INDIAN INSTITUTE OF TECHNOLOGY DELHI

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NPTEL ONLINE CERTIFICATION COURSE

Stochastic Processes

Module 8: Renewal Processes

Lecture-04

Non Markovian Queues

With

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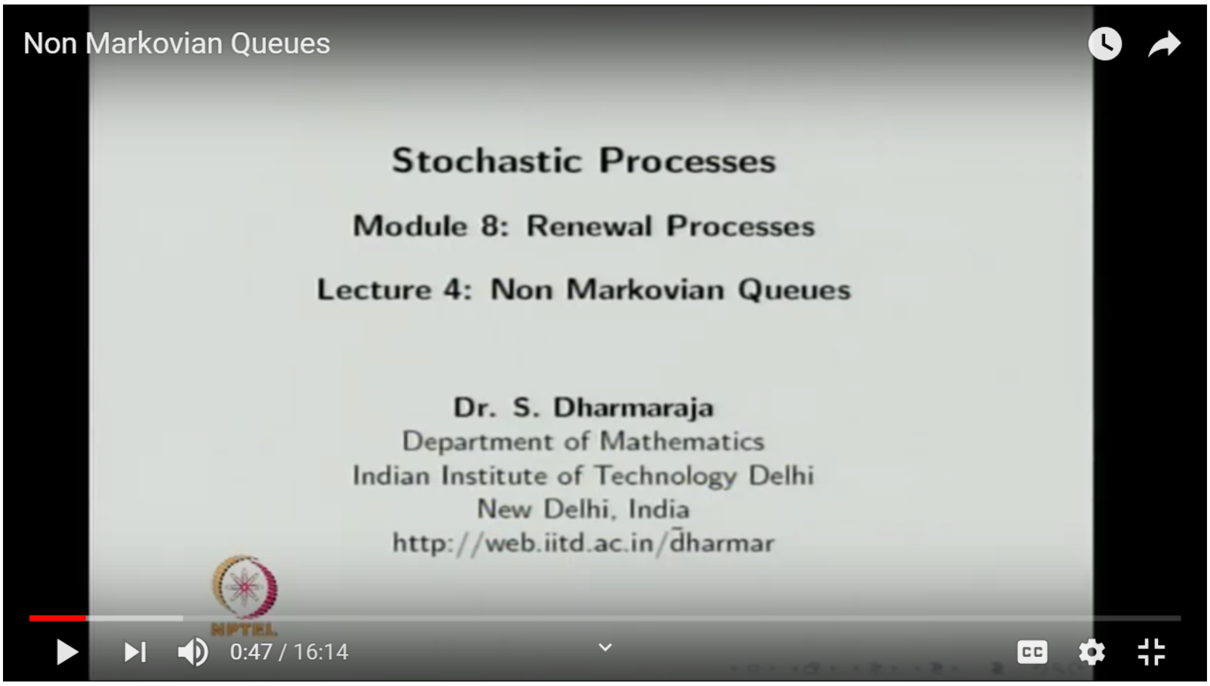
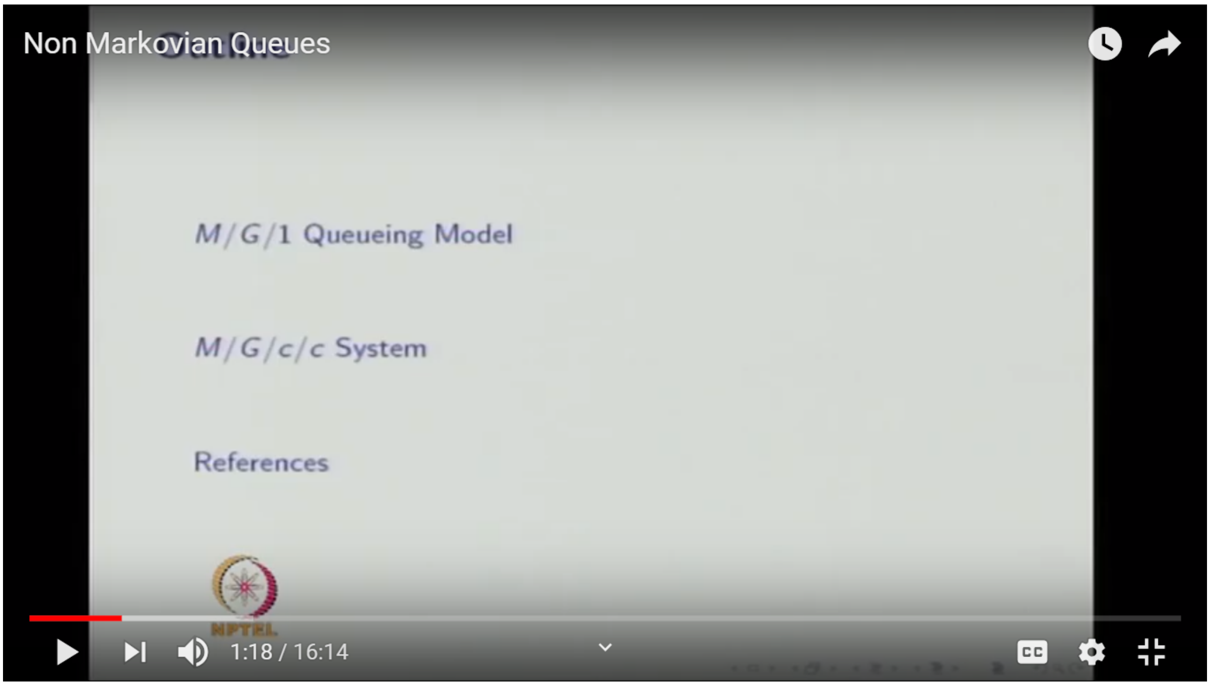
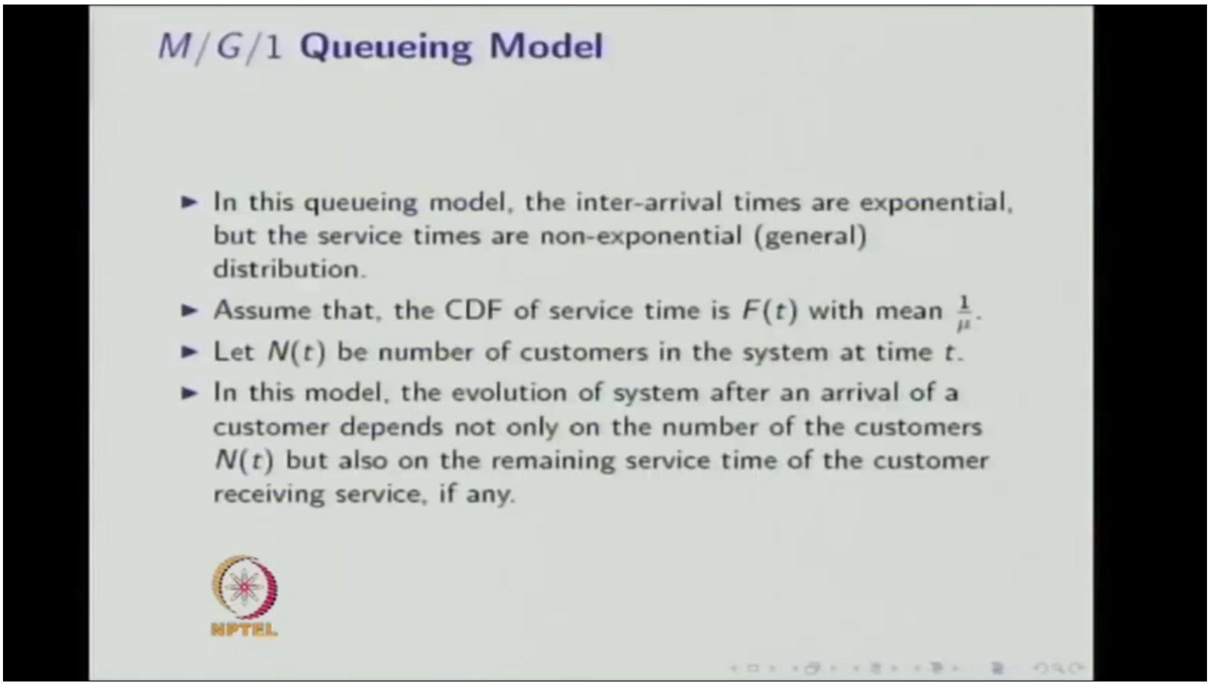


This is Stochastic Processes, Module 8, Renewal Processes. In the first three lectures, we have discussed the renewal process and its properties. Then later, we have discussed the important limiting theorems on renewal processes, then we have discussed the Markov renewal process and Markov regenerative process. In between we have discussed their reward renewal processes also.



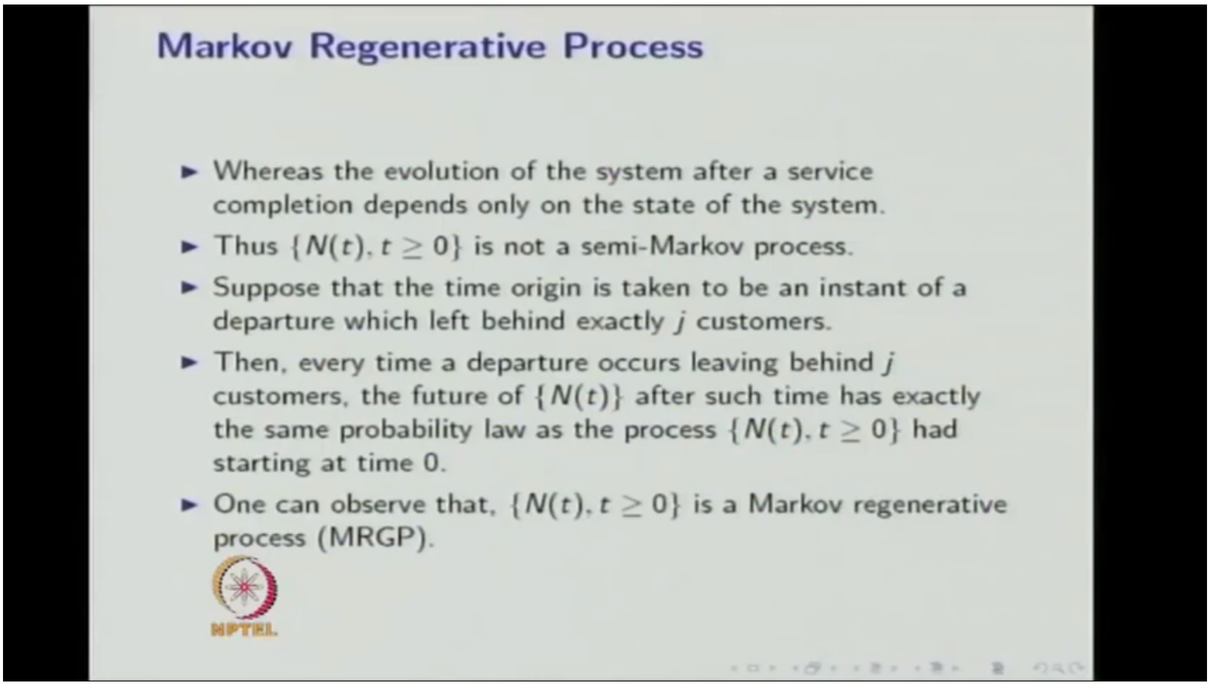
In this lecture, we are going to cover the non-Markovian queues with respect to the service time distribution. In particular we are going to discuss the M/G/1 queue M/G/c/c system.



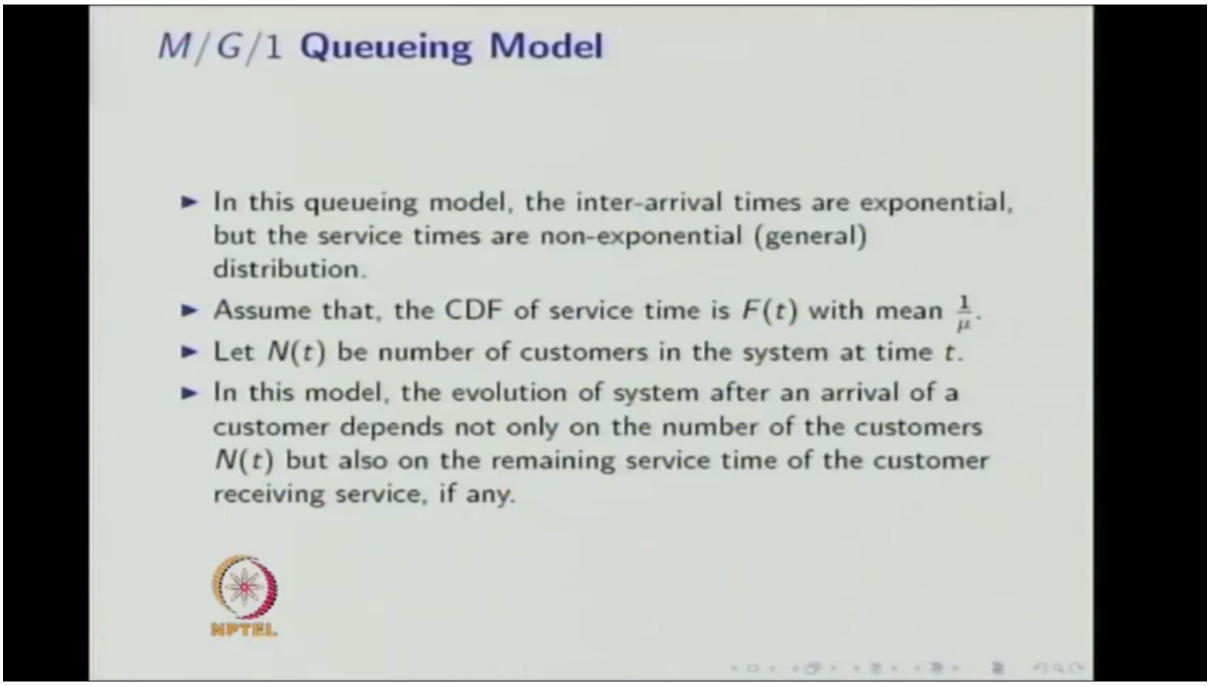
What is M/G/1 queue? In this queuing model, the inter-arrival times are exponential, but the service times are non-exponential, nothing but general distribution. Assume that the CDF of the service time is F(t) with the mean 1/µ. So suppose you assume that service time is exponential distribution with the mean 1/µ then it is a M/1 queuing model.

Let N(t) be the number of customers in the system at time t. In this model, the evaluation of the system after the arrival of a customer depends not only on the number of customers N(t) but also the remaining service time of the customers receiving service. The remaining service is nothing, but since the service times are random variable, the remaining service time is the elapsed service time or remaining or residue service time. Whenever the service times are exponential distribution, the remaining time or elapsed service time or residual service time all our exponential distribution.

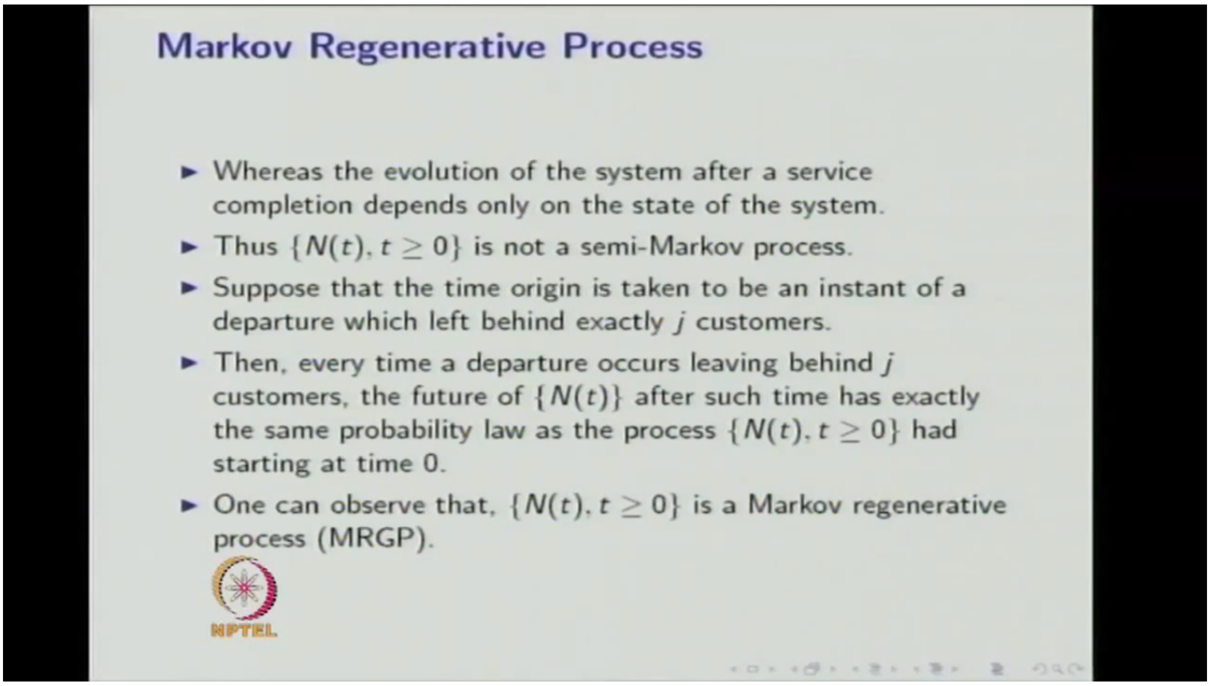
Whenever the service times are not exponential distribution, then the remaining service time will be some other distribution.



The evaluation of the system after the service completion depends only on the system of the state. In this queuing model, the system changes the state based on two types of transition; one is arrival epochs and the other one is the service epochs.

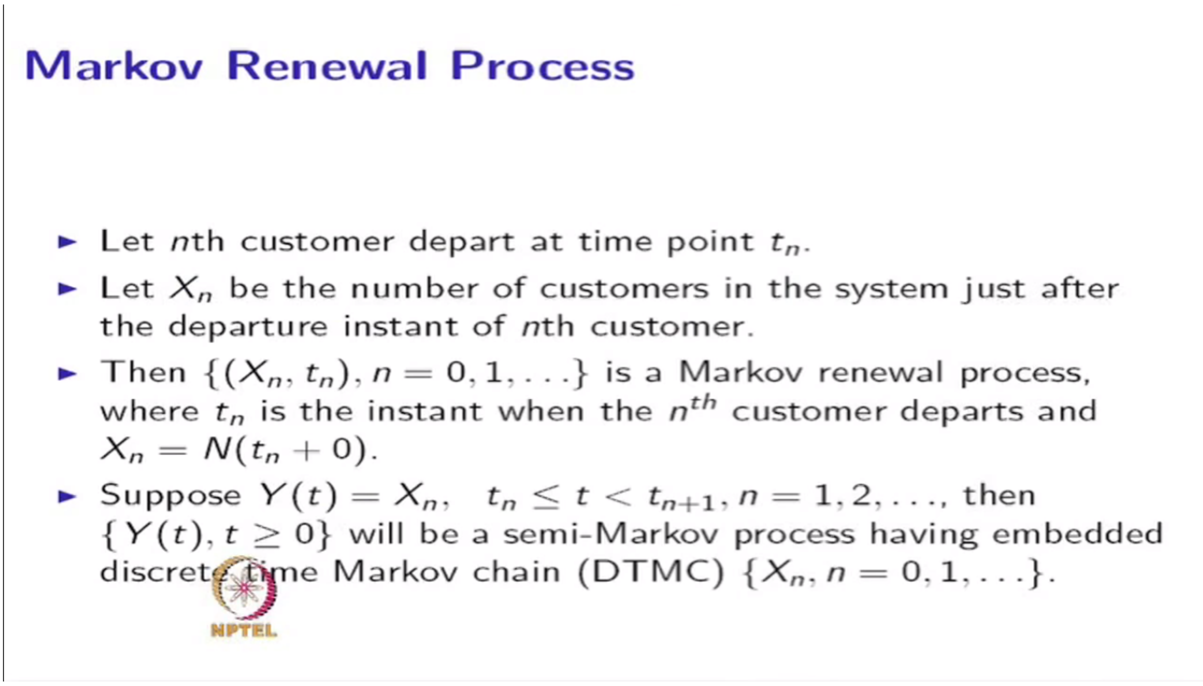


First we are discussing the arrival scenario. The evaluation of the system after the arrival of a customer depends not only the number of customers N(t), but also the remaining service time, whereas the evaluation of the system after service completion depends only on the state of the system. Therefore, N(t) is not a semi-Markov process because it depends on the remaining service time. Since after the arrival of a customer, the remaining service time also play a role whereas after the service completion, it depends only on the state. If you recall the definition of semi-Markov process, the memoryless property will be satisfied at all time transition instants. The memoryless property will be satisfied at all time transition instants then only the stochastic process is called a semi-Markov process, therefore, here the N(t) is not a semi-Markov process.



Suppose the time origin is taken to be instant of departure which left behind exactly j customers then every time the departure occurs leaving behind j customers, the future of N(t) after such time has exactly the same probability law as the process had started starting at time 0. So this is so-called the probabilistic replica with the Markov property as well as time homogeneity.

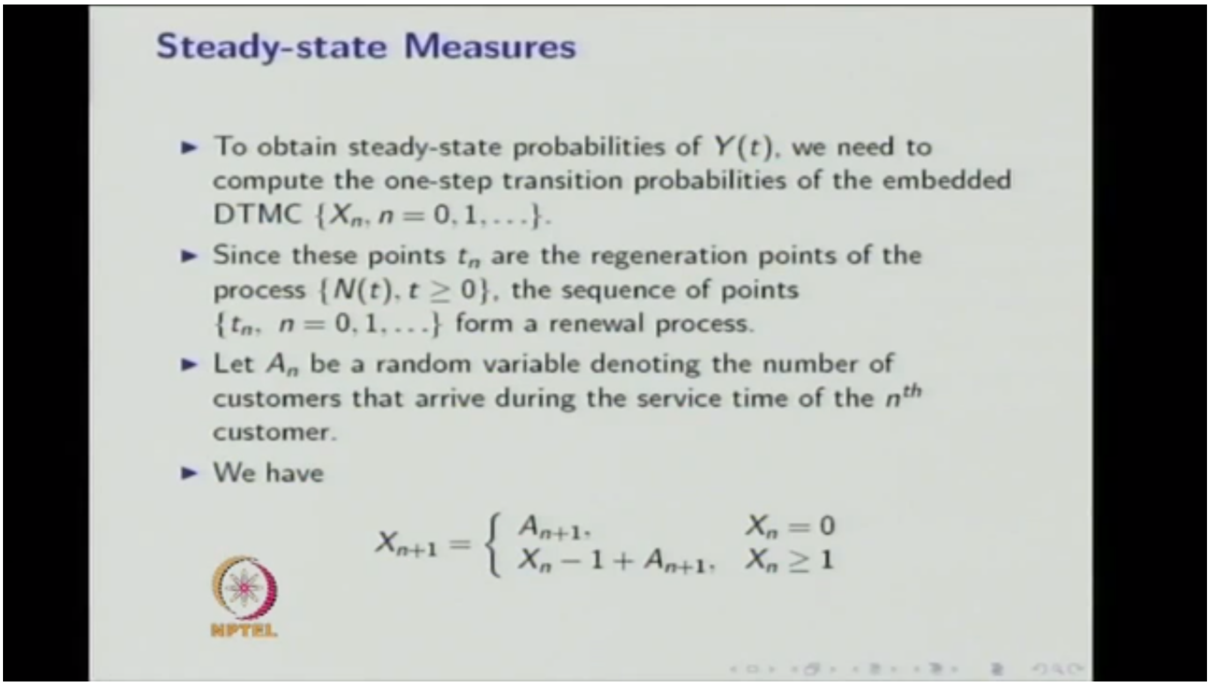
If you recall the definition of a Markov regenerative process, a stochastic process has the property of a probabilistic replica with the few time instants it has the Markov property with the time homogeneity at few states. Therefore, N(t) is a Markov regenerative process. Please refer the previous lecture for understanding the definition of a Markov regenerative process. So by using the definition of Markov regenerative process, the N(t) will be the Markov regenerative process if you consider the time epochs are departure time epochs.



Let nth customer departure at the time instant t_n. Let X_n be the number of customers in the system just after the departure instant of nth customer. That means X_n is nothing but the $N(t_n+0)$ just after the departure instant of nth customer. That will be treated as the random variable X_n. Then (X_n, t_n) will form a Markov renewal process. The t_n are the time instant at the departure epochs not the arrival epochs. Only the departure epochs that collection of time points along with X_n from a Markov renewal process where X_n is $N(t_n+0)$. That in words, it is a number of customers in the system just after the departure instant of nth customer leaves.

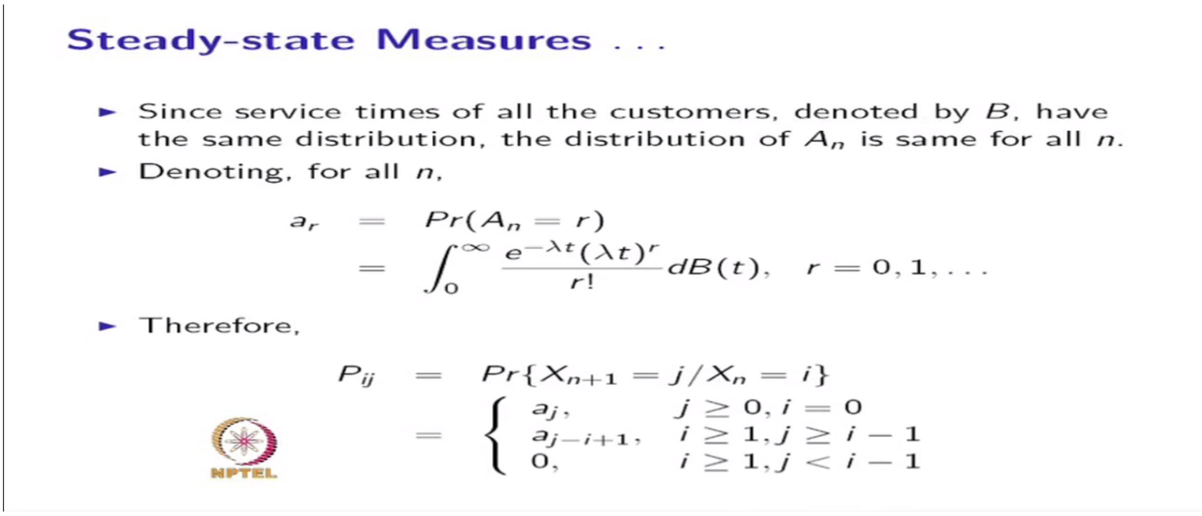
We can create a semi-Markov process by way of $Y(t) = X_n$ if that value will be between t_n to t_n+1, if it is the same then it will form a semi-Markov process. That means the system is not moving any other states in between the time duration, then the Y(t) will be a semi-Markov process. As such, N(t) is not a semi-Markov process whereas in (X_n, t_n) will form a Markov renewal process and the N(t) will be a Markov regenerative process.

With the embedded, the Y(t) will be a semi-Markov process if I make $Y(t) = X_n$ where ts lies between t_n to t_n+1 and having the embedded DTMC, X_n.



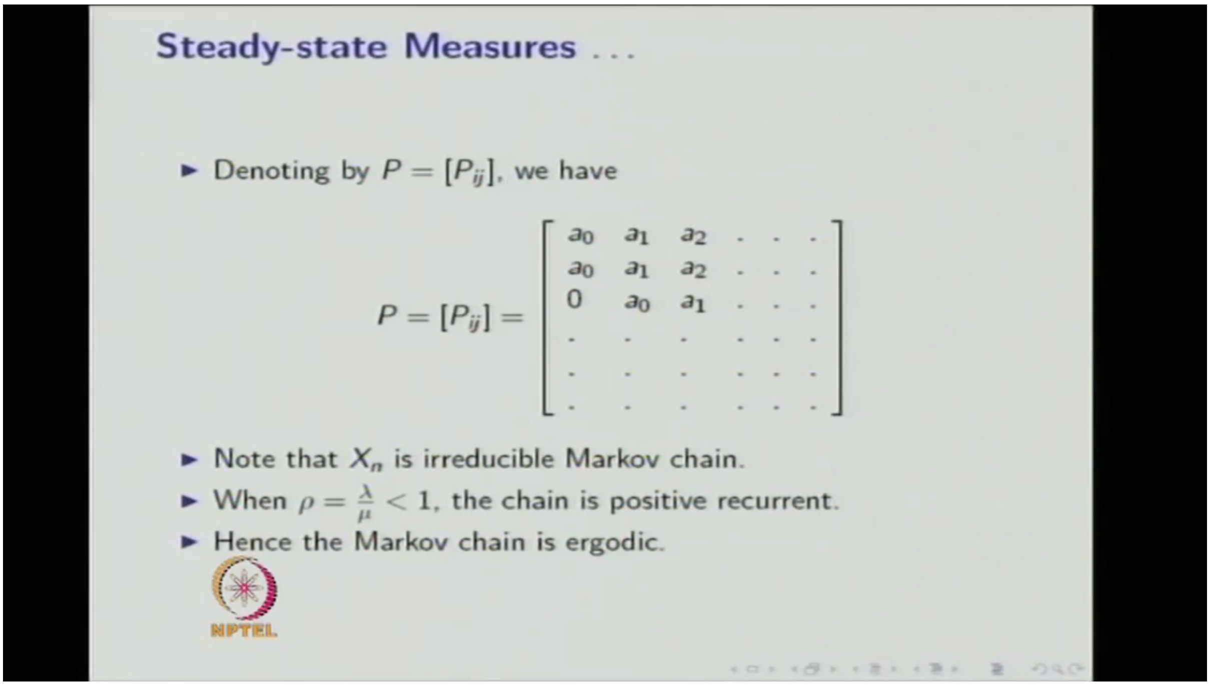
Our interest is to find out the steady state measures. To obtain the steady state probabilities of Y(t), we need the one-step transition probabilities of embedded DTMC. Suppose these time point, t_n, are the regeneration points then the t_n will form a renewal process. Let A_n be the random variable denoting the number of customers that arrive during the service time of nth customer. That means X_{n+1} will be A_{n+1} if X_n was 0. If X_n was greater or equal to 1, X_{n+1} will be X_n-1+A_{n+1}. That is nothing but how many customers in the system when the (n+1)th customer leaves that is same as when the nth person leaves, how many customers in the system. -1 is for the (n+1)th customer himself and how many customers enter into the system during his service. That is A_{n+1} whenever the X_n was greater or equal to 1.

If X_n was 0, then the number of customers will be in the system when (n+1)th customers leaves will be same as number of customers who enter into the system during his service time.

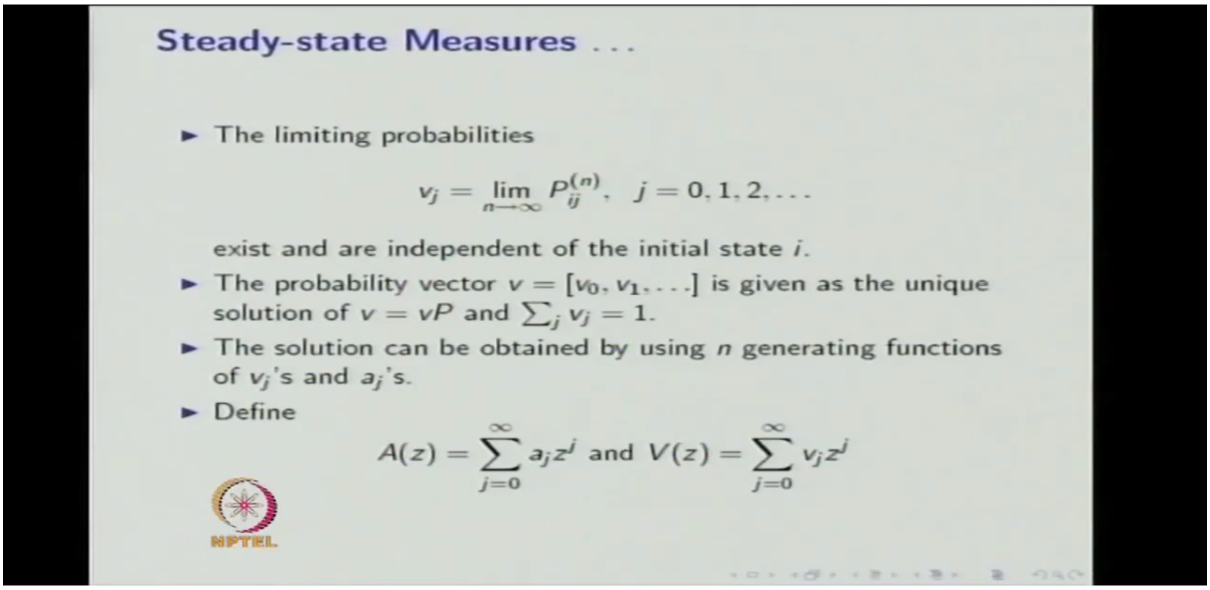


Since the service times of all the customers have the same distribution, the distribution, the distribution of A_n is same for all n. Therefore, you can go for, for all n, we can go for a_r is the r customers enter into the system during any customer. So that is probability mass function of A_n.

Now we can find out the transition probability of system moving from the state i to j with respect to the stochastic process, X_n. X_n is embedded Markov chain in the N(t). So that will be a_j if j≥0 and i=0. Similarly, a_{j-i+1} or 0 according to the values of i and j.

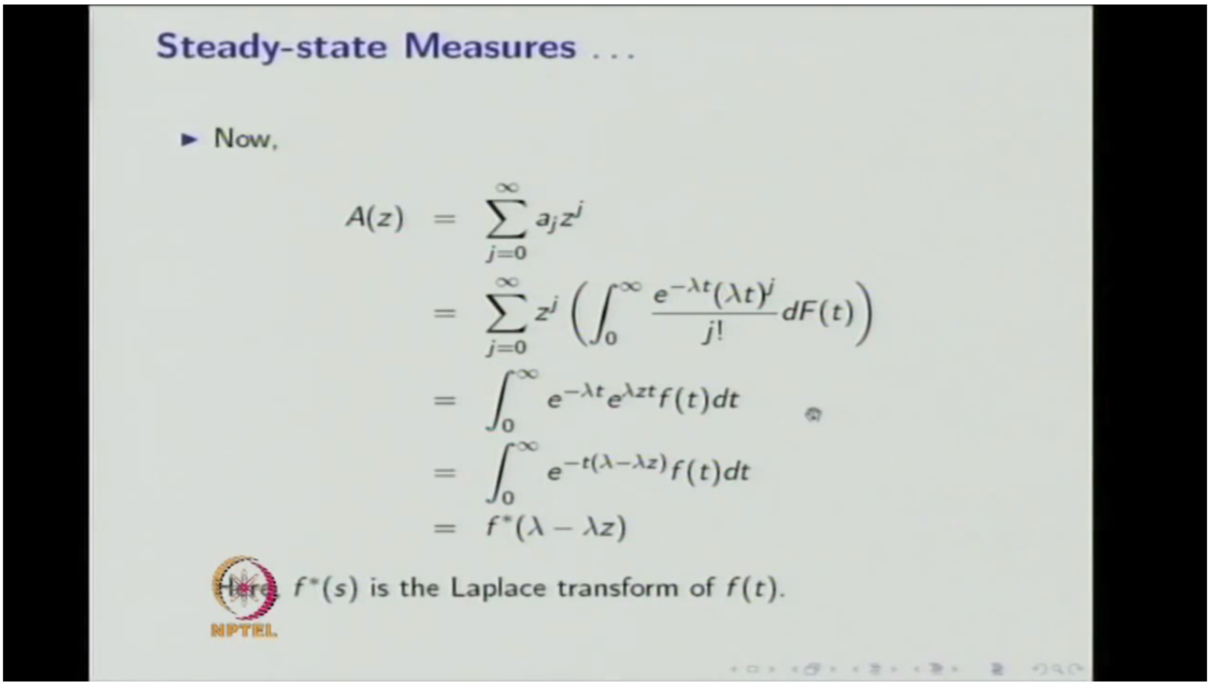


Now we can make a one-step transition probability matrix that is nothing but the P, the rho(ρ) sums are going to be 1. Note that X_n is a irreducible Markov chain because each state is reachable by every other states. If you make a ρ that is nothing but $\lambda/\mu<1$, λ is the parameter for the inter-arrival time which is exponential distribution. So whenever ρ<1, the chain is a positive recurrent. ρ as less than one is recurred because only then the mean recurrence time will be finite. Hence, the Markov chain will be a positive recurrent. Therefore, it is ergodic. Irreducible positive recurrent gives that Markov chain is ergodic, therefore the embedded Markov chain is ergodic.

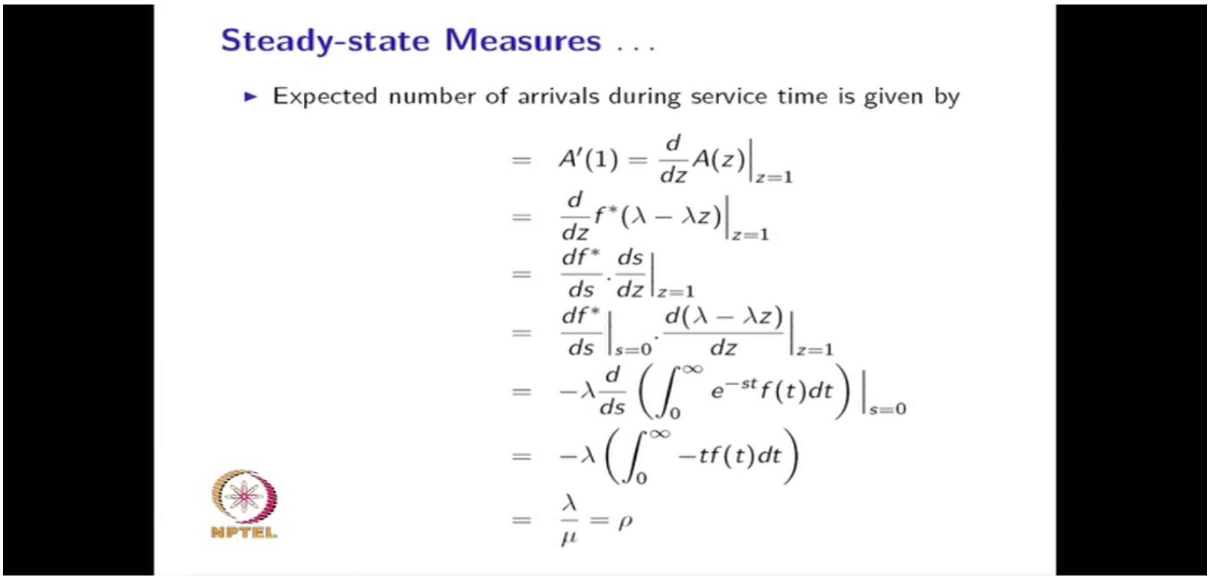


So we can find out the limiting probabilities that will exist as well as it is independent of initial state i. You can find out the limiting probabilities by solving v=vP and summation of v_j=1. The solution can be obtained by using the probability generating function of v's and a's.

v's are steady state probability vector and a's are the number of customers arrived during the service of any customer. So a's is nothing but the probability generating function for a_j's and V(z) is nothing but probability generating function for the probabilities v_j's.



Now you can find out A(z) in terms of the Laplace transform of the function f(t) because A(z) is nothing but the summation of $a_j z^j$ and you can replace a_j by the integration. Now integration and summation can be interchanged and the f(t) is nothing but the distribution of service time. f (t) is nothing but the distribution of service time. Therefore, the f(t) is the probability density function of the service time. So the f* is nothing but the Laplace transform of f(t). So the probability generating function of a_j's is nothing but the Laplace transform of this probability density function of service time distribution.



Now we can find the expected number of arrivals during the service time that is nothing but since we got the probability generating function, differentiate with respect to z, and substitute z=1 will be the average number of arrivals during any customers service time. So you can simplify that by using the chain rule.

Steadу-state Measures . . .

- Expected arrival is given by

$$
\begin{aligned}
E(Arrival) &= A'(1) = \frac{d}{dz}A(z)\Big|_{z=1} \\
&= \frac{d}{dz}f^*(\lambda - \lambda z)\Big|_{z=1} \\
&= \frac{df^*}{ds}.\frac{ds}{dz}\Big|_{z=1} \\
&= \frac{df^*}{ds}\Big|_{s=0}.\frac{d(\lambda - \lambda z)}{z}\Big|_{z=1} \\
&= -\lambda\frac{d}{ds}\left(\int_0^\infty e^{-st}f(t)dt\right)\Big|_{s=0} \\
&= -\lambda\left(\int_0^\infty -tf(t)dt\right) \\
&= \frac{\lambda}{\mu} = \rho
\end{aligned}
$$

So that will be nothing but λ/μ, λ/μ is nothing but the ρ. So the expected number of arrivals is ρ, ρ is nothing but λ/μ.

Steady-state Measures . . .

- Now,

$$v_j = v_0 a_j + \sum_{i=0}^{j+1} v_j a_{j-i+1}, \quad j = 0, 1, 2, \ldots$$

- Multiplying by z_j on both the sides and taking the sum, we get

$$
\begin{aligned}
\sum_{j=0}^{\infty} V_j z^j &= \sum_{j=0}^{\infty} v_0 a_j z^j + \sum_{j=0}^{\infty}\left(\sum_{i=1}^{j+1} v_j a_{j-i+1}\right) z^j \\
V(z) &= v_0 \sum_{j=0}^{\infty} a_j z^j + \sum_{i=1}^{\infty}\left(\sum_{j=i-1}^{\infty} v_j a_{j-i+1} z^j\right) \\
&= v_0 A(z) + \frac{1}{z}[V(z) - v_0]A(z) \\
V(z) - \frac{1}{z}A(z)V(z) &= v_0 A(z) - \frac{1}{z}v_0 A(z) \\
V(z) &= \frac{v_0 A(z)(z-1)}{z - A(z)}
\end{aligned}
$$

Our interest is to find out the steady-state probabilities, v_j's. So the v_j's you can relate in terms of by expanding $v = vP$, you will get $v_j = v_0 a_j$+summation form. So multiply z^j in both sides, taking the summation, you will get after simplification, you will get v_z in terms of a_z and v_0.

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