Generalized Renewal Processes and Renewal Limit Theorems contd.

by

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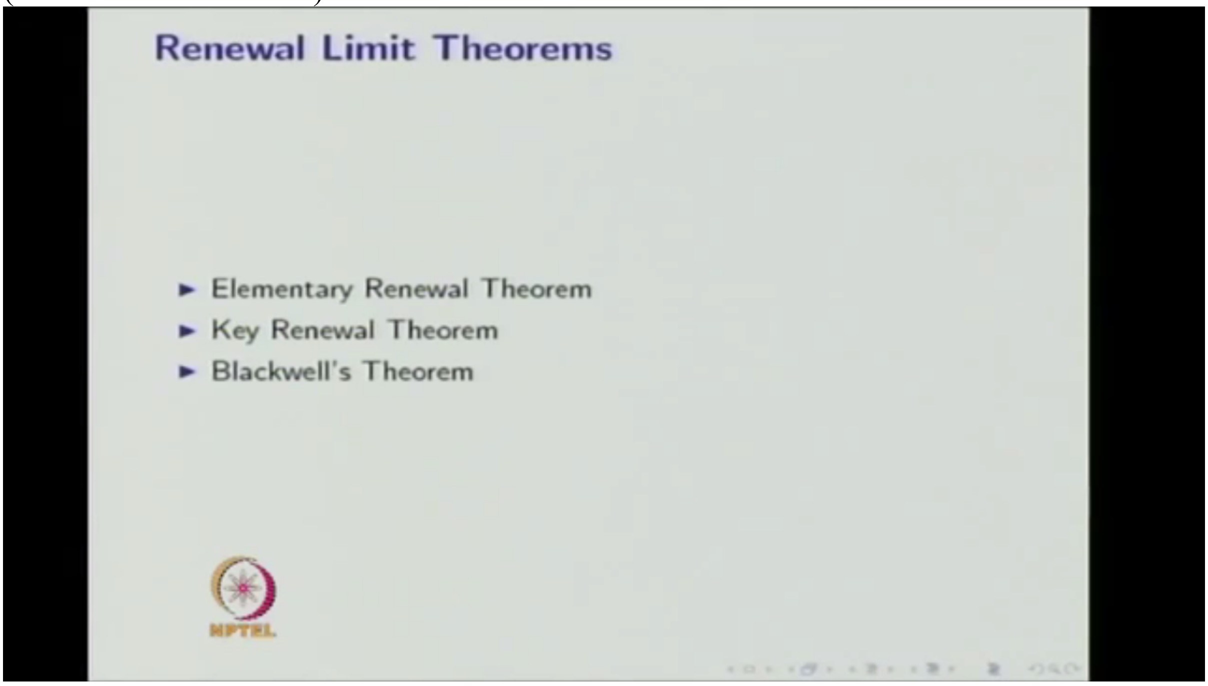

This is the last part of in this lecture that is Renewal Limit Theorems. We are going to discuss only three important Renewal Theorems without proof.

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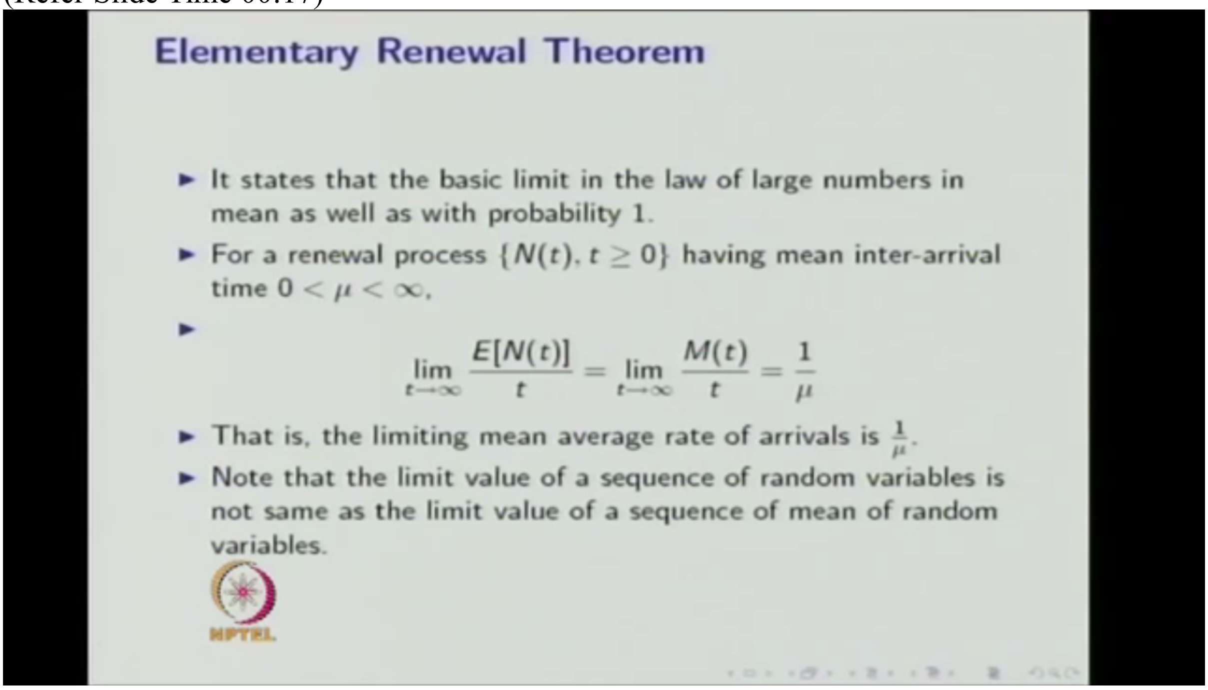

The first one is Elementary Renewal Theorem. It states that the basic limit in the law of large numbers in mean as well as with the probability 1. For a renewal process having the mean inter-arrival time where μ is lies between 0 to infinity, the limit t tends to infinity, the expectation of N(t) divided by t that is nothing but limit t tends to infinity of expectation of N(t) is nothing but the renewal function capital M(t) divided by t that is same as 1 divided by μ.

Since N(t) goes to infinity almost surely and by the strong law of renewal process we get this result. This result shows that for large t, the number of renewals per unit time converges to 1 divided by μ. That is the limiting mean arrival rate of arrivals, mean average rate of arrivals is 1 by μ. That is the limiting mean average rate of arrivals is 1 by μ.

Note that the limit value of a sequence of random variables is not same as the limit value of a sequence of mean of random variable.

This is true in general also. Here we discuss particularly for renewal process.

The N(t) is a random variable. Expectation of N(t) for fixed t is a constant. So here we are finding limit t tends to infinity, the expectation of N(t) divided by t, that is 1 by μ.

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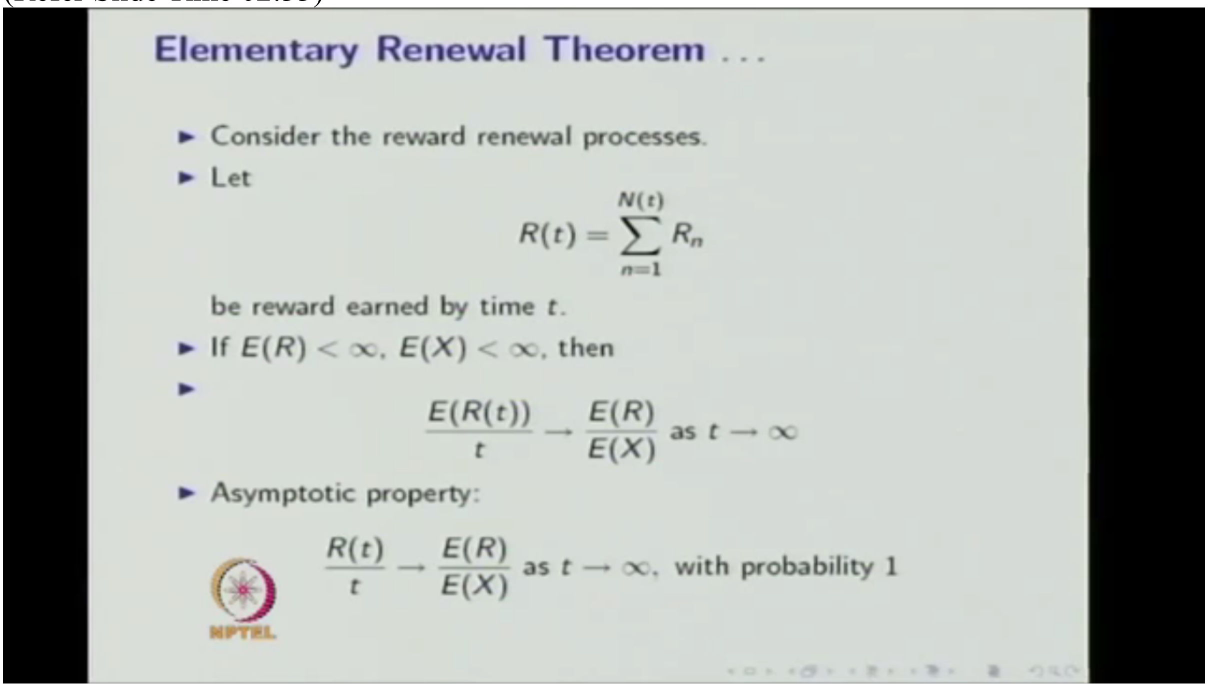

If you consider the reward renewal process R(t) when the expectation of R is finite as well as expectation of X is finite, limit t tends to infinity, the expectation of R(t) divided by t, that will tends to expectation of R divided by expectation of X using the Elementary Renewal Theorem. If you know the value of expectation of R and if you know the value of expectation of X as t tends to infinity, expectation of R(t) divided by t will be expectation of R divided by expectation of X. So this result we have used it in one example.

The asymptotic property of reward renewal process is as follows: as t tends to infinity, the R(t) divided by t will converge to expectation of R by expectation of X with the probability 1. This is called asymptotic property of reward renewal process. Both will be used in applications.

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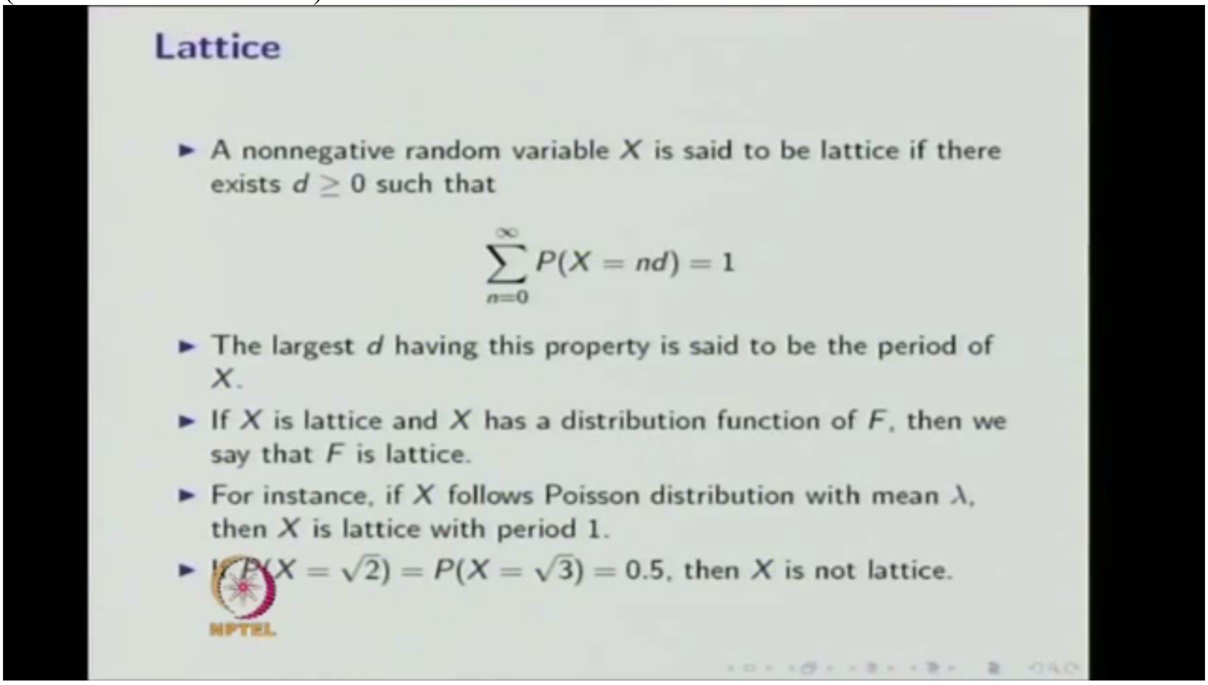

Before moving into the next Renewal Limit Theorem, we need the definition of a lattice. A non-negative random variable X is said to be a lattice if there exists d greater than or equal to 0 such that the probability of X equal to n times d for n is varies from 0 to infinity, that will be 1. If this condition is satisfied, then we say the d is called the period and the corresponding random variable is called a lattice.

The largest d having this property is said to be a period of X.

If X is lattice and X has a distribution F, then we say that F is lattice.

For instant, X follows a Poisson distribution with the mean λ, then X is lattice with the period 1 because the summation n is equal to 0 to infinity, the probability of X is equal to n is equal to 1. That means d will be 1 whereas in the second example, if P of -- probability of X equal to square root of 2 is equal to 0.5 and the probability of X equal to square root of 3 is 0.5, then you cannot find d such that summation of n is equal to 0 to infinity probability of X equal to n times d is equal to 1. This condition is not satisfied by the second example. Therefore, X is not lattice.

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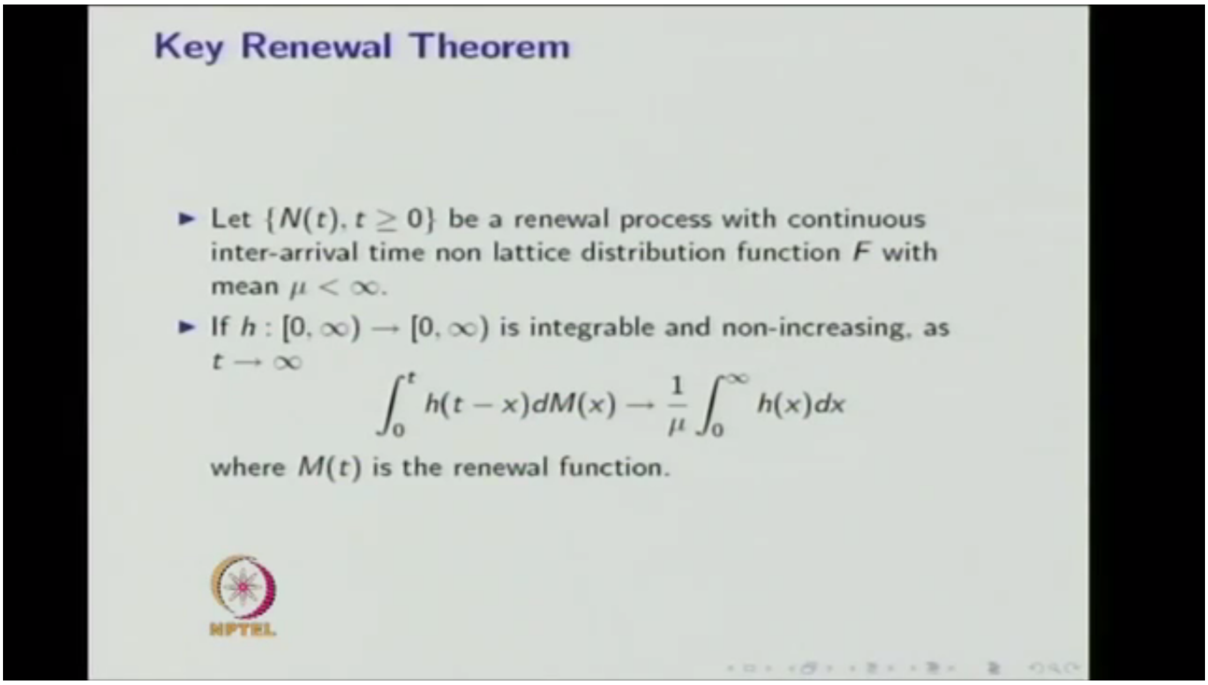

So using lattice we are going to discuss the second Renewal Limit Theorem that is Key Renewal Theorem. Let X(t) be a renewal process with the continuous inter-arrival time non lattice distribution function F with the mean μ. So this is a renewal process and has a inter-arrival time which is the distribution is non lattice and it has a finite mean.

If h is integrable and non increasing function, as t tends to infinity, the integration of 0 to t, h(t - x) integration with respect to the renewal function capital M(t) will tends to 1 divided by μ, the integration 0 to infinity h(x)dx. So this type of integral comes into the -- comes in the integral equation, which we discuss in the renewal equation.

So as t tends to infinity, whenever the integrant is integrable and non-decreasing function, and integration is with respect to the renewal function and the corresponding renewal process has a non lattice distribution for the inter-arrival time, then as t tends to infinity, this integral will be 1 divided by μ times integral 0 to infinity h(x)dx whenever h is integrable and non-decreasing function.

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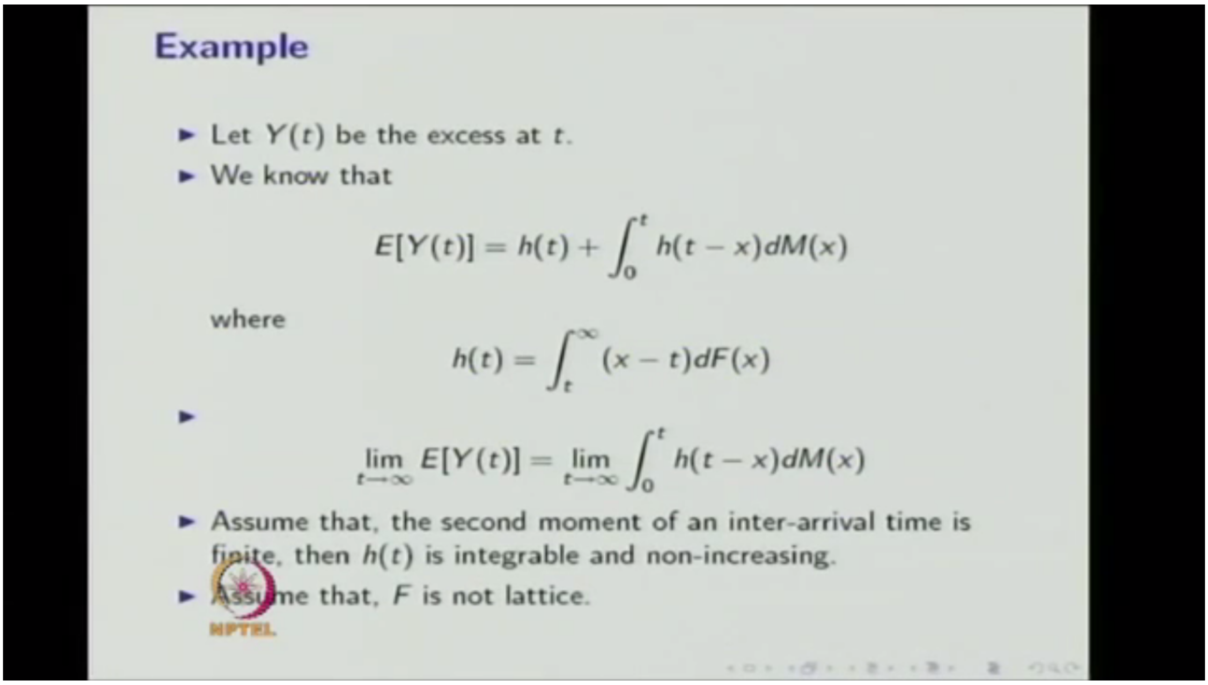

We are going to consider one simple example for that. Let Y(t) be the excess at time t and we know that the expectation of excess is nothing but h(t) plus integration 0 to t, h(t - x) integration with respect to the renewal function where h(t) is nothing but t to infinity (x-t) integration with respect to the distribution function F.

Therefore, limit t tends to infinity, the expectation of excess will be limit t tends to infinity of the integration 0 to t, h(t - x)dM(x) since the first value will be 0 as t tends to infinity.

Assume that the second moment of the inter-arrival time is finite, then h(t) is a integrable and non-increasing function. Therefore, you can use the Key Renewal Theorem by making additional assumption F is non lattice distribution.

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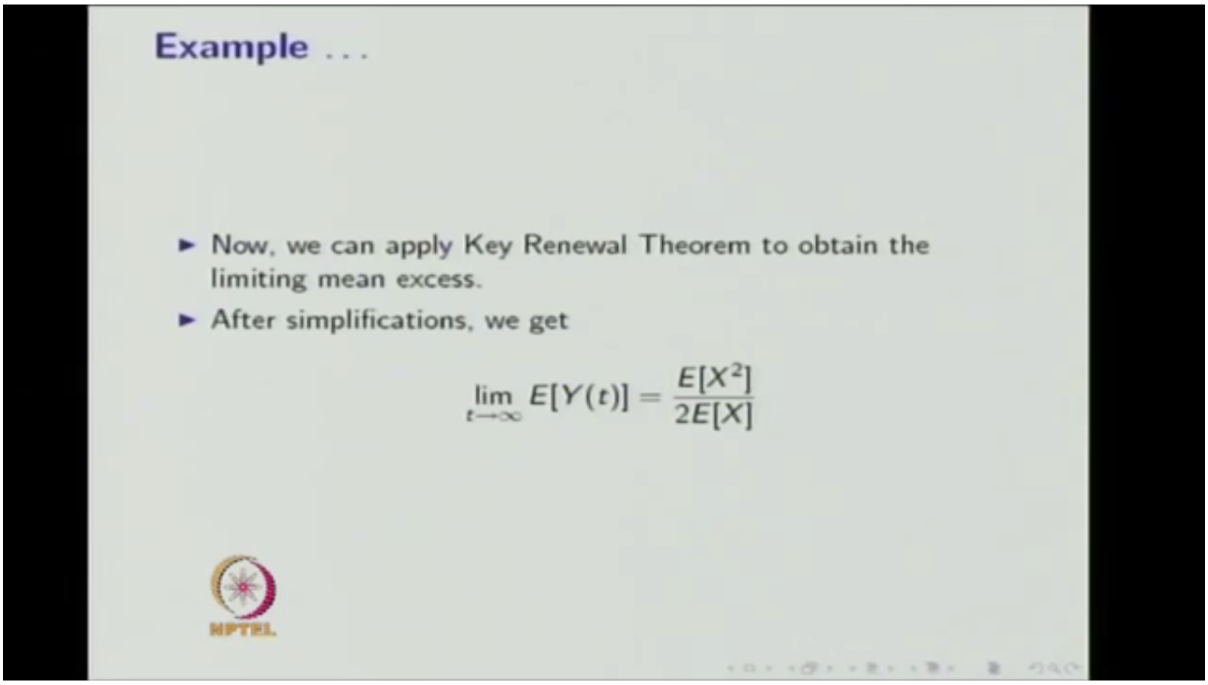

Now we can apply the Key Renewal Theorem to obtain the limiting mean excess. After simplification one can get limit t tends to infinity expectation of Y(t) is same as expectation of X^2 divided by 2 times expectation of X.

So to get this value, we made an assumption, the second moment of inter-arrival time is finite. Hence, h(t) is integrable and non-increasing function and also we made an assumption F is a non lattice distribution.

Hence, we are able to apply the Key Renewal Theorem and getting the limiting mean excess as expectation of X^2 divided by 2 times expectation of X.

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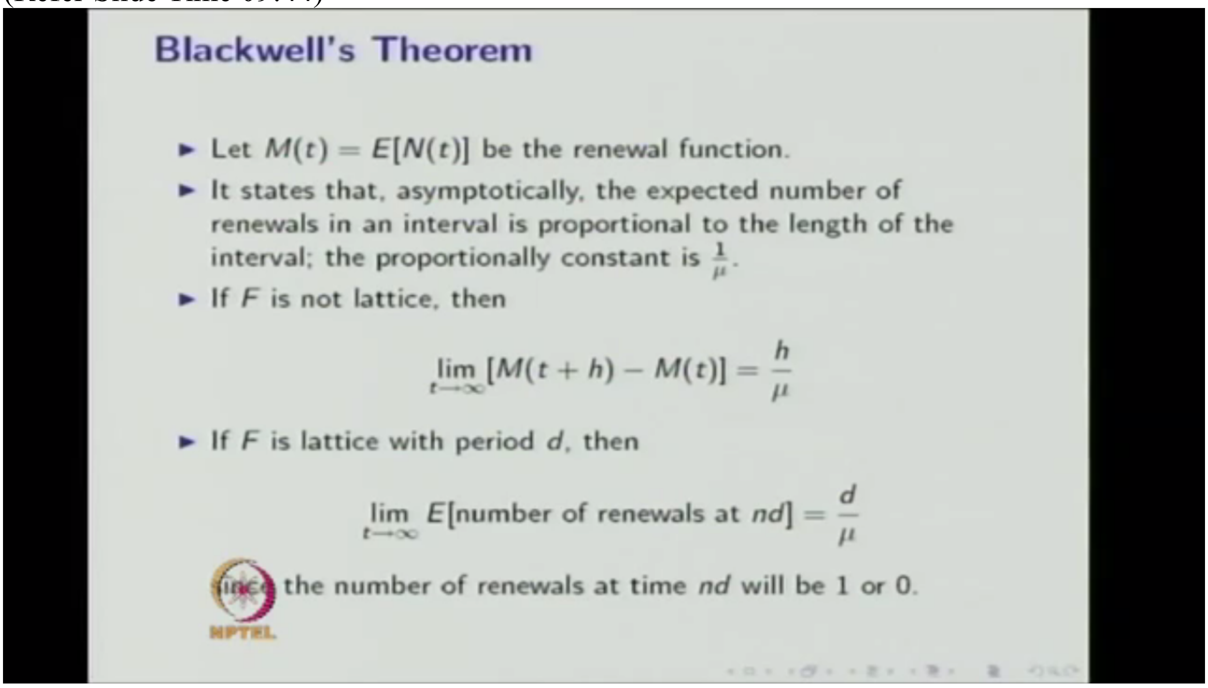

The last Renewal Limit Theorem is Blackwell's Theorem.

Let M(t) be the renewal function and Blackwell theorem says asymptotically the expected number of renewals in the interval is proportional to the length of the interval. That is the proportionality constant is 1 divided μ.

If F is not lattice, then limit t tends to infinity the renewal function evaluated at (t + h) minus renewal function at t is same as the interval length multiplied by 1 by μ. 1 by mu is a proportionality constant. The length is h because you are evaluating the renewal function between the interval t to (t + h).

If F is a lattice distribution with the period d, then the limit t tends to infinity the expected number of renewals at nd will be t divided by μ because the probability of number of renewals at nd will be d by μ and the number of renewals will be either 1 or 0.

Therefore, the limit t tends to infinity expected number of renewals at nd is d by μ.

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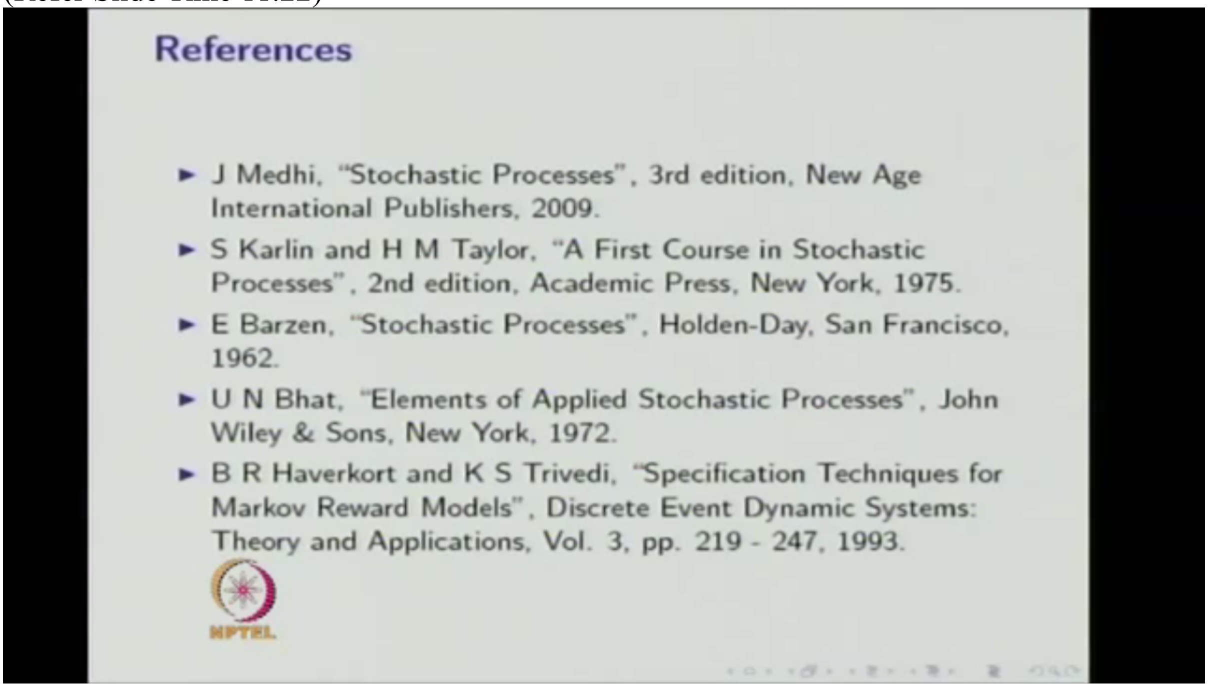

Here is the reference for Lecture 2 in Module 8.

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