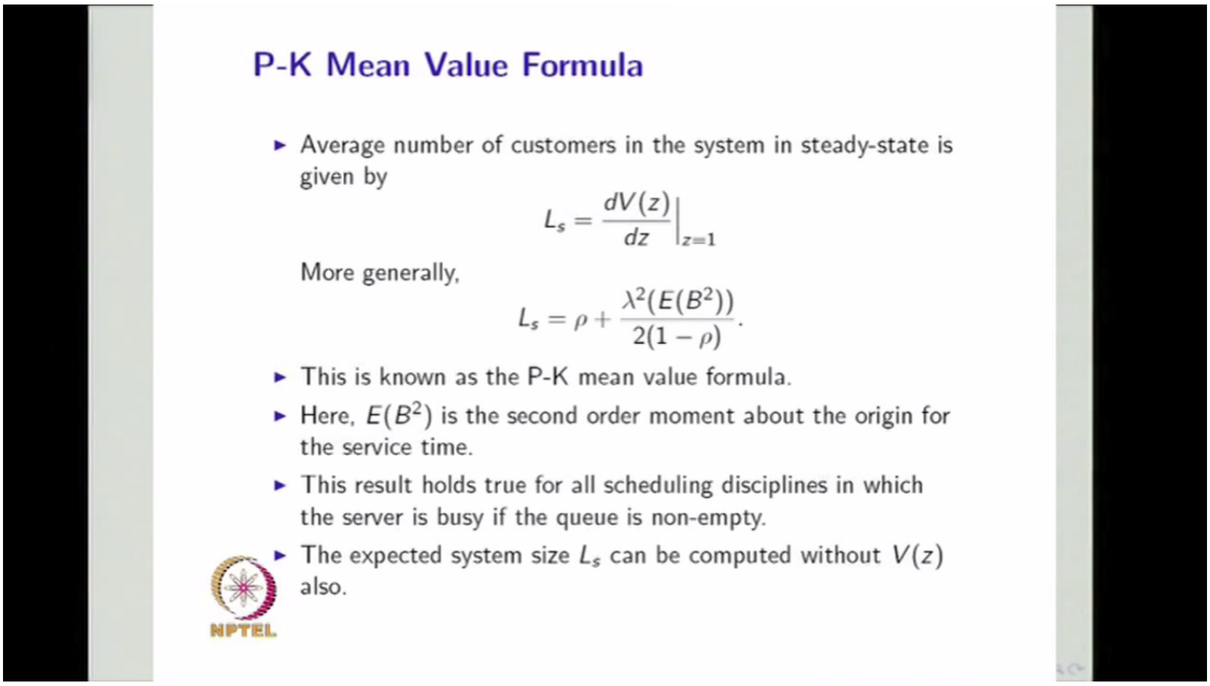


You know that since the V(z) and A(z)s are probability generating functions so V(1) and A(1) are equal to be 1. So using that relation you can find out what is the value of v_0. So v_0 is nothing but 1-A'(1) and A'(1) is nothing but the expected arrivals. Just now we got expected arrival is equal to ρ, therefore, v_o = 1- ρ. So earlier we got V(z) in terms of A(z) with the v_0 and already we have the relation, A(z) is a Laplace transform of a probability density function of service time distribution and just now we got v_0 in terms v_o = 1-ρ. Therefore, the probability generating function of the steady-state probabilities will be in terms of Laplace transform of a probability density function of service time distribution with ρ and this equation is known as Pollaczek-Khinchin or (P-K) formula. So this is the formula used to find out the steady-state probabilities because if you get the coefficient of Z that is the steady-state probabilities.



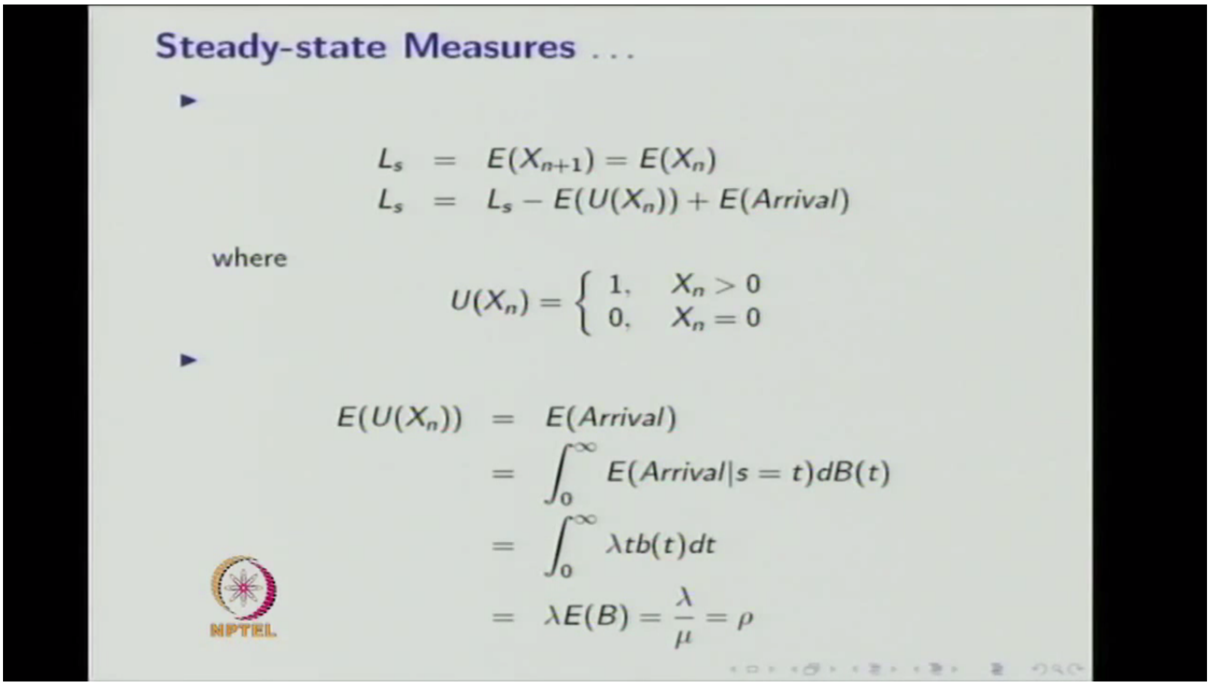
Note that the quantities of interest are the steady-state probabilities at a different set of points, namely the arrival epochs. From the point of view of arriving customer, the number of customers that he finds in the system, not the number he leaves behind, is the quantity of interest. Since the arrival follows Poisson process, the equilibrium distribution of the number of customers are found by the arrival and the equilibrium distribution of the number of customers left behind by a departure in this queuing model are the same. Note that, in the M/G/1 queue, the limiting distributions of the number of customers in the system at the arrival epochs, at the departure reports, and at the arbitrary time points are the same only since the arrival occurs as a Poisson process.

So we found that the limiting probabilities are the departure epochs, but since the arrival follows a Poisson process, the limiting distribution of number of customers in the system at the arrival epochs and the departure epochs and at the arbitrary time points all are same. This is a standard result which we shall be using without proof.

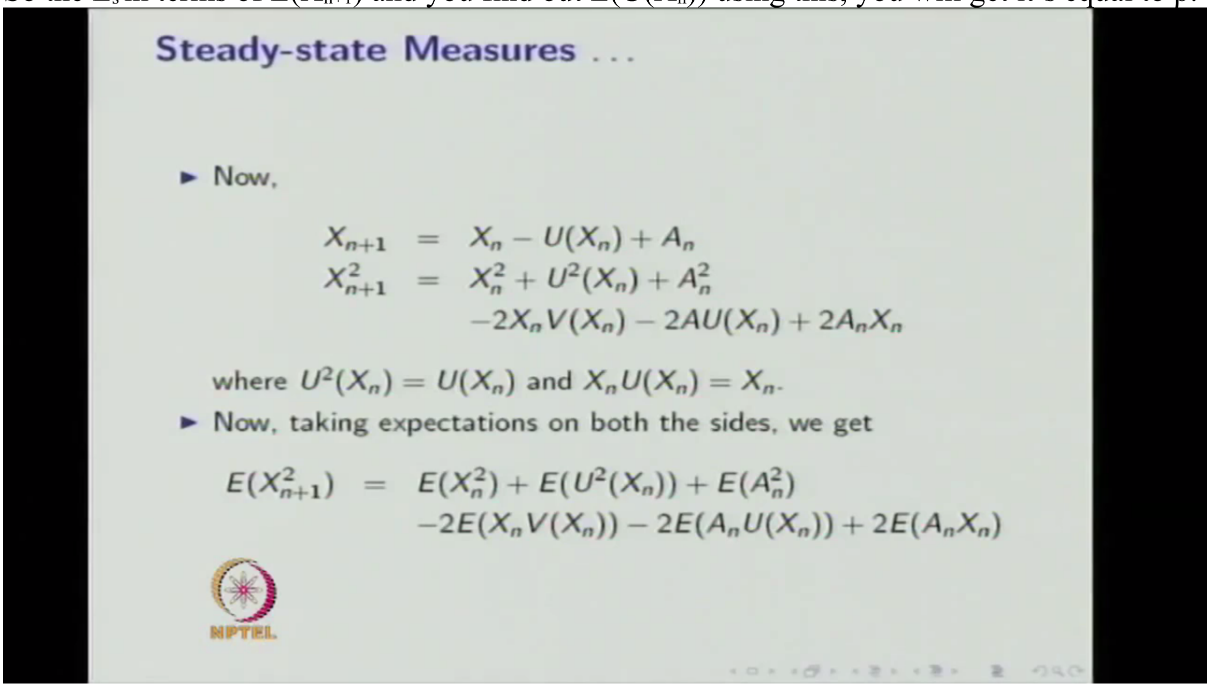


Now we can find out the average measures. Average number of customers in the system in steady state is given by differentiator, the probability generating function, and substitute, z=1, will be the average number of customers. If you do the simplification, you will get rho times, rho plus lambda square, expectation of the service time whole square, expectation of B^2 divided by 2(1- ρ). $Ls=\rho+\frac{\lambda^2(E(B^2))}{2(1-\rho)}$. And this equation is known as a P-K mean formula. The previous one was the P-K formula because that gives the steady-state probabilities, whereas this gives the average measures. Therefore, this is called the P-K mean formula. Here, expectation of B^2 is the second order moment about the origin for service time. This result holds true for all scheduling discipline in which the server is busy if the queue is non-empty.

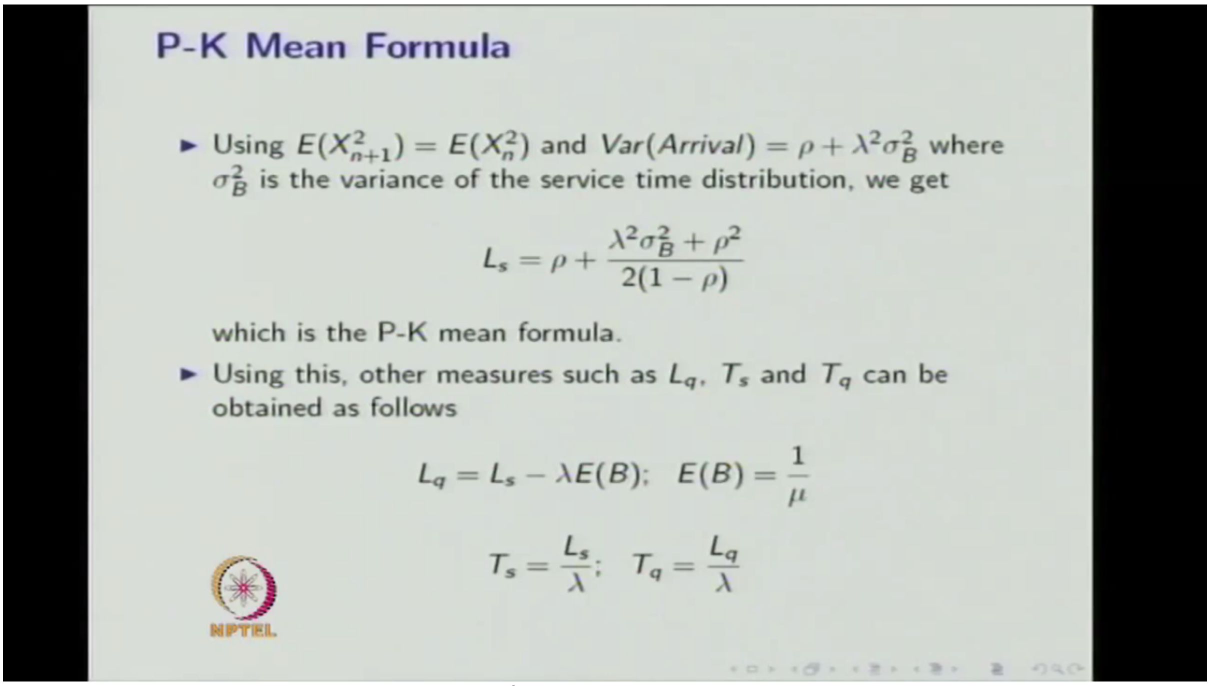
The expected system size, L_s, can be computed without V(z) because if you know ρ as well as if you know the expectation of B^2, you can find out the expected system size. Now we are deriving the L_S in a different way, not via the P-K formula. So the derivation is as follows:



So the L_s in terms of $E(X_{n+1})$ and you find out $E(U(X_n))$ using this, you will get it's equal to ρ.

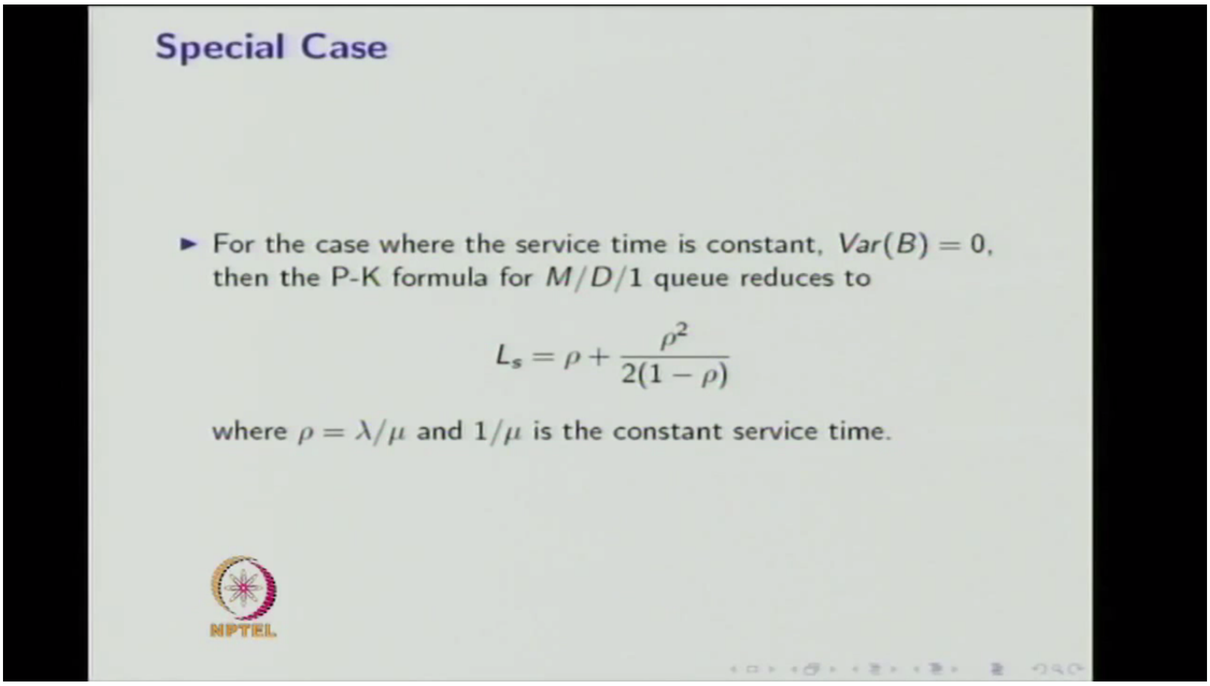


And after you do the simplification, you can get the expectation of X^2_{n+1} also in terms of expectation of X^2_n as well as expectation of A^2_n.

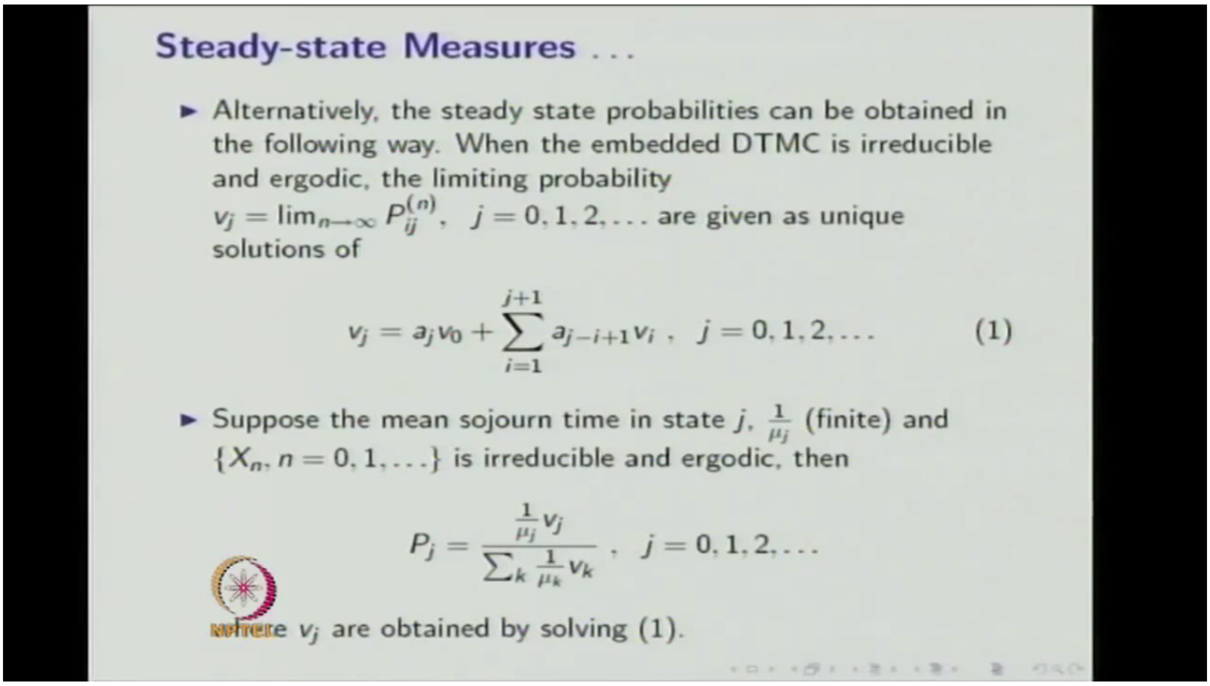


So once you know the expectation of X^2_{n+1} and the variance of arrival where variance of arrival means the variance of number of arrivals during a service time, you can get the expected number of customers in the system in steady-state. So this is called the P-K mean formula without using the P-K formula.

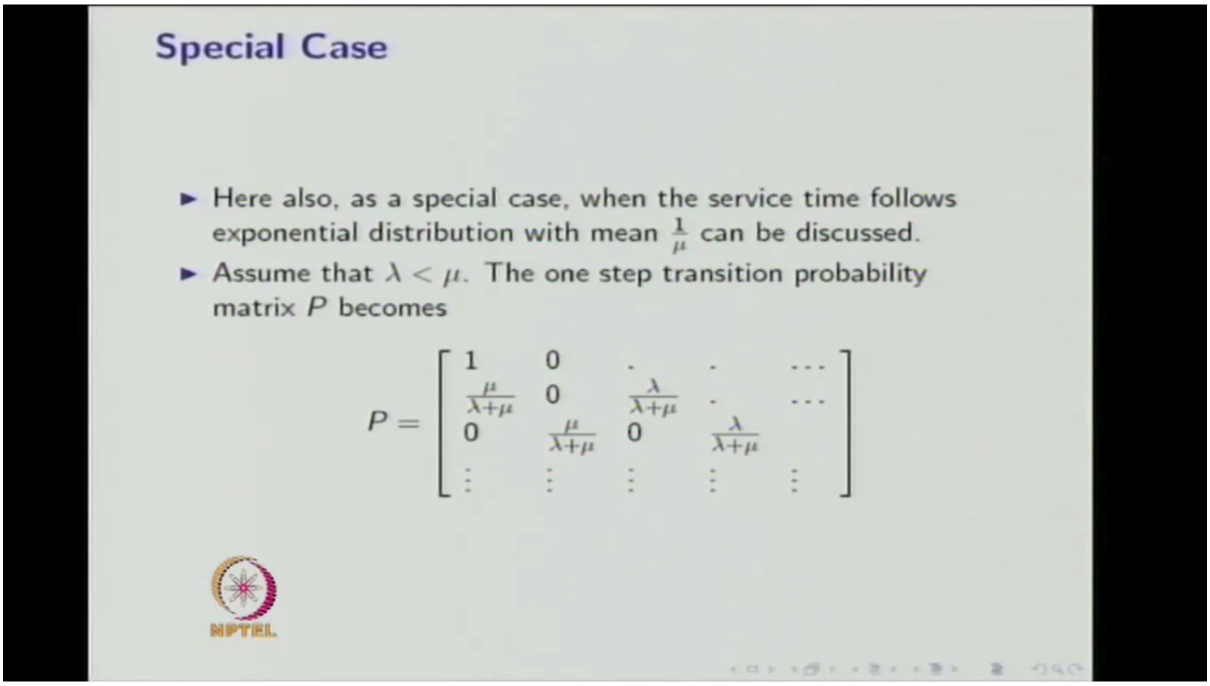
Once we know the L_S that is average number of customers in the system, you can find out the L_q that is average number of customers in the queue and T_s is nothing but the average time spent in the system and the T_q is the average time spent in the q. So using Little's formula you can find all other measures.



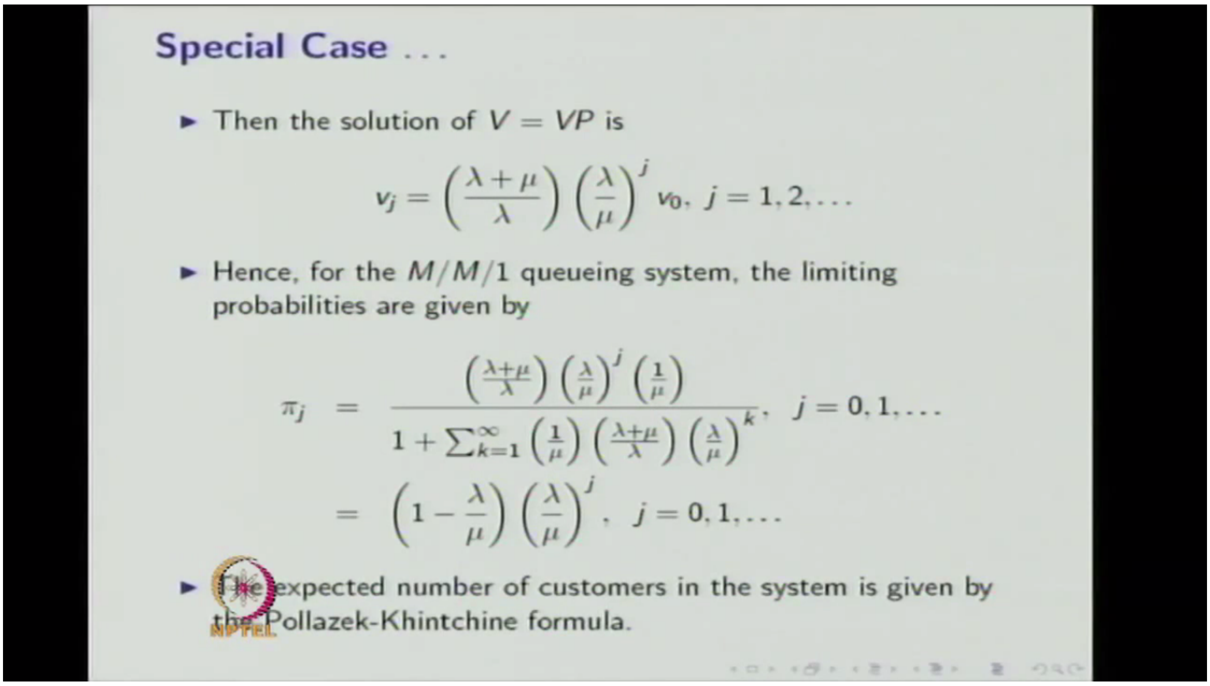
As a special case, if the variance is 0, variance of service time is 0 that means it is a M/D/1 queue, then you can get the average number of customers in steady state will be $Ls = \rho + \frac{\rho^2}{2(1-\rho)}$. Here the ρ is nothing but λ/μ where 1/ μ is a constant service time.



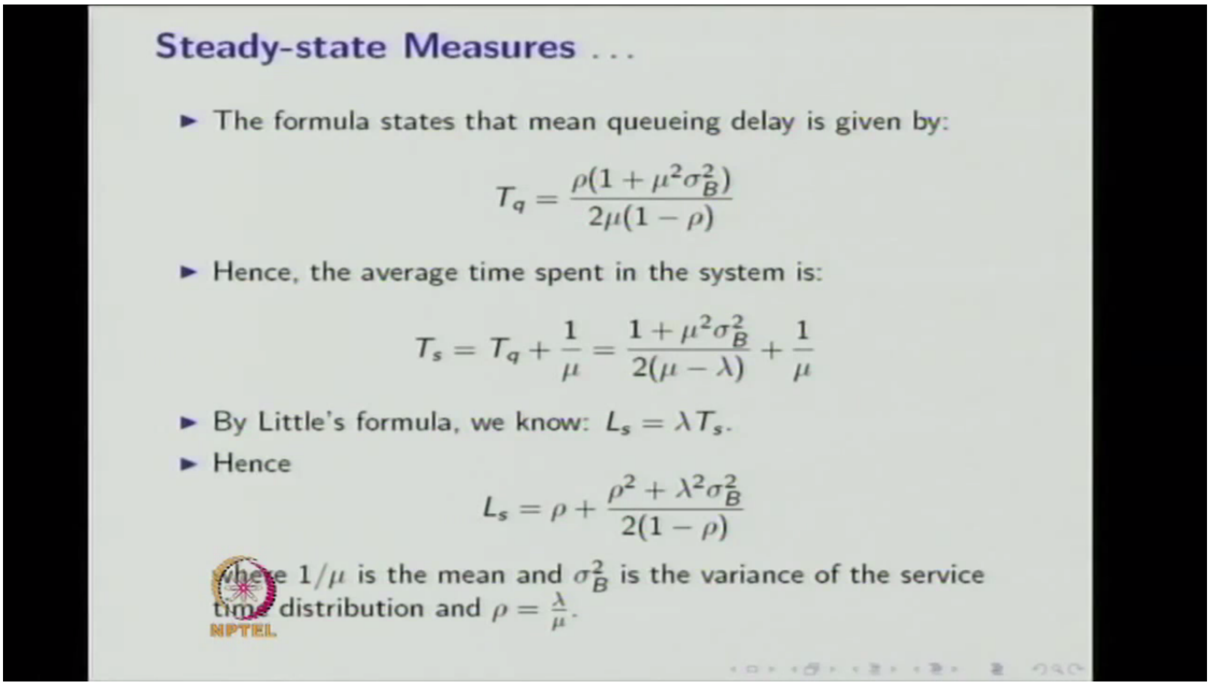
Alternatively, you can find out the steady-state probabilities by solving a V=v_p you can use the mean sojourn time that is $1/\mu_j$ for the state j, then you can find out the steady-state probabilities, p_j's nothing average sojourn time multiplied by the steady-state probabilities of embedded Markov chain, the way we have done it in the semi-Markov process.



As a special case, you can verify when this service time follows exponential distribution with the mean 1/ μ, you can get the one-step transition probability matrix in this form for the embedded Markov chain,

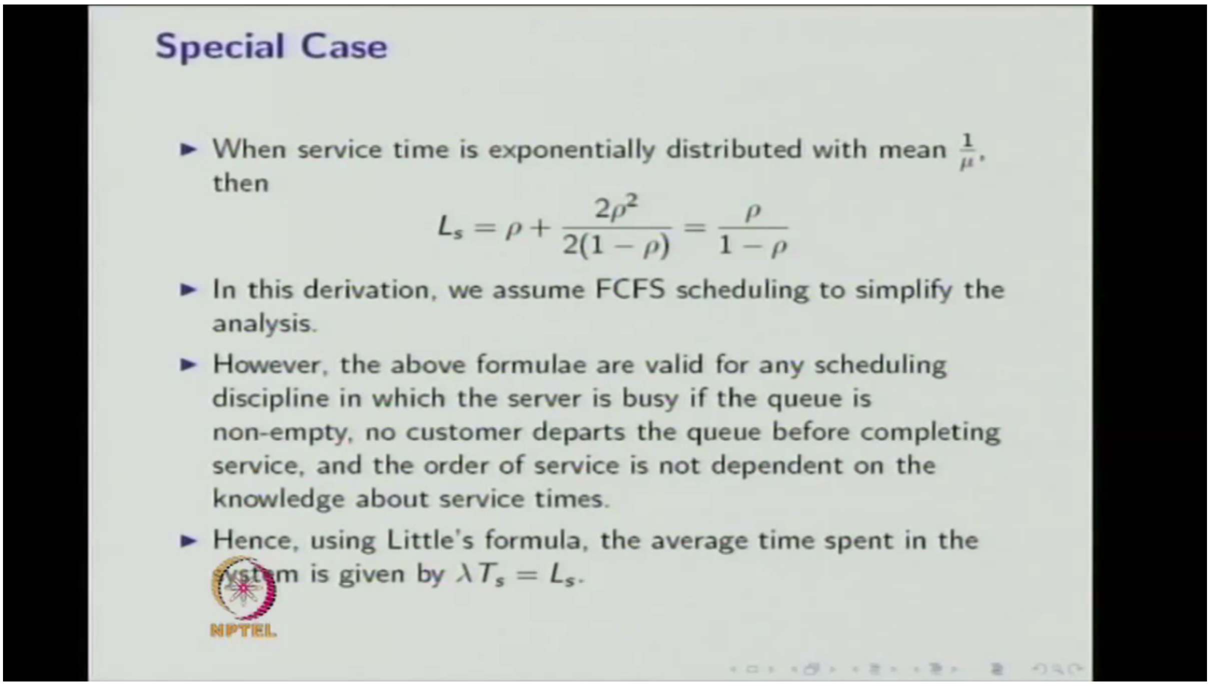


Then you can solve V=VP with the summation of v_i=1, you will get v_j's. Then if you substitute the v_j's then you can get the steady-state probabilities for the M/M?1 queue that is same as a 1- $\rho(\rho)^j$.



For any M/G/1 queue, you can find out the average time spent in the queue that is a mean queuing delay and once you know the mean queuing delay, you can get the average time spent in the system by adding average service time that is a 1/ μ. So the T_s will be T_q+1/ μ that will be the average time spent in the system.

By Little's formula, you can get L_s. L_s= $\lambda(T_s)$. You know the T_s, so from that you can get the λ_s which is same as what you got it in the P-K mean formula.

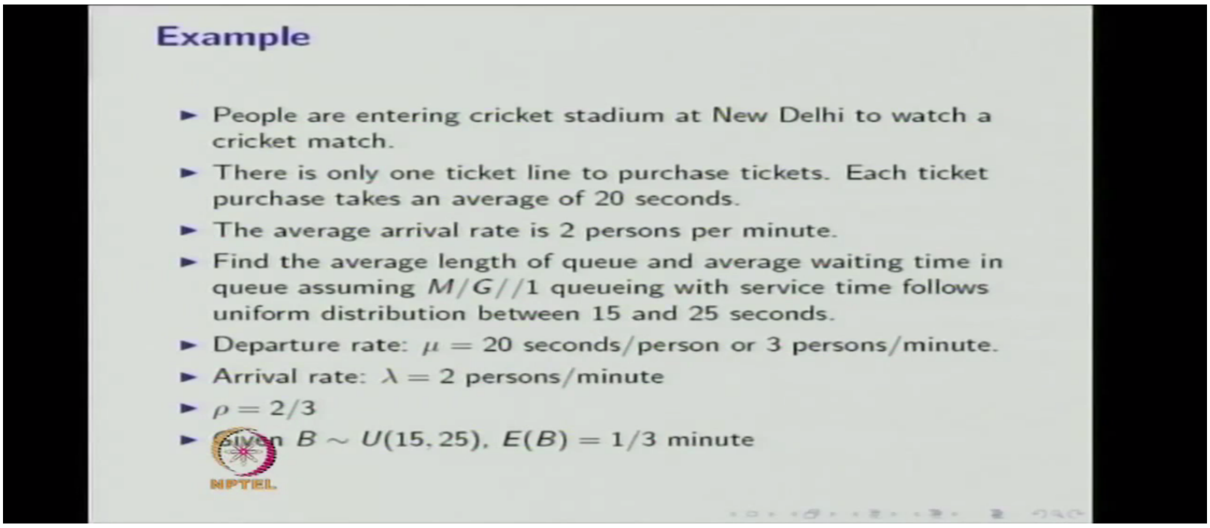


As a special case, when the service time is exponential distribution, with the mean 1/ μ, you can get the average time spent in the average number of customers in the system will be ρ/1-ρ. This is same as the average number of customers in the M/M/1 queue.

In this derivation, we assume first-come first-served scheduling to simplify the analysis. But the above formula are valid for any scheduling discipline in which the server is busy if the queue is non-empty and no customer departs queue before completing the service and the order of service is not dependent on the knowledge about the service time.

If these conditions are satisfied then for any scheduling discipline, you can use the above formula of average number of customers in the system.

So using Little's formula you can find out the average time spent in the system, $\lambda T_s = L_s$.



As a simple example, consider the people entering cricket stadium at New Delhi to watch the match. There is only one ticket line to purchase tickets. Each ticket purchase takes an average of 20 seconds. The average arrival rate is two persons per minute. So the question is find the average length of queue as well as average waiting time in queue assuming the queuing model is a M/G/1 with the service follows uniform distribution between 15 to 25 seconds.

With this given information, you can get their departure rate, arrival rate. Because it is 20 seconds per person therefore the rate will be three persons per minute, and the arrival rate is two persons per minute. Therefore, you can get λ that is 2/3. So it is irreducible positive recurrent Markov chain. Therefore, the steady-state probabilities exist and given the service time distribution is uniform distribution between the interval 15 to 25, you can get the measures of steady-state probabilities as well as all the average measures.

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