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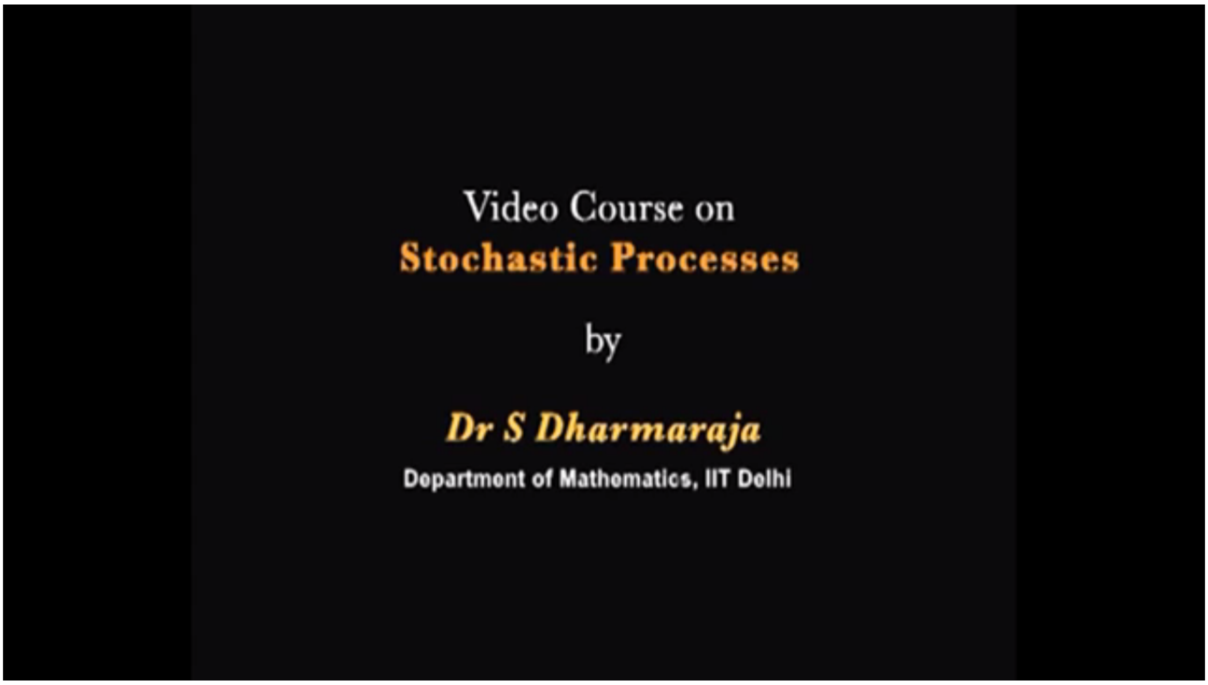

Video Course on
Stochastic Processes

by

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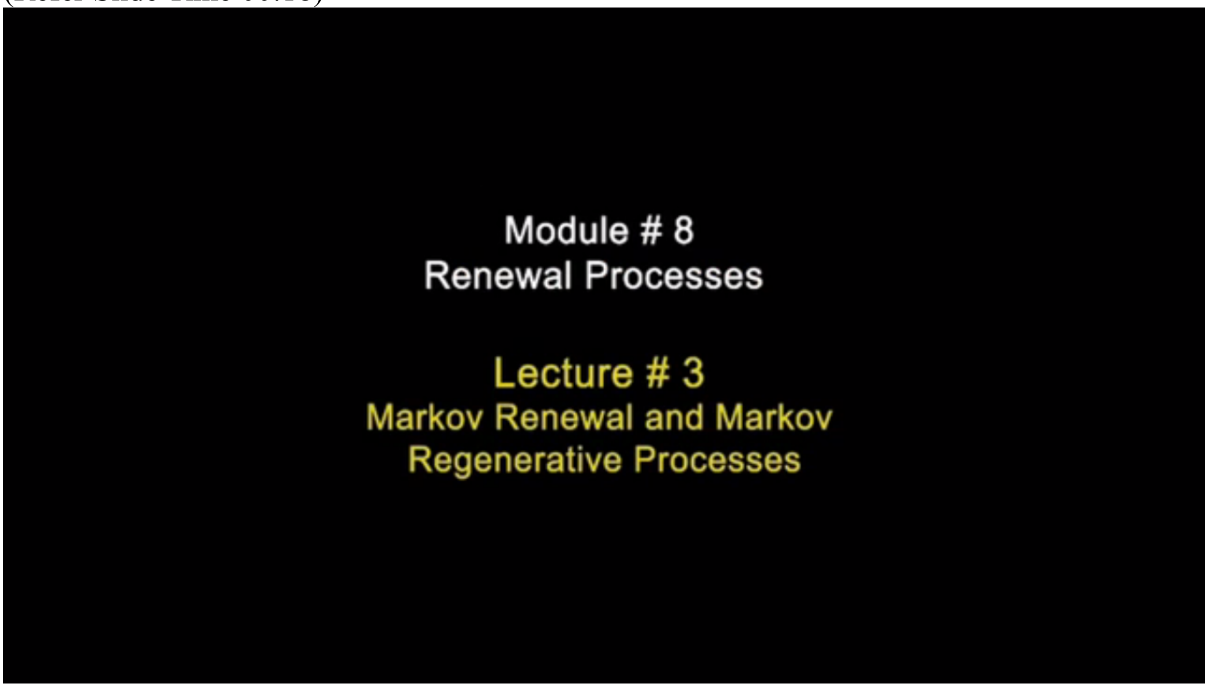

Module # 8
Renewal Processes

Lecture # 3
Markov Renewal and Markov Regenerative Processes

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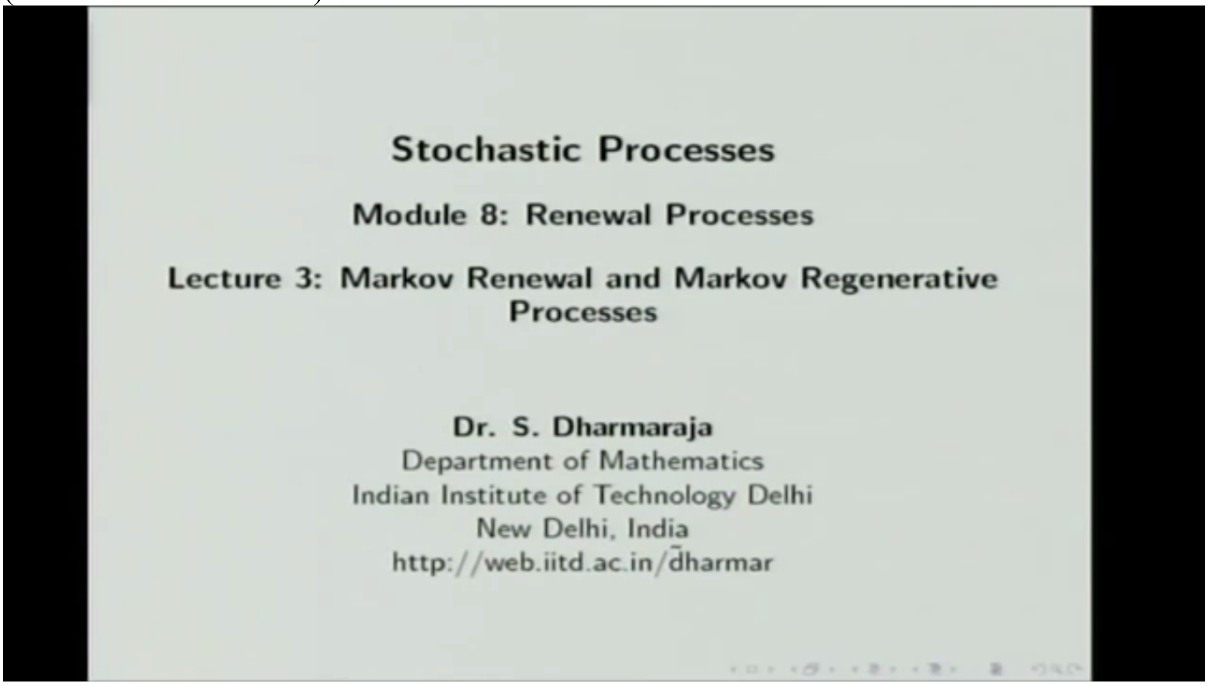

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This is Stochastic Processes, Module 8, Renewal Processes.

In the first two lectures, we have discussed renewal functions and its properties, and then we have discussed renewal theorems. There are three important theorems we have discussed in the Lecture 2.

Today's lecture is Lecture 3: Markov Renewal and Markov Regenerative Processes.

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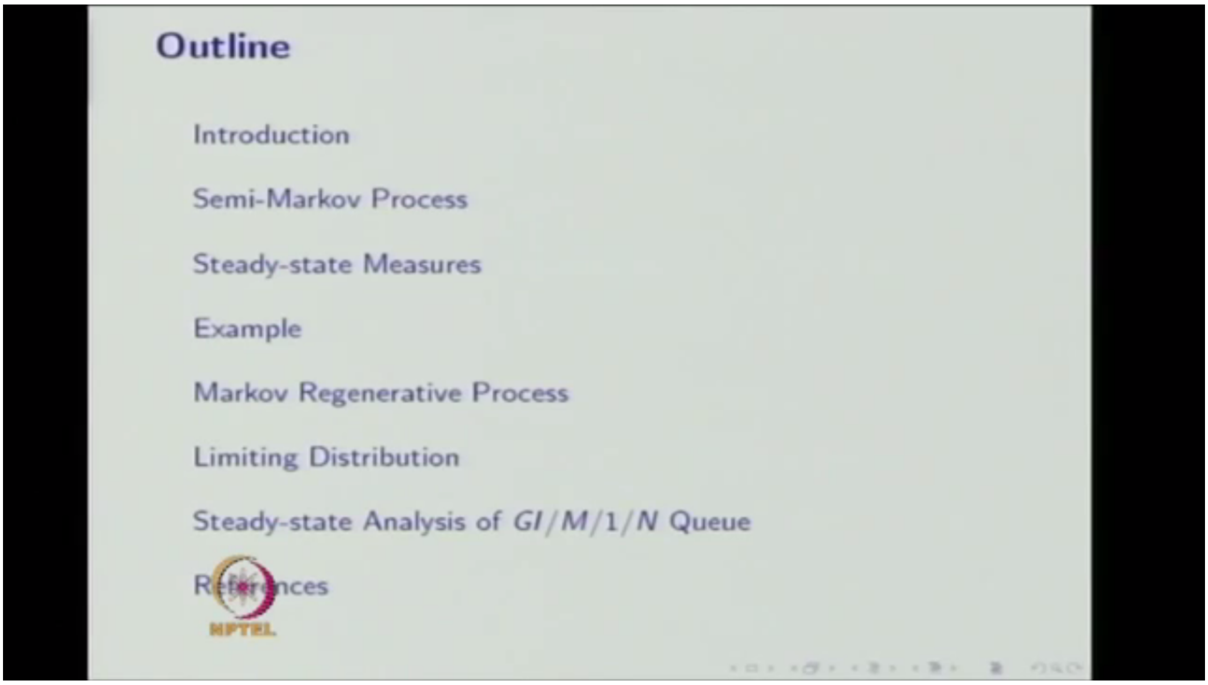

In this lecture, I am going to cover Markov Renewal Process or Semi-Markov Process, the definition and its properties followed by the definition and the properties, I am going to discuss the steady-state measures, and I am going to discuss one simple example for the Semi-Markov process.

Then the second part of today's lecture, I am going to cover Markov Regenerative Process. The definition and the properties I am going to discuss followed by the definition and properties, I am going to discuss the limiting distribution. As an example, we are going to discuss the steady-state analysis of G/M/1/N Queue.

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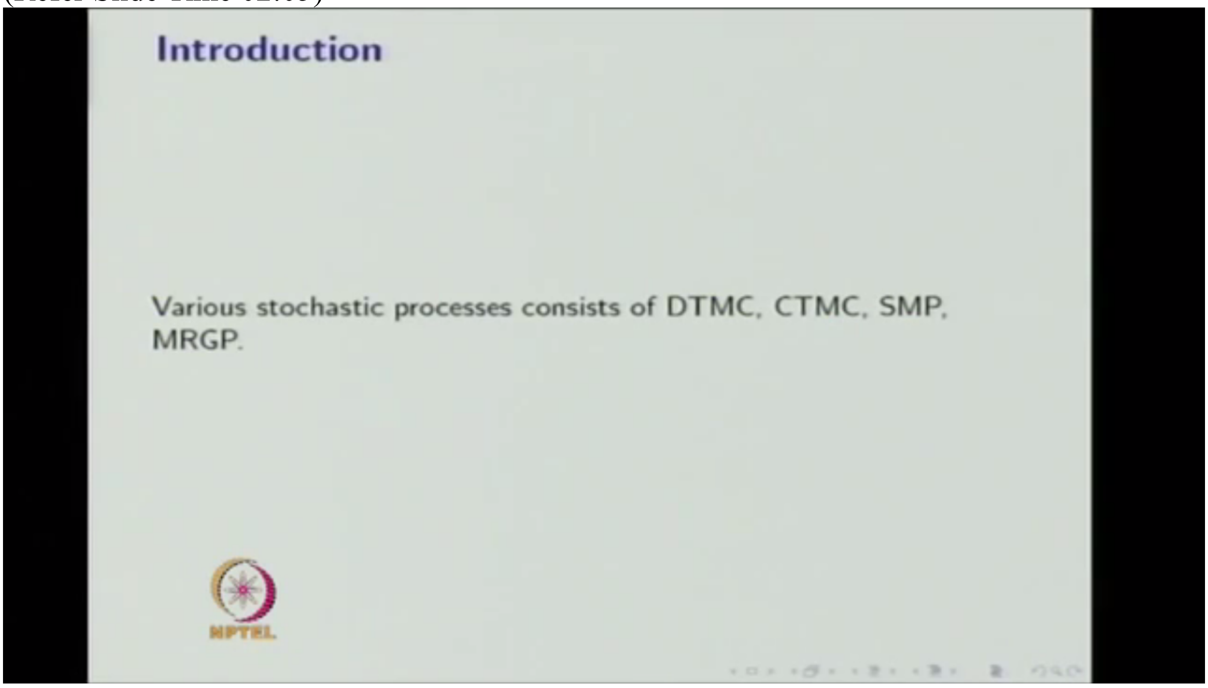

First we are going to discuss the various stochastic processes, which consists of DTMC, CTMC, semi-Markov process and Markov Regenerative Process.

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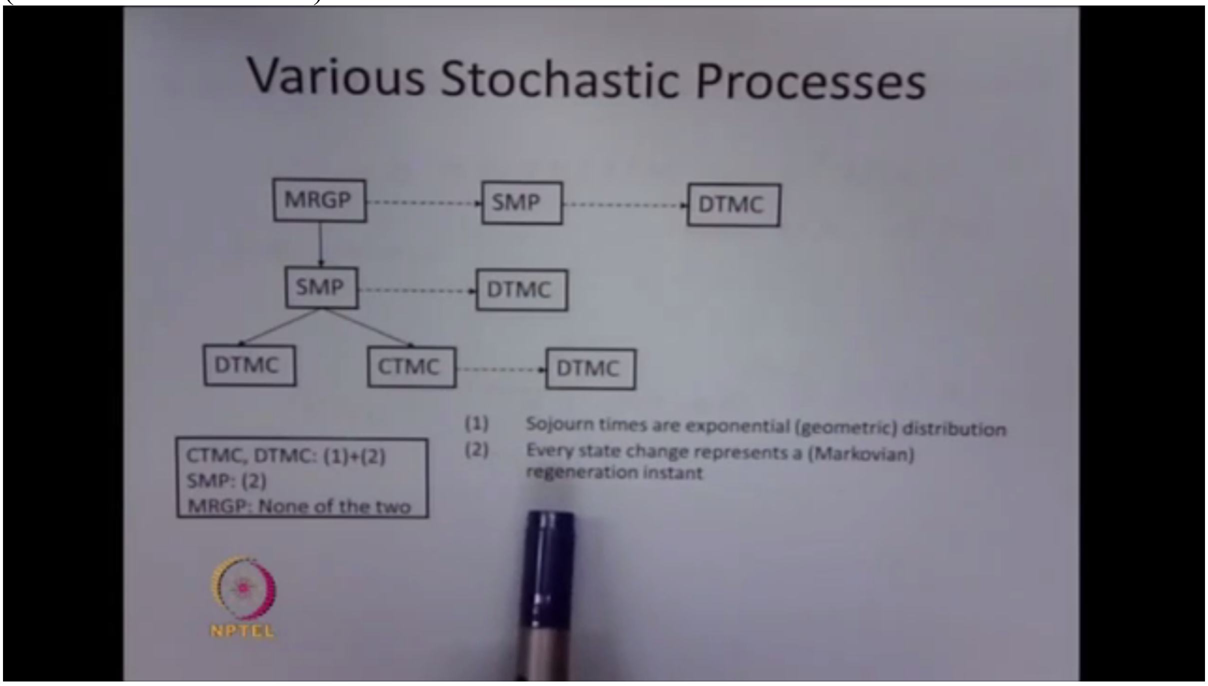

First you consider, number one, the sojourn times are exponential distribution. Number two, every state change represents a regeneration instant.

The epochs or points t_1, t_2 and so on in which the process probabilistically restarts from scratch that is from epoch t_1 the process is independent of its past and the stochastic process behaviour from epoch t_1 is the same as it had from $t_0 = 0$. These epochs, these epochs or points are called regenerative epochs or regenerative time points or regenerative instance.

So these are all the two important properties. One is the sojourn times are exponential distribution. The second one is every state change represents a regeneration instant.

Accordingly, we can classify the stochastic process. The first one is if both the properties are satisfied, then the corresponding stochastic process are either it is -- it is a Markov process. So based on the time-space is a discrete or continuous, we have a discrete-time Markov chain or continuous-time Markov chain. Whenever the two properties are satisfied, the stochastic process is said to be a Markov process and the state spaces are discrete, then the Markov process is called a Markov chain and based on the time space or parameter space is a discrete or continuous, accordingly, we have a discrete-time Markov chain or continuous-time Markov chain.

If any stochastic process satisfies the property number two only, not the property number one, that means sojourn times are exponential distribution, if that property is not satisfied, then that stochastic process is called a Semi-Markov Process. We are going to discuss in detail about the semi-Markov process. That is a stochastic process satisfying the property number two only. If both the properties are not satisfied, but still not every state change represents a regeneration instant instead of this, there are few state change represents a regeneration instant, then the stochastic process is called a Markov Regenerative Process.

So if two properties are satisfied, then it is a CTMC or DTMC. If only the property number 2 satisfies, then it is a SMP. If both the properties are not satisfied, but few state change represent a regeneration instant, then the stochastic process is called a Markov regenerative process.

So that we have shown it in the diagram. From the stochastic -- from the semi-Markov process, you can have an embedded DTMC by making proper assumptions that the sojourn times are exponential distribution, then it will be a CTMC. From the CTMC, you can create the DTMC with the embedded. Similarly, from the SMP, you can make an embedded DTMC. By proper assumptions, you can get the DTMC or CTMC.

The general case of a semi-Markov process is a Markov regenerative process. If you make some assumptions, then it will be a semi-Markov process. The down arrow means if you -- if you make additional assumptions in the Markov regenerative process, then you will get the semi-Markov process.

If you make additional assumptions in the semi-Markov process, then you will get a DTMC or CTMC whereas these dotted arrows means embedded stochastic process. So from the DTMC, from the CTMC, we can have an embedded DTMC. From the SMP, you can have a embedded DTMC. From the MRGP, you can have an embedded SMP and from the SMP, you can have an embedded DTMC.

So this is the pictorial representation of various stochastic process based on the these two properties.

Now we are moving into the semi-Markov process introduction.

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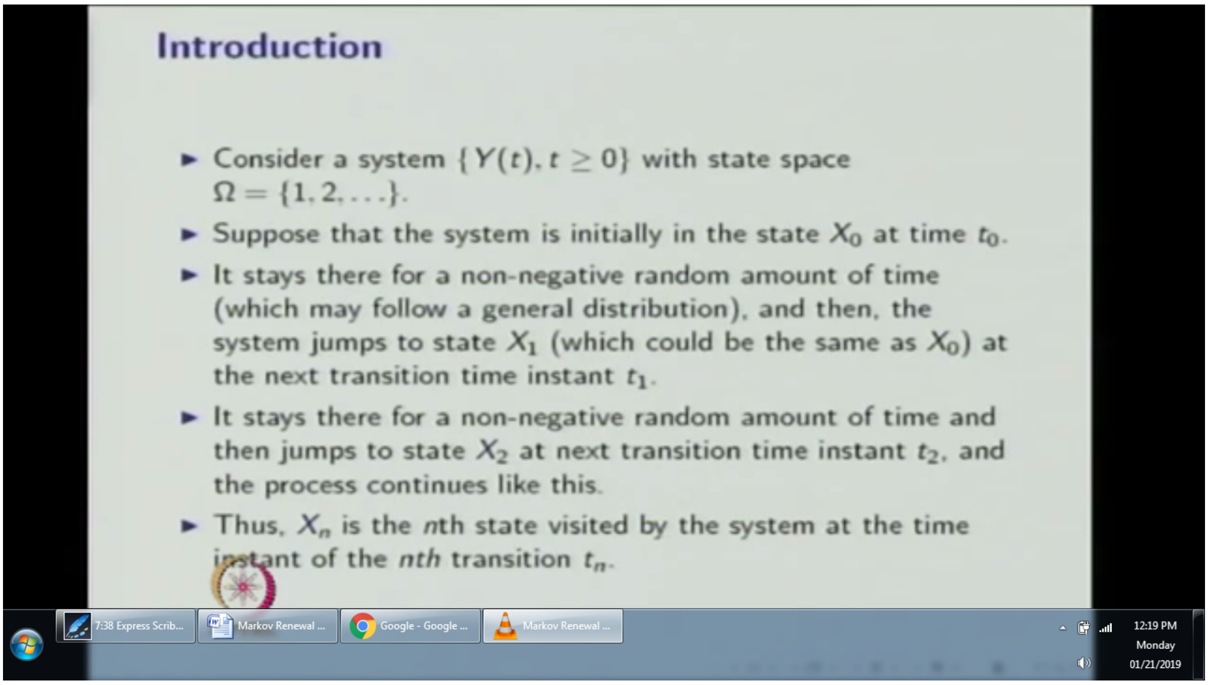

Consider a system Y(t) with the state space Ω. That means the Y(t) is a discrete state continuous-time stochastic process.

Suppose that the system is initially in the state X_0 at time t_0. It stays there for a non-negative random amount of time (which may follow a general distribution), and then the system jumps to the state X_1 at the next transition time instant t_1. It stays there for a non-negative random amount of time and then jumps to the state X_2 at the next transition time instant t_2, and the process continues like this.

Thus X_n is the n^{th} state visited by the system at the time instant of n^{th} transition t_n.

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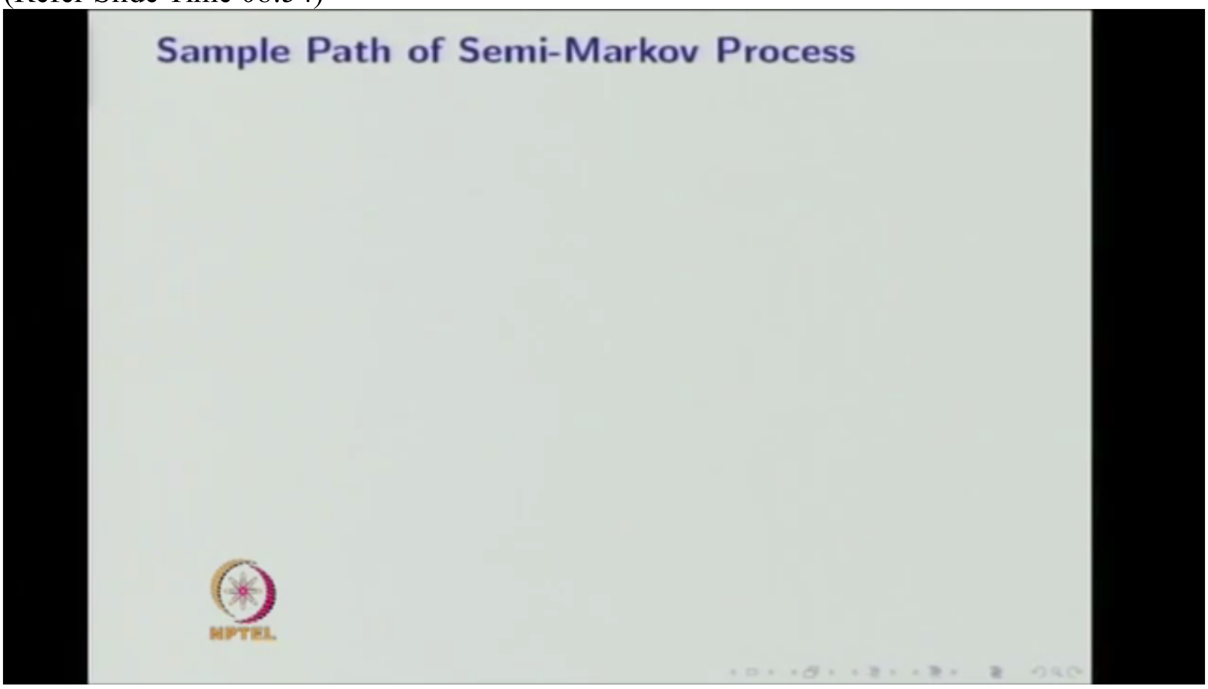

This can be depicted through the sample path. So sample path means it's a trace. Consider this sample part of Semi-Markov process.

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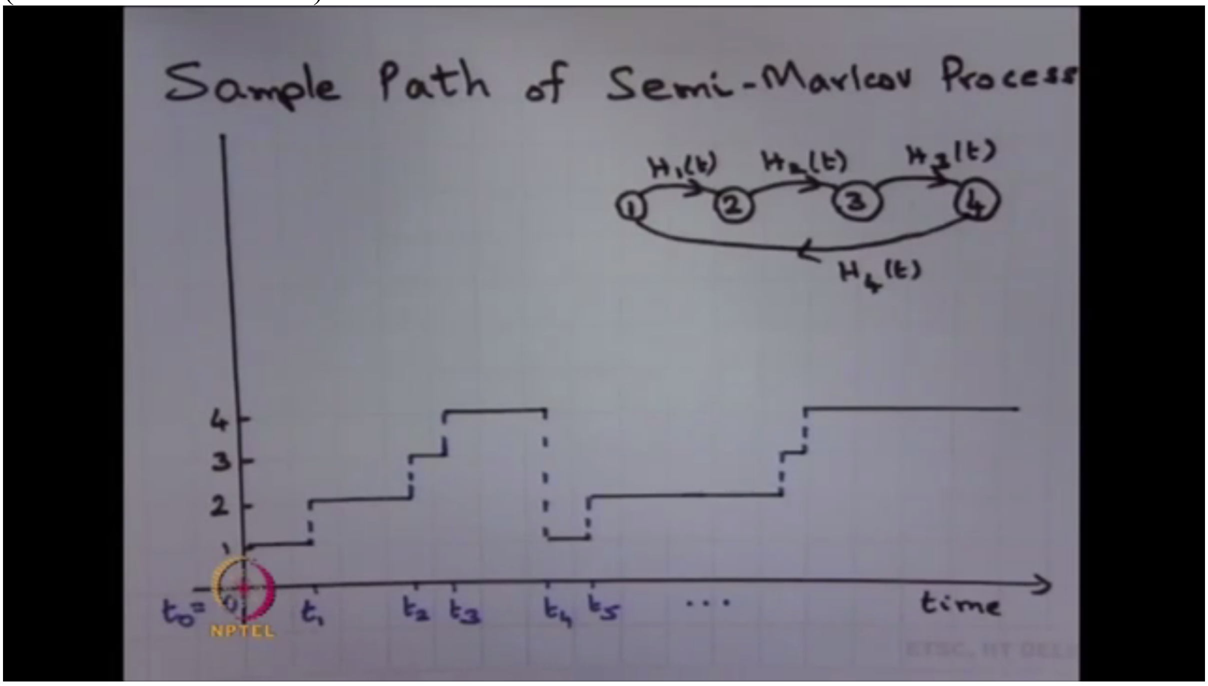

The corresponding Y(t) is in the y-axis. The time is in the x-axis. Y axis is Y(t). The corresponding stochastic process has the four states Ω is 1, 2, 3, 4. So at time 0, the system is in the state 1 and the system is in the state 1 till time t_1, t_1 time instant. At the time t_1 instant, the system moved to the state 2.

So the system was in the state 0 at time t_0 that is equal to 0 and at the time t_1, it moved to the state 2. The system is in the state 2 till time instant t_2 and then it moves to the state 3 at time instant t_2. So from the state -- from the state 2, the system moves to the state 3 at the time point t_3, sorry, t_2.

The system moves to the state 4 at the time point, time instant t_3. Now the system is in the state 4 and the system was in the state 4 till the time point t_4. Then it moved to the state 1 and so on, and the t_1, t_2, t_3, t_4 are the time instant. In those time instant, the system is moved from one state to the other states.

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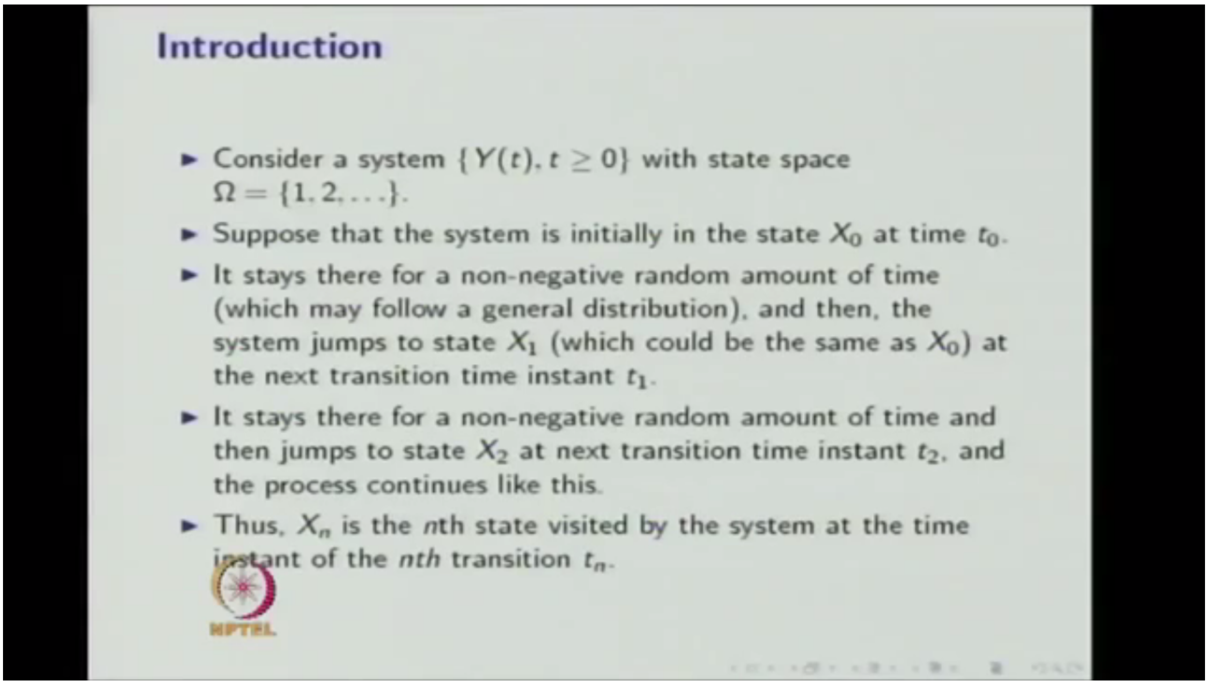

The system was in the state X_0 at time t_0 and at time t_1, the system moved to the state X_1 and so on. Thus X_n is the n^{th} state visited by the system at the time instant of n^{th} time transition t_n.

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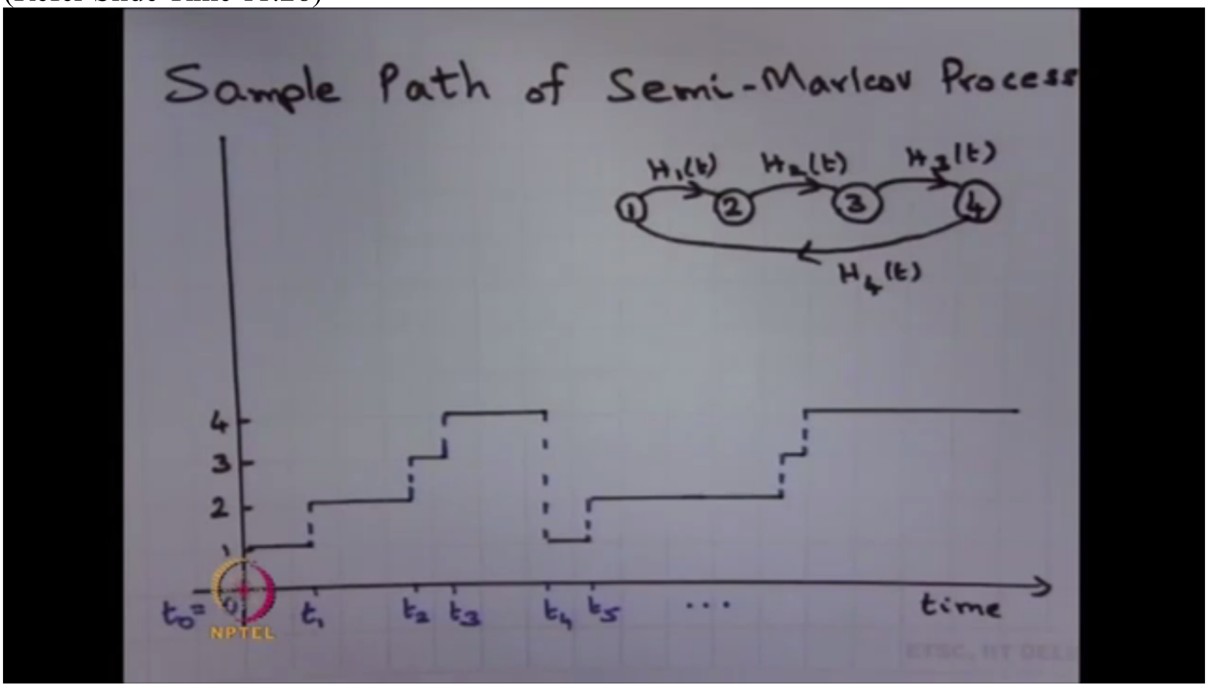

So in this sample path, t_1, t_2, t_3, t_4, t_5 are the time instants and the system is visiting the states. So here it is a four-state model. The Ω is 1, 2, 3, 4. Therefore, the system is keep moving into the from one state to other states according to the -- this state transition diagram.

We will come back to the same example again. So this is the illustration of a sample path of Semi-Markov Process and t_1, t_2 are the time instant.

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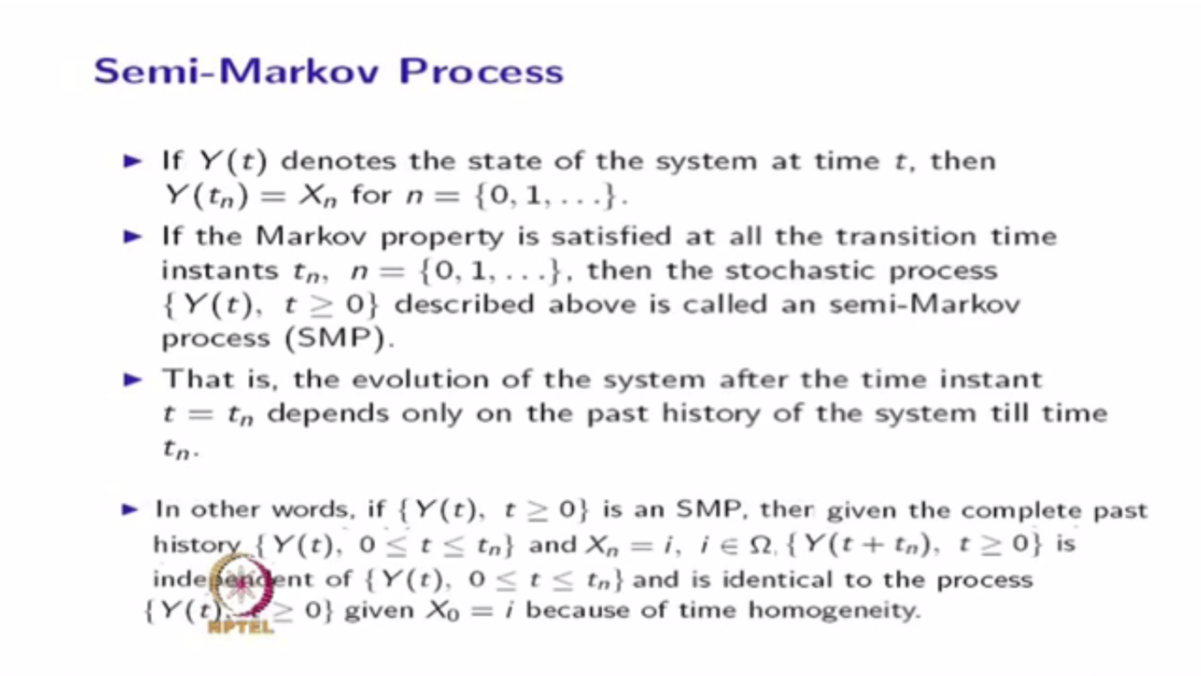

If Y(t) denotes the system, if Y(t) denotes the state of the system at time t, then $Y(t_n) = X_n$. Whenever you observe at the time instant the system, then that is the possible values of X_n.

If the Markov property is satisfied at all the time -- all the transition time instants t_n, then the stochastic process described above is called the semi-Markov process. Whenever the Markov process is satisfied at the time instant t_0, t_1, t_2, t_3, t_4 and so on all the time instants, the time instants are nothing but the system is moving from one state to another state at those time instants.

So if the Markov property is satisfied at these time instants, then the stochastic process Y(t) is called the semi-Markov process or in other words it is called the Markov Renewal Process. That is the evaluation of the system after the time instant t equal to t_n depends only on the past history of the system till the time t_n.