Video Course on
Stochastic Processes

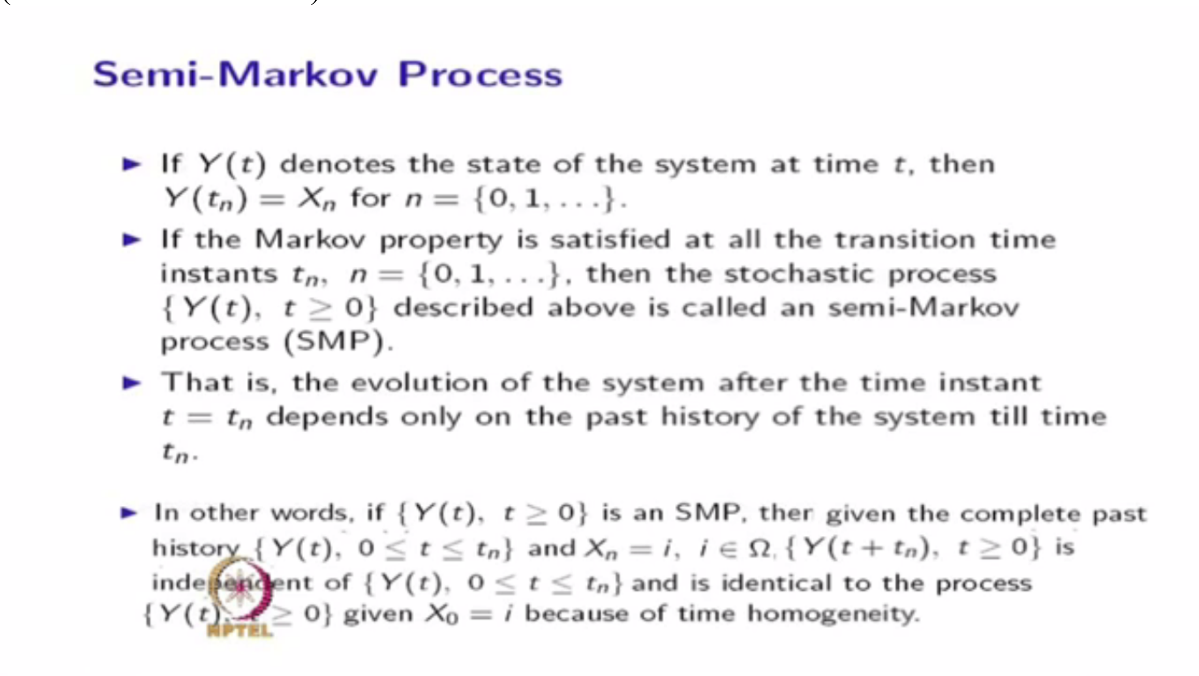
by

Dr. S Dharmaraja
Department of Mathematics, IIT Delhi

Module # 8
Renewal Processes

Markov Renewal and Markov Regenerative Processes (contd)

(Refer Slide Time 00:00)



In other words, if Y(t) is a Semi-Markov Process, then the process $Y(t + t_n)$ is independent of...

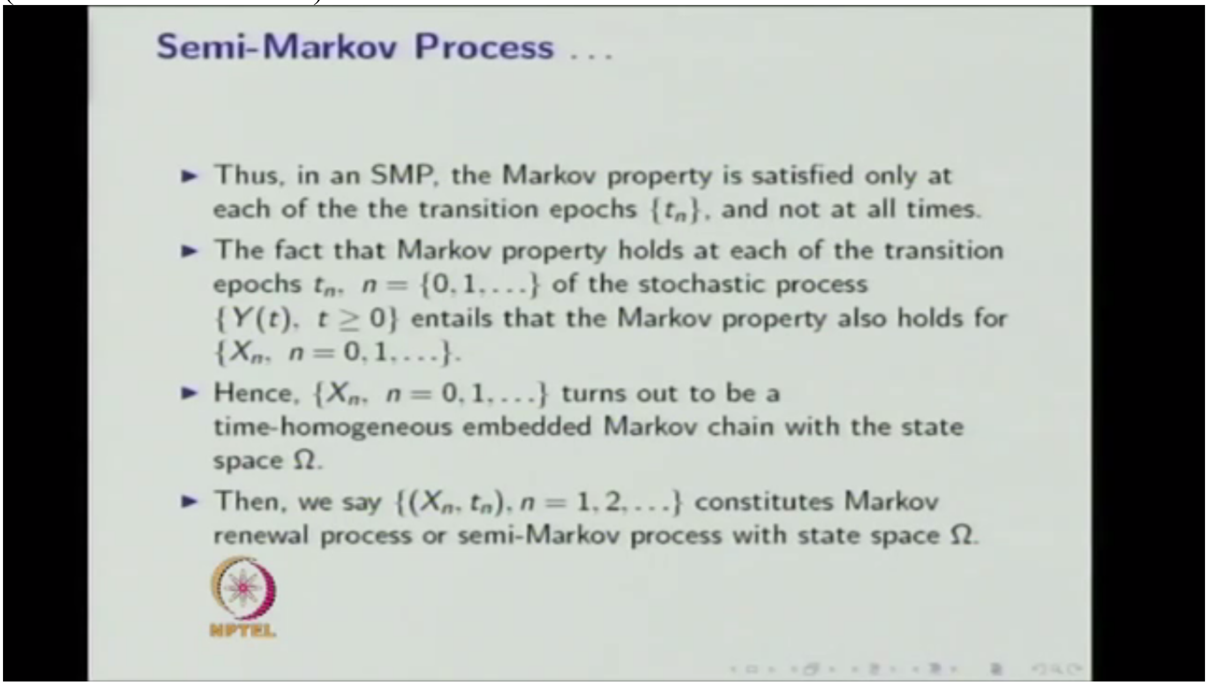
A Semi-Markov Process is thus a stochastic process in which changes of state occur according to a Markov chain and in which the time interval between two successive transitions is a random variable whose distribution may depend on the state from which the transition takes place as well as on the state to which the next transition takes place.

Y(t).

Y(t). $Y(t + t_n)$ is independent of Y(t) given the complete past history Y(t) and $X_n = i$ and is identical to the process Y(t) given $X_0 = i$ because of time homogeneity. Not only $Y(t + t_n)$ is independent of Y(t), it is also identical to the process Y(t) given $X_0 = i$ because of time homogeneity. It is satisfying the time invariant property also.

So it is very button when the Markov property is satisfied at the time in transition time instants t_n, then the stochastic process is called a Semi-Markov Process.

(Refer Slide Time 01:40)



Thus, in a semi-Markov process, the Markov property is satisfied only at the each of the transition epochs and not at all times.

This is very important. The Markov property is satisfied only at the each of the transition epochs t_n, not at all the times. If the Markov property is satisfied at all the time -- all times, then the stochastic process is called the Markov process. Since it satisfies only at each of the time transition instance, the Markov process is called a semi-Markov process.

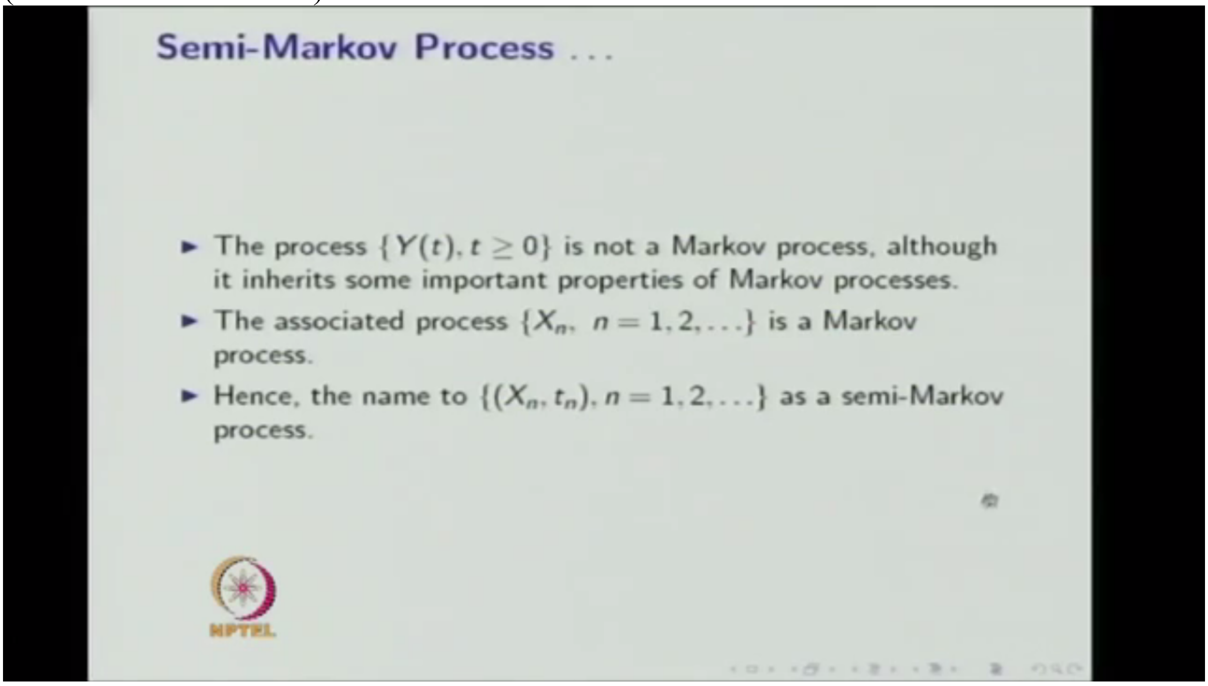
The fact that the Markov property holds at each of the transition epochs t_n of the stochastic process Y(t) entails the Markov property also holds for X_n. Since X_n is nothing but the $Y(t_n)$, therefore, the Markov property is satisfied for the stochastic process X_n, not for the Markov property is satisfied for the Y(t) for all times.

Hence, X_n turns out to be the time-homogeneous embedded Markov chain with the state space Ω. Y(t) is a stochastic process and the X of n is nothing but the $Y(t_n)$ where t_n is the transition time instance and the Markov property is satisfied only at all the time, all the transition time instants. Therefore, X_n form a discrete-time Markov chain. Since X_n is $Y(t_n)$, this X_n stochastic process is called an Embedded Markov Chain.

X_n stochastic process is embedded in this stochastic process Y(t). Therefore, X_n is a time-homogeneous embedded Markov chain because it satisfies the time invariant property as well as Markov property. Therefore, X_n is the time-homogeneous embedded discrete-time Markov chain.

Now we can say (X_n, t_n) constitute a Markov renewal process or semi-Markov process with the state space Ω. X_n is an embedded Markov chain where X_n is $Y(t_n)$ and T_n is nothing but the transition time instance. Therefore, the together (X_n, Y_n) constitutes Markov renewal process because these are all the time points in which the system is moving into the different states and the Markov property is satisfied only at those time points. Therefore, it is called the semi-Markov process or Markov renewal process with the state space Ω.

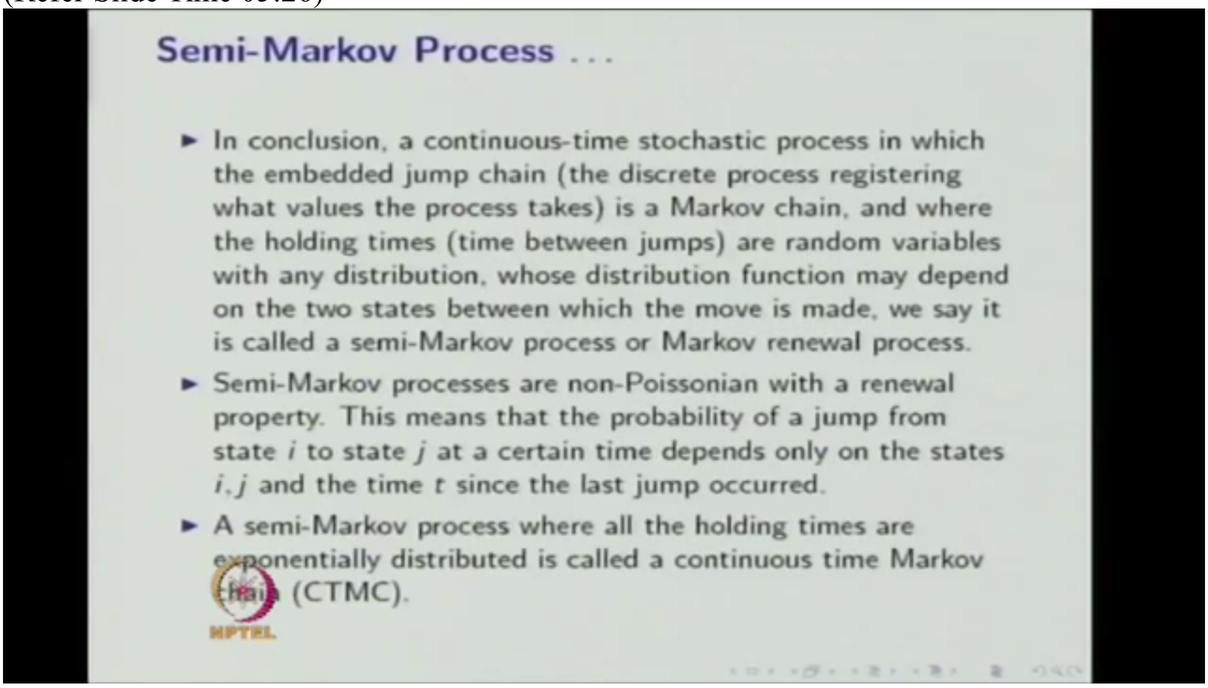
(Refer Slide Time 04:53)



The stochastic process -- the stochastic process Y(t) is not a Markov process although it inherits some important properties of Markov processes.

The associated process that is X_n is a Markov process. Hence, the name (X_n, Y_n) as a semi-Markov process.

(Refer Slide Time 05:26)



In conclusion, a continuous-time stochastic process in which the embedded jump chain (that is nothing but the discrete process registered what values the process takes), so the embedded jump chain is a Markov chain.

In conclusion, a continuous-time stochastic process which embedded as a Markov chain and where the holding times are random variables with any general distribution whose distribution function may depend on the two states between which the move is made, we say it is called -- we say it is semi-Markov process or Markov renewal process. Whenever these properties are satisfied, we say the stochastic process is a semi-Markov process or Markov renewal process.

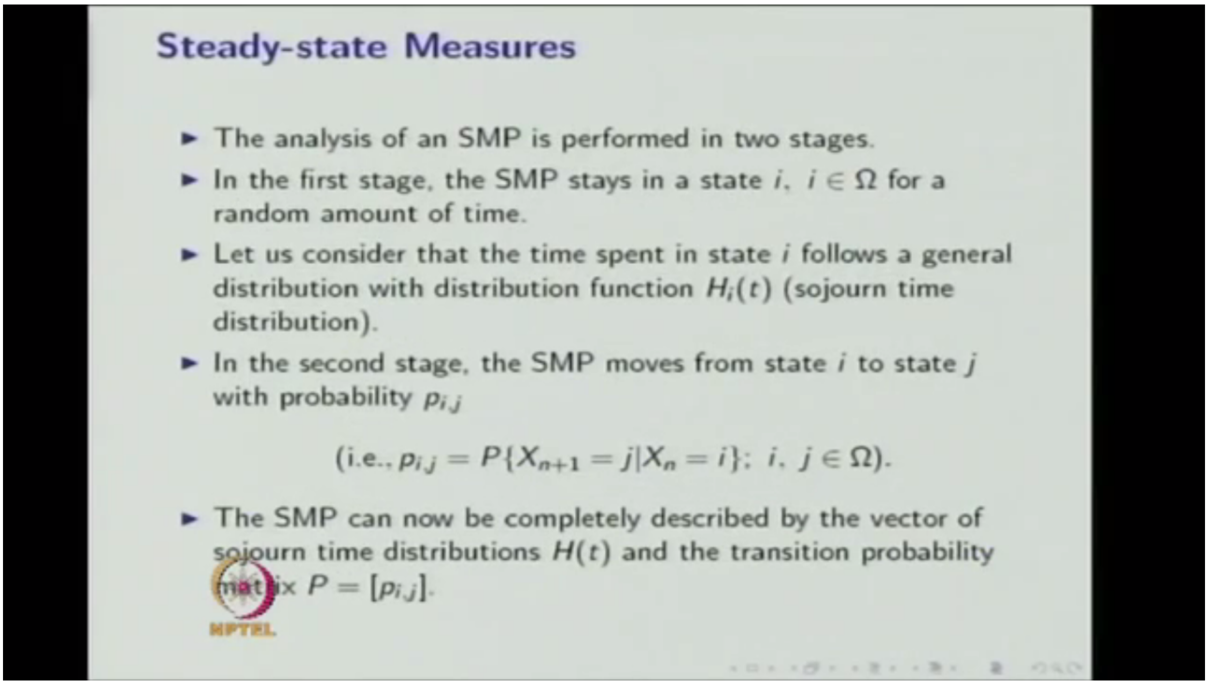
The semi-Markov processes are non-Poissonian with the renewal property. This means that the probability of a jump from a state i to j at a certain time depends only on the states i, j and the time t since the last jump occurred.

If you restrict the holding times are exponential distribution and each time transition instance are the renewals, then the special case of the semi-Markov process is a Poisson process, but in general, semi-Markov processes are non-Poissonian. If you make assumptions of holding times are exponential distribution with the same parameter and each time transitions are nothing but the renewals, then the special case of semi-Markov process is the Poisson process.

A semi-Markov process where all the holding times are exponential distribution is called a continuous-time Markov chain. So if I restrict only the holding times are exponential distribution, the each transition need not be the renewals, then a semi-Markov process is a continuous-time Markov chain.

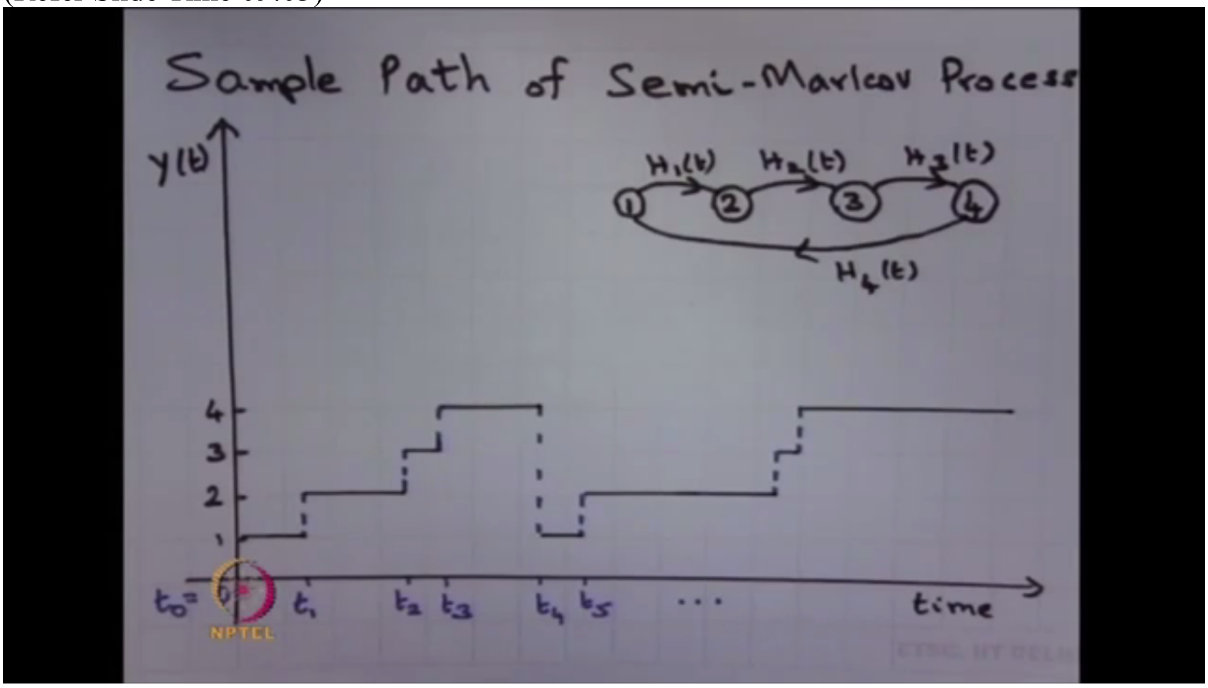
Now we are going to discuss the steady-state measures of semi-Markov process.

(Refer Slide Time 08:27)



The analysis of a semi-Markov process is performed in two stages. In the first stage, the semi-Markov process stays in a state i for some random amount of time.

(Refer Slide Time 09:03)



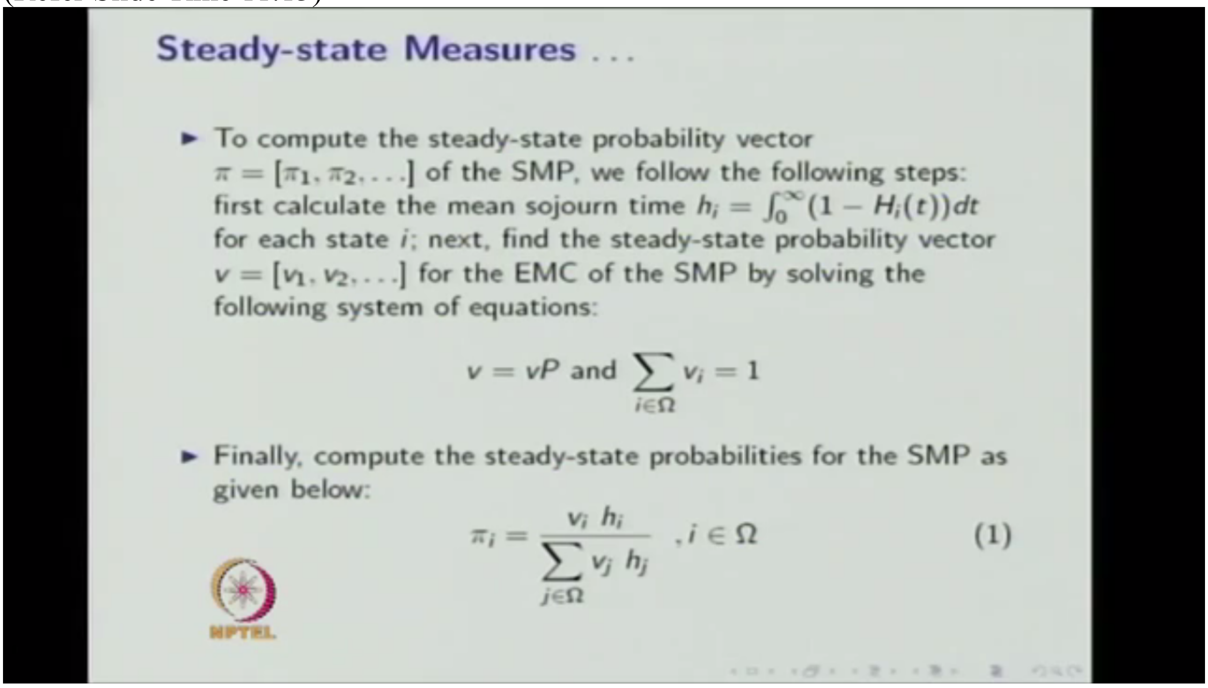
For example, in the sample path of a semi-Markov process, in this model, we have four straight semi-Markov process. So in the each state, the system stays a random amount of time.

Let us consider that the time spent in state i follow a general distribution with the distribution function $H_i(t)$. That means in this example, $H_1(t)$, that is the time spent in this system spent in the state 1. $H_2(t)$ is the distribution of the system staying in the state 2 and so on.

In the second stage, the SMP moves from the state i to j with the probability $P_{i,j}$ where $P_{i,j}$ is defined what is the probability that the system was in the state i at the n^{th} time instance. The system will be in the state j at the $(n+1)^{th}$ time instant; i, j belonging to Ω. So that is a conditional probability. The conditional probability of P X_{n+1} is equal to j given X_n is equal to i for i, j belonging to Ω.

So the SMP can now be completely described by the vector of sojourn time distributions H(t) and the transition probability matrix P, that is $P_{i,j}$. So the transition probability matrix is the transition probability. Therefore, all the row sums are equal to be 1 and the values are lies between 0 to 1, and you have to supply the sojourn time distribution for each state. So if these two information are given, then we are known with semi-Markov process.

(Refer Slide Time 11:13)



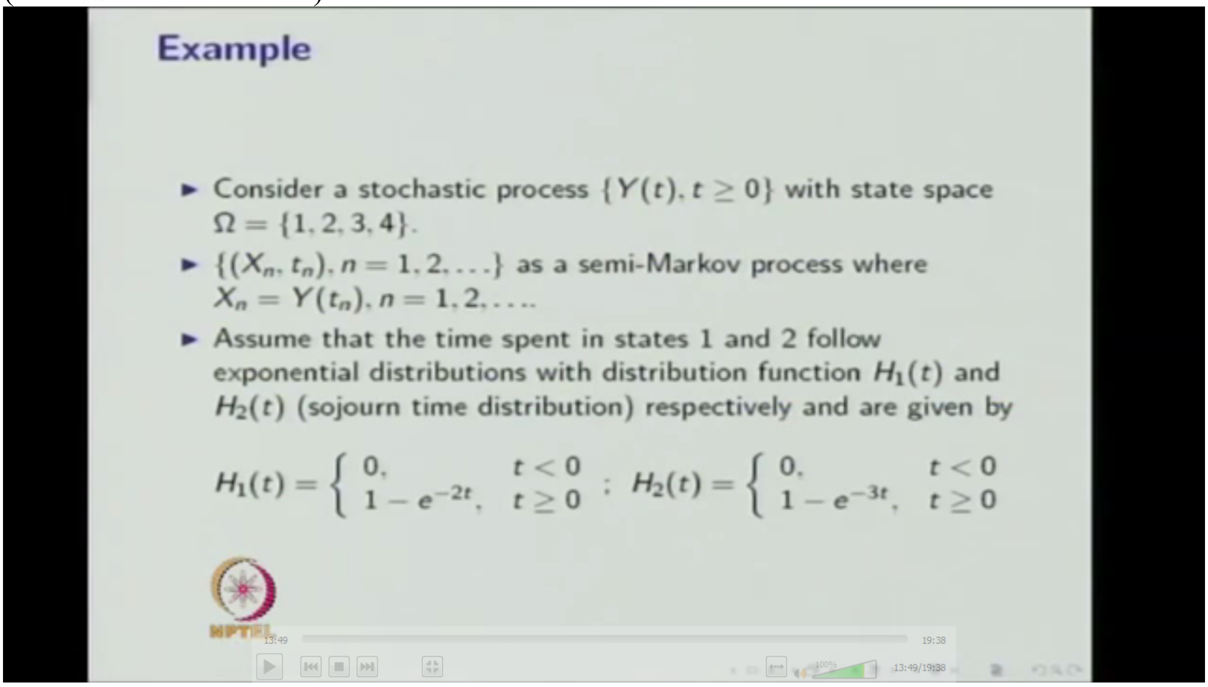
To compute the steady-state probability vector, let us assume that π is the vector; π_1, π_2 are the elements of a semi-Markov process. First calculate the mean sojourn time that is nothing but the small h_i. Since $H_i(t)$ is the distribution function, so 1 minus $H_i(t)$ the integration between 0 to infinity will be the mean sojourn time because each -- each random variable is a non-negative random variable. Therefore, the mean will be 0 to infinity 1 minus the CDF integration for each state i.

Next find the steady-state probability vector v_i's for the embedded Markov chain of the semi-Markov process. First you have to find out the steady-state probability vector for the embedded Markov chain of the semi-Markov process and using the steady-state probability vector and the mean sojourn time you can get the steady-state probability vector of -- steady-state probability vector π.

So how to find the steady-state probability vector of embedded Markov chain? So you know P is the transition probability matrix. So solve v = vP and summation of v_i is equal to 1. You will get v_i's. The v = vP is the homogeneous equation and including summation of v_i is equal to 1, you will have a non-homogeneous system of equation. So you can get the non-trivial solutions satisfying these two conditions.

So once you know the v_i's, you can compute the steady-state probabilities of the semi-Markov process. That is nothing but v_i's multiplied by h_i's divided by summation of all the v_j's h_j's where j is belonging to Ω. So the v_i's are nothing but the steady-state probability vector of embedded Markov chain and h_i's are nothing but the mean sojourn time and π_i's you will get by using this formula.

(Refer Slide Time 13:48)

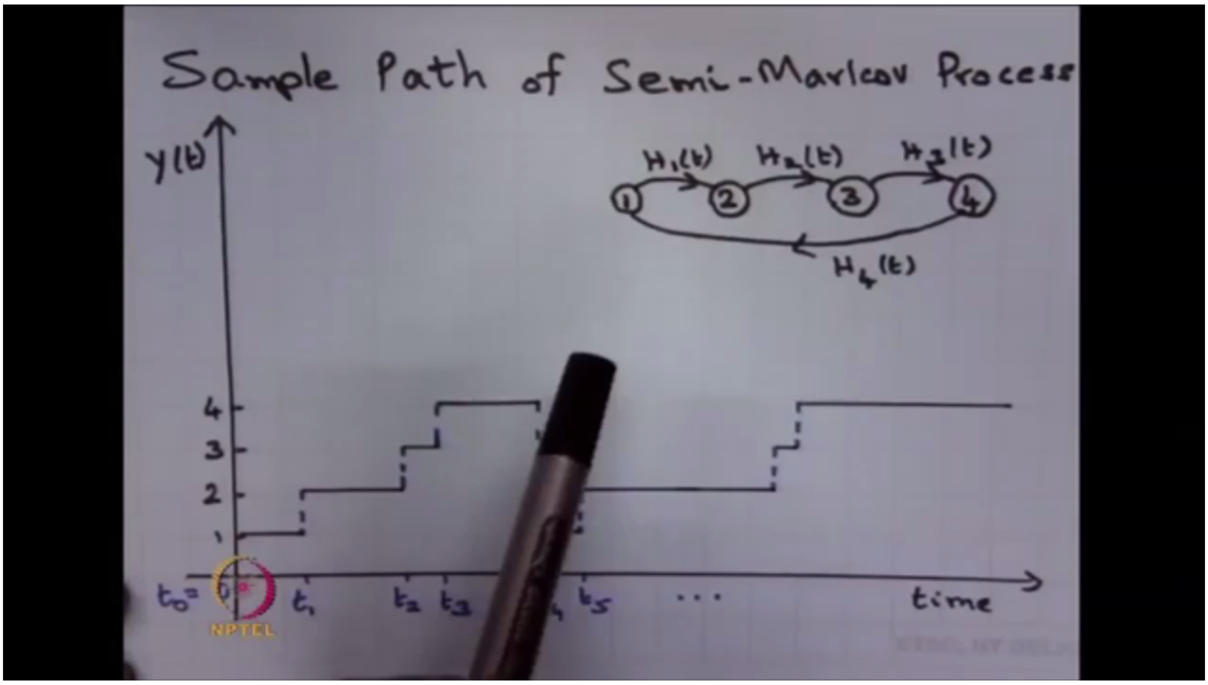


Now let us consider the simple example the stochastic process with the state space $\Omega = \{1, 2, 3, 4\}$.

In the previous -- previous steady-state measures with the assumption that the steady-state probability vector probability -- probabilities exist, we are giving the how to compute the steady-state probability measures. So here the assumption is steady-state probabilities are exist.

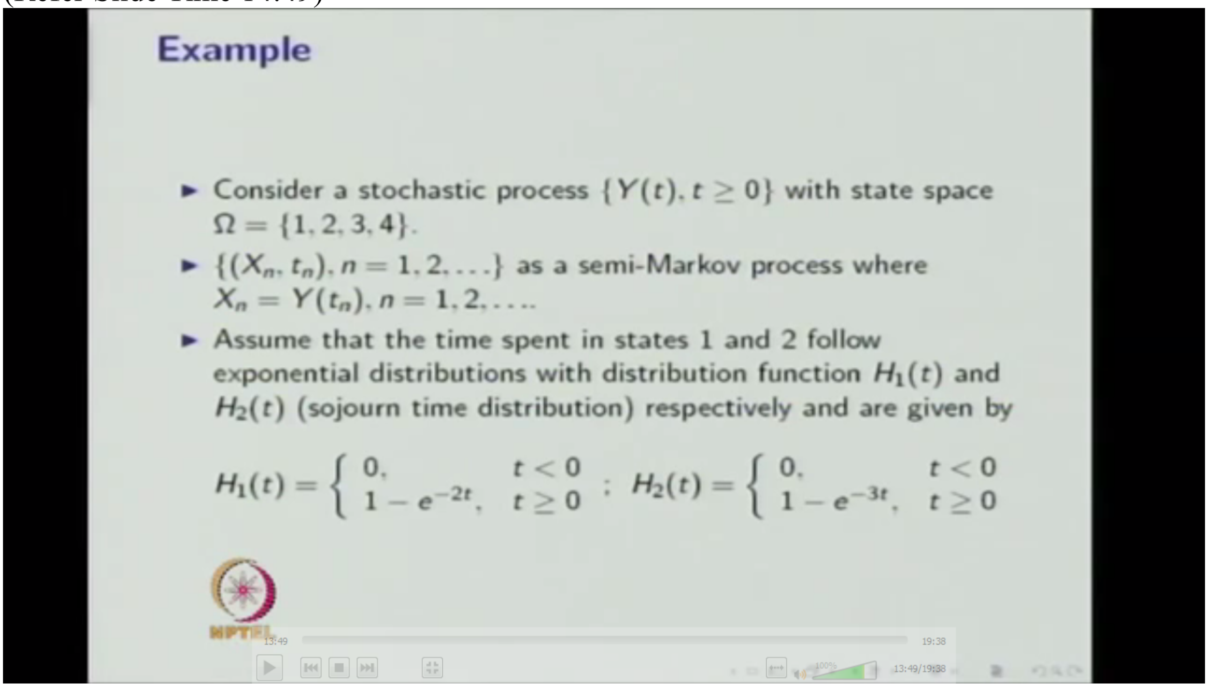
Now we come to the example.

(Refer Slide Time 14:30)



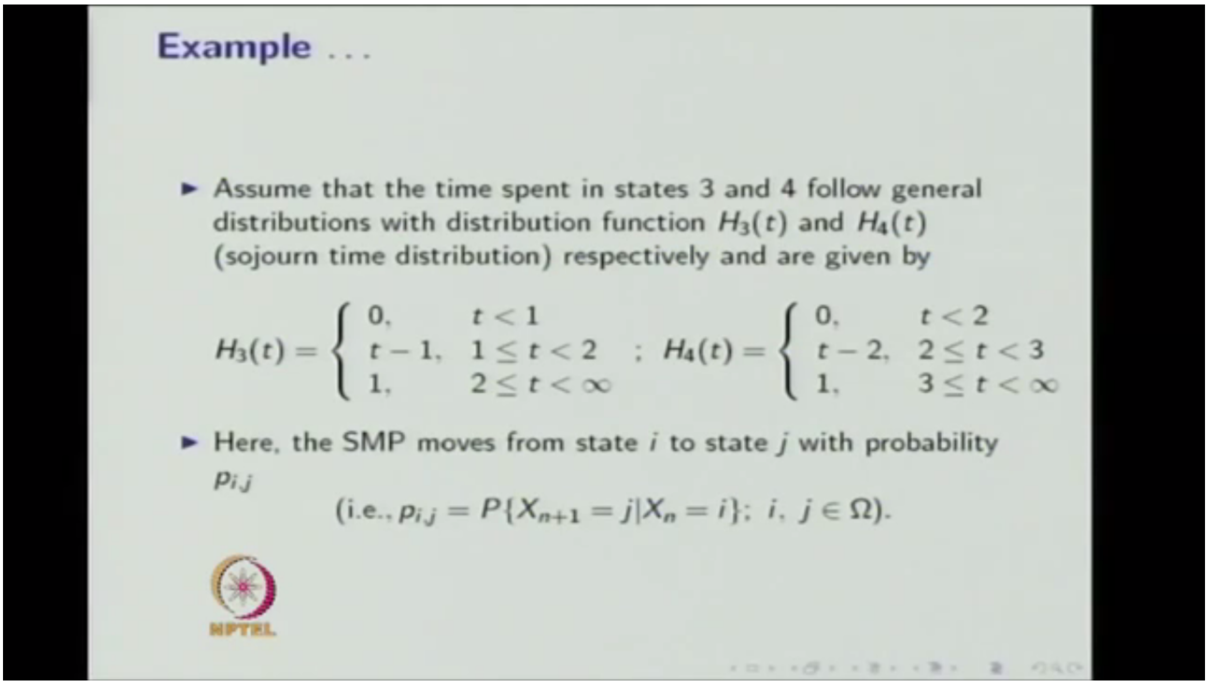
So this is a four state stochastic process with the states 1, 2, 3, 4 and H_i's are nothing but the CDF of sojourn time in each state and X_n is nothing but $Y(t_n)$ and (X_n, t_n) will form a Markov renewal process or semi-Markov process.

(Refer Slide Time 14:49)



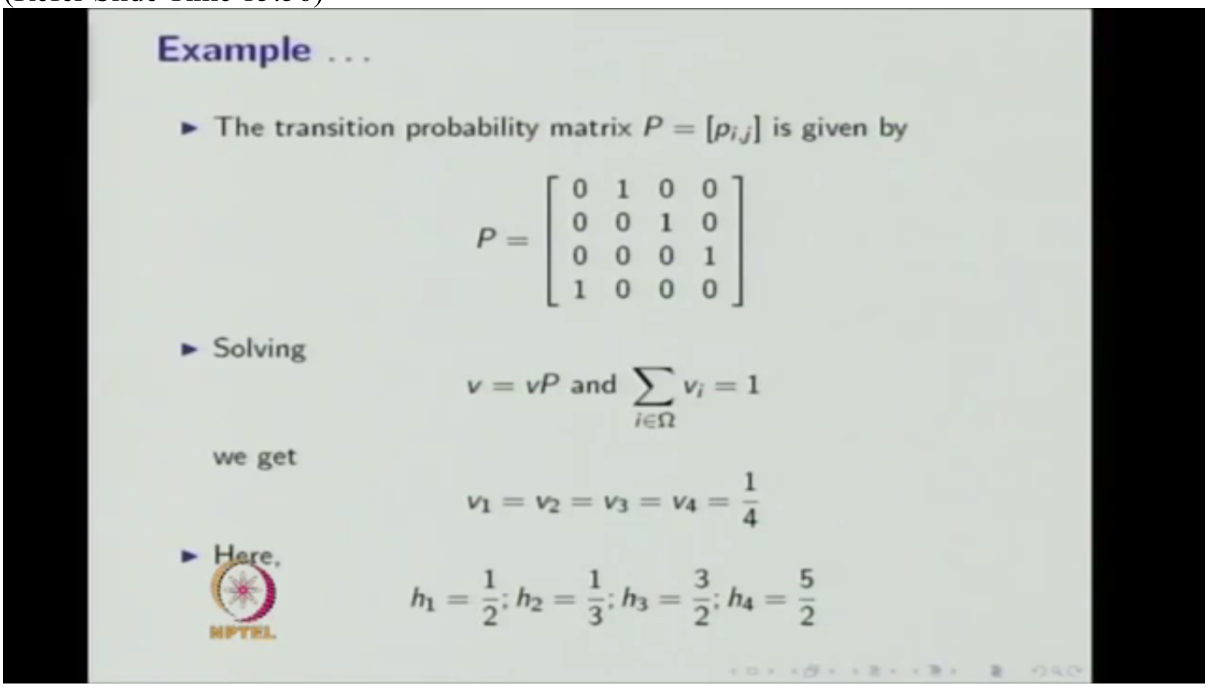
Assume that the time spent in the states 1 and 2 follow exponential distributions with the CDF H_1 and H_2 whereas the time spent in the states 3 and 4 follow uniform distribution with the CDF $H_3(t)$ and $H_4(t)$.

(Refer Slide Time 15:16)



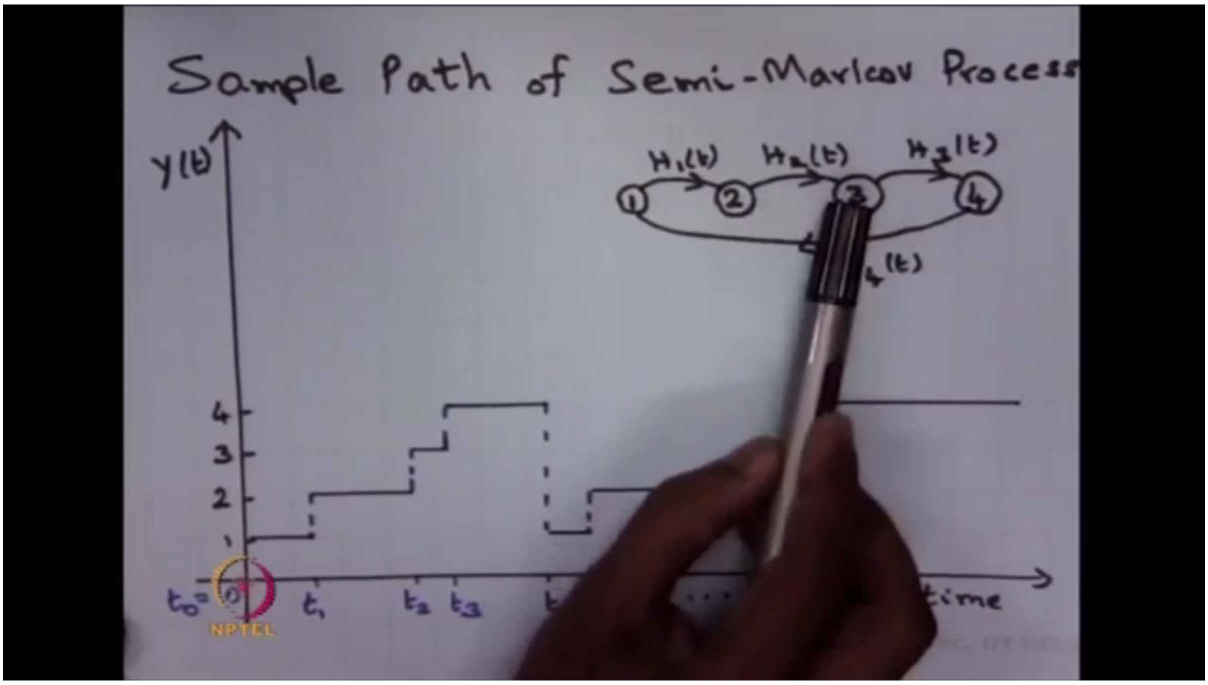
So the sojourn time in the state 3 is uniformly distributed between the intervals 1 and 2, and the sojourn time spent in the state 4 is also uniform distribution between the intervals 2 and 3. Therefore, the CDFs are in this form $H_3(t)$ and $H_4(t)$.

(Refer Slide Time 15:56)



The semi-Markov process moves from the state i to j with the probability P_{ij}. That is nothing but the transition probability matrix can be in the form the states are 1, 2, 3, 4. Therefore, in the one-step transition probability of system is moving from 1 to the state 2 is sure, therefore, that probability is 1 and all other probabilities are zeros.

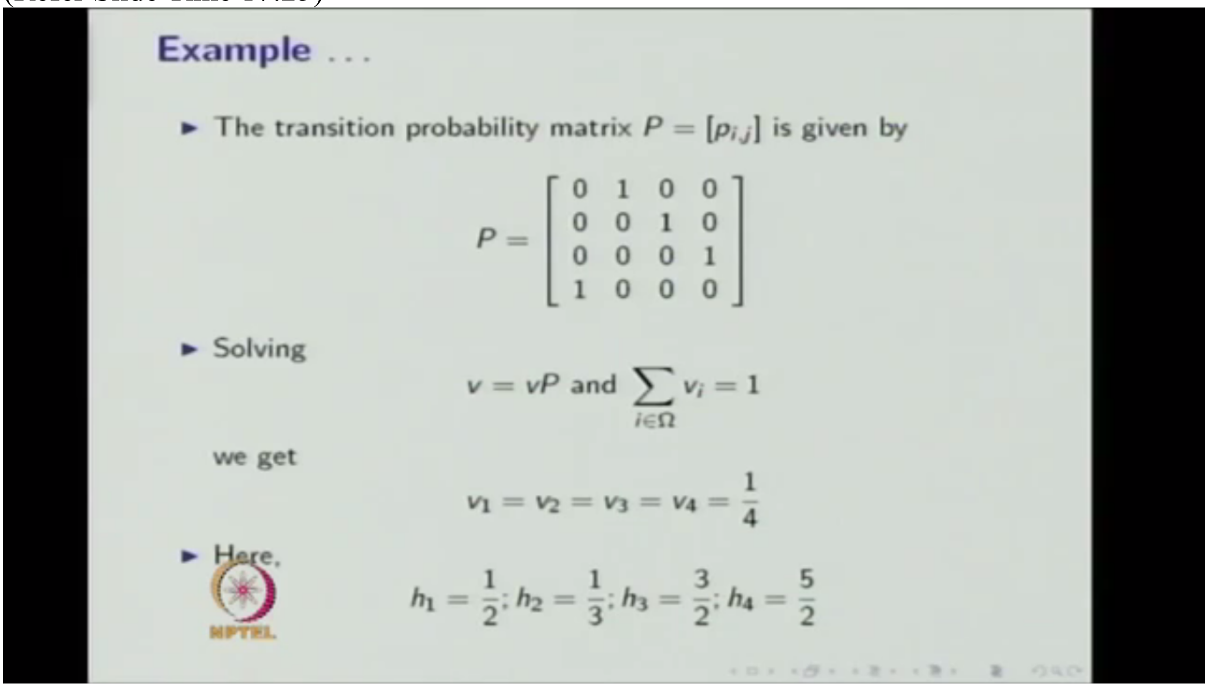
(Refer Slide Time 16:28)



Similarly, the system moved from the state 2 to 3. That probability will be 1 and all other states moving probabilities are 0. Therefore, all other transition probabilities from the state 2 to 1, 2 to 4 are zero whereas 2 to 3 will be -- 2 to 3 will be will be 1. Similarly, 3 to 4 will be 1 and all other states, all other transition probabilities are zeros. Then the system moving from the state 4 to 1 is 1 and all other states are 0.

So you know the transition probability matrix as well as you know the mean -- you know the sojourn time distribution. So using the transition probability matrix, by solving you can get the steady-state probability vector of embedded Markov chain.

(Refer Slide Time 17:25)



If you see the transition probability matrix, since it is a transition probability matrix, the values are lies between 0 to 1 and the row sum is 1, but in this particular transition probability matrix, it satisfies the one more additional condition, the column sum is also 1.

Therefore, if you solve the v is equal to vP and the summation of v_i is equal to 1, the solution will be 1 divided by the number of states. So the steady-state probabilities are uniformly distributed, uniform distribution. So number of states are 4. Therefore, the steady-state probability of the embedded Markov chain is 1/4.

So you know the steady-state probabilities of embedded Markov chain and from the sojourn time distribution you can find out the mean sojourn time. Since the first two random variables sojourn -- first two states sojourn times are exponential distribution, therefore, the mean sojourn time is 1/2, 1/3 respectively for the states 1 and 2 and the sojourn time in the state 3 is uniform distribution between the interval 1 to 2. Therefore, the mean sojourn time is 3/2 and for the state 4, the sojourn time distribution is uniform distribution between the intervals 2 to 3. Therefore, the mean sojourn time is 5/2.

So using transition probability, sorry, using the steady-state probability -- steady-state probabilities of embedded Markov chain and the mean sojourn time -- using the steady-state probabilities of embedded Markov chain and the mean sojourn time -- times, you can get the steady-state probabilities of semi-Markov process.