(Refer Slide Time 00:00)

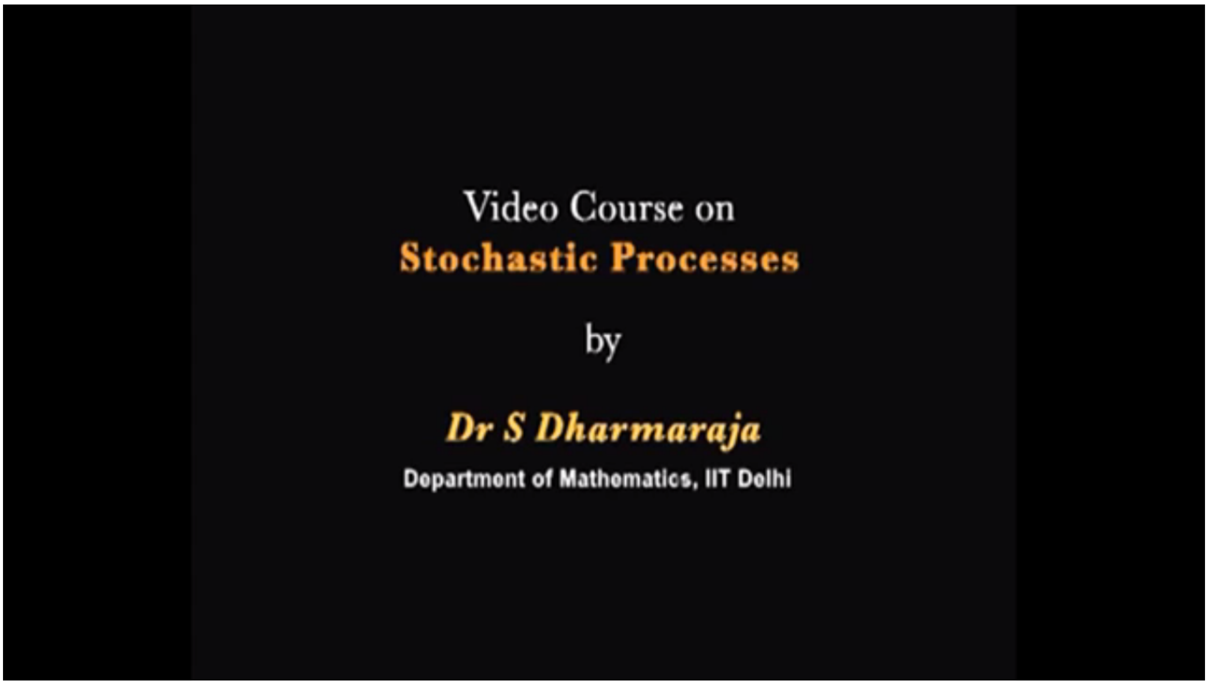
Indian Institute of Technology Delhi

(Refer Slide Time 00:06)

NPTEL

National Programme on
Technology Enhanced Learning

(Refer Slide Time 00:12)

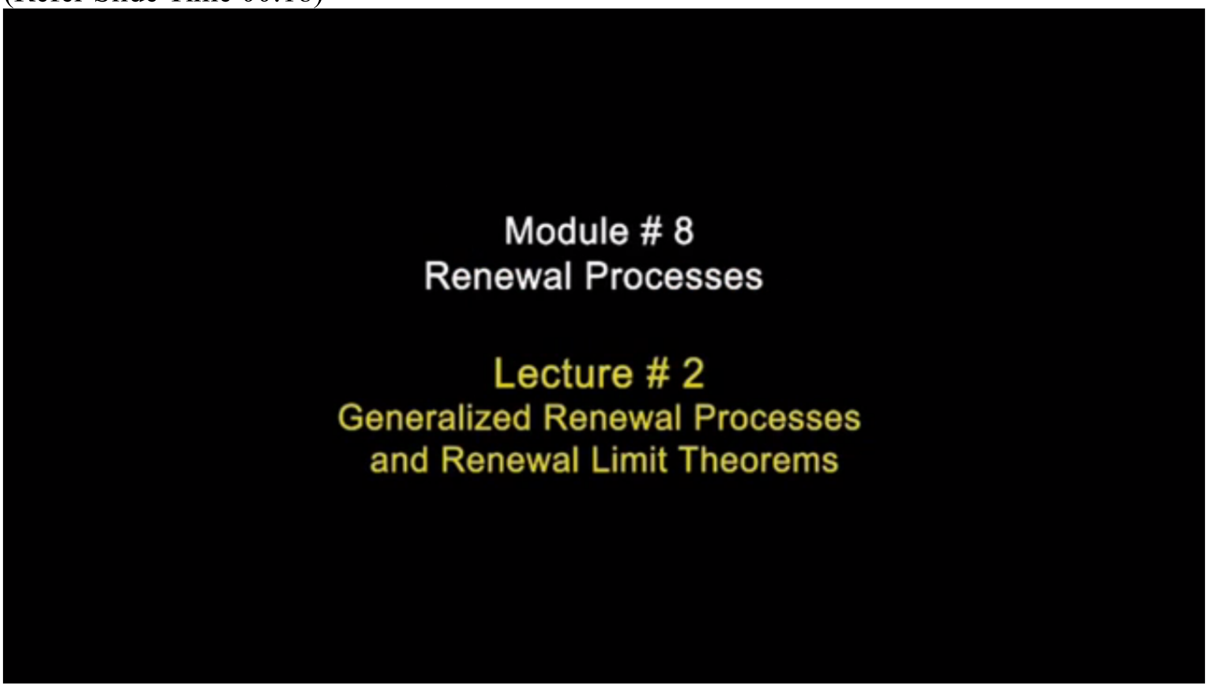


Video Course on
Stochastic Processes

by

Dr. S Dharmaraja
Department of Mathematics, IIT Delhi

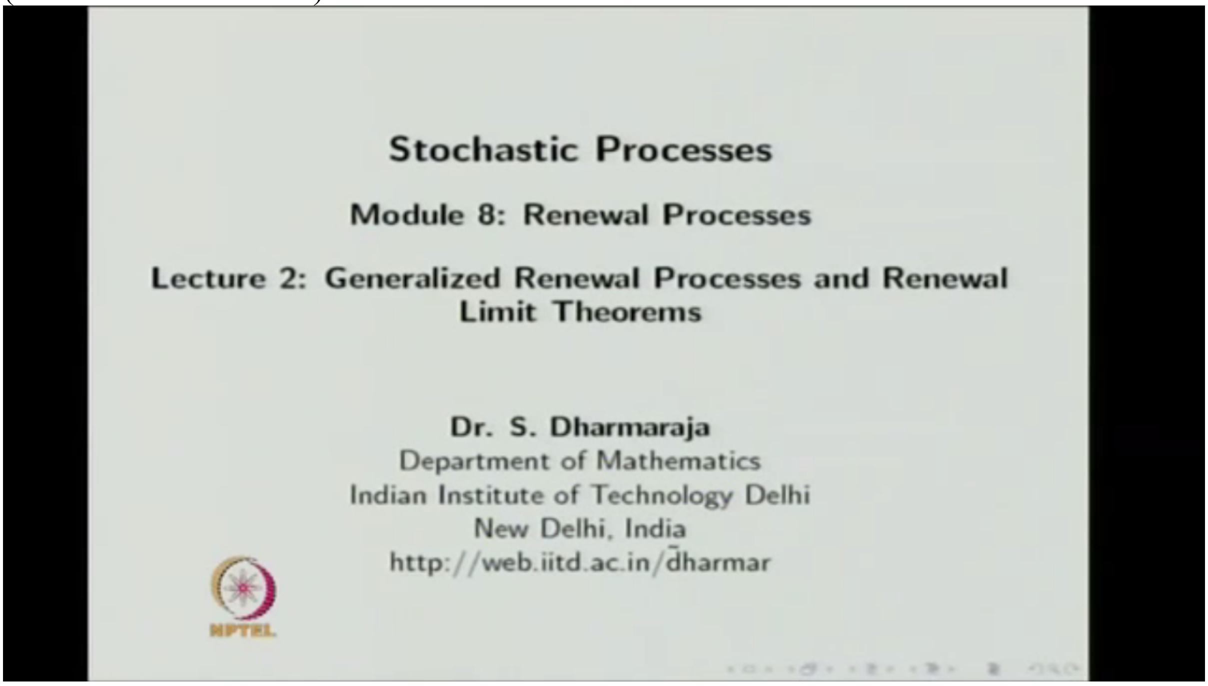
(Refer Slide Time 00:18)



Module # 8
Renewal Processes

Lecture # 2
Generalized Renewal Processes and Renewal Limit Theorems

(Refer Slide Time 00:24)

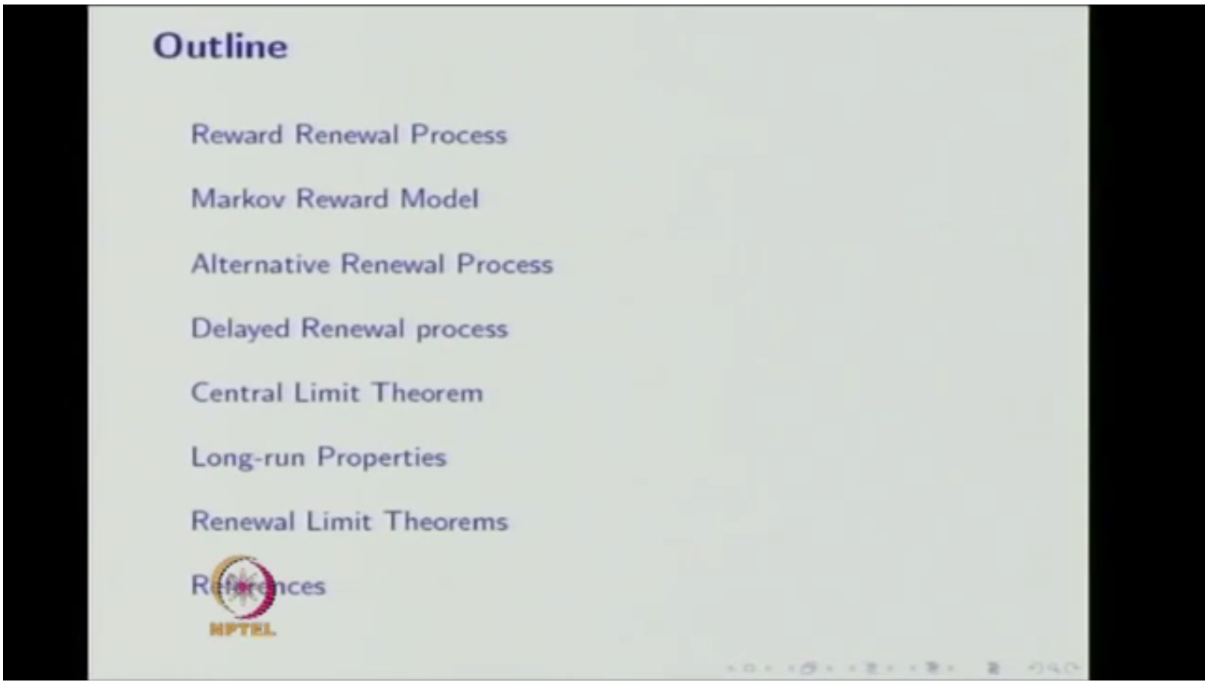


[Music]

This is Stochastic Processes, Module 8: Renewal Processes, Lecture 2: Generalized Renewal Processes and Renewal Limit Theorems.

In the Lecture 1, we have discussed the renewal processes definition and its properties. Followed by renewal process definition, we have discussed the renewal function, and then we have discussed the renewal equation, and also we have seen few examples in the Lecture 1.

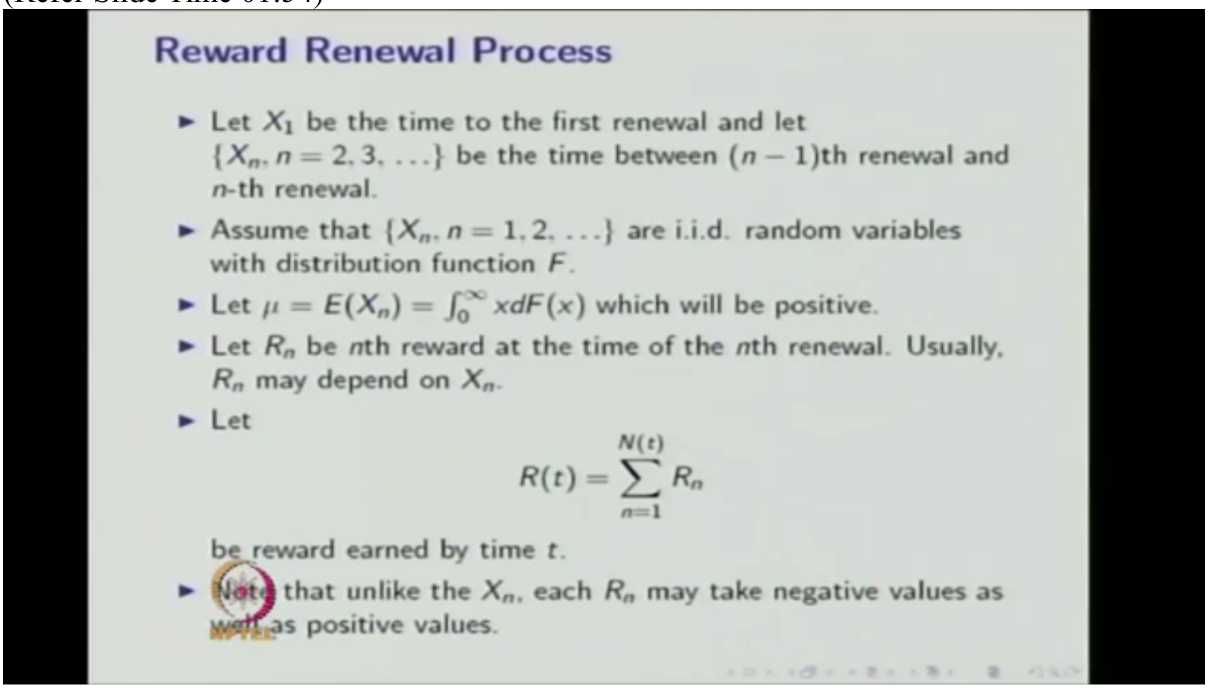
(Refer Slide Time 01:07)



In the Lecture 2, we are planning to discuss the reward renewal process. As a special case, we are going to discuss the Markov reward models. Then we are going to discuss two different renewal processes, that is alternative renewal process and delayed renewal process. With this we are completing the generalized renewal processes.

The second half, we are going to discuss the Central Limit Theorem, Long-run properties and three important renewal limit theorems.

(Refer Slide Time 01:54)



What is a reward renewal process?

Let X_1 be the time to the first renewal and let X_n be the time between $(n-1)^{th}$ renewal and n^{th} renewal. The same definition which we have used for the renewal process and also you assume that X_i's are i.i.d. random variables with the CDF capital F of x.

Since X_i's are the inter-arrival times, the mean will be a positive. Mean exists and it will be positive.

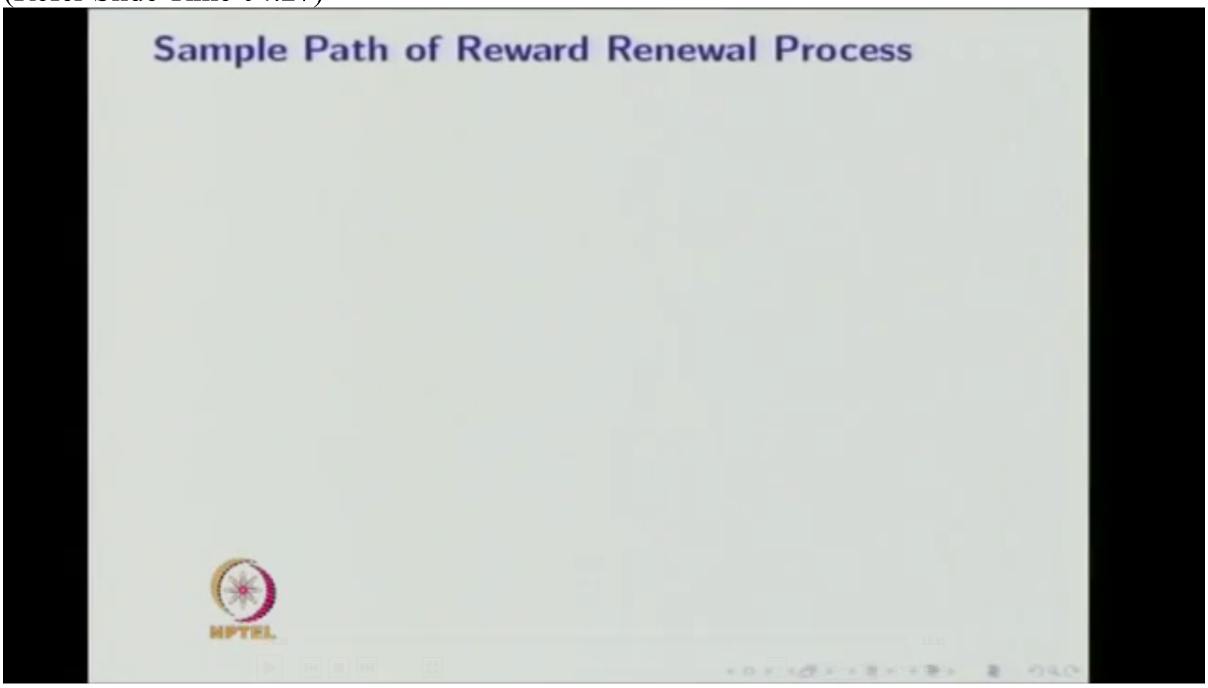
Let R_n be the n^{th} reward at time t of the n^{th} renewal. For each renewal, you are attaching the reward at the time of the renewal. Usually, R_n may depend on X_n.

Now we are defining a new random variable R(t) function of t, that is nothing but the summation of R_n's where n is running from 1 to N(t) where N(t) is a renewal process. The collection of R(t) for t greater than or equal to zero will be called it as a Reward Renewal Process. That R(t) is nothing but a reward earned by time t.

So from the renewal process, we are attaching the reward for each renewal and by defining R(t) is equal to summation of n is equal to 1 to N(t) R_n, that will be call it as a Reward Renewal Process.

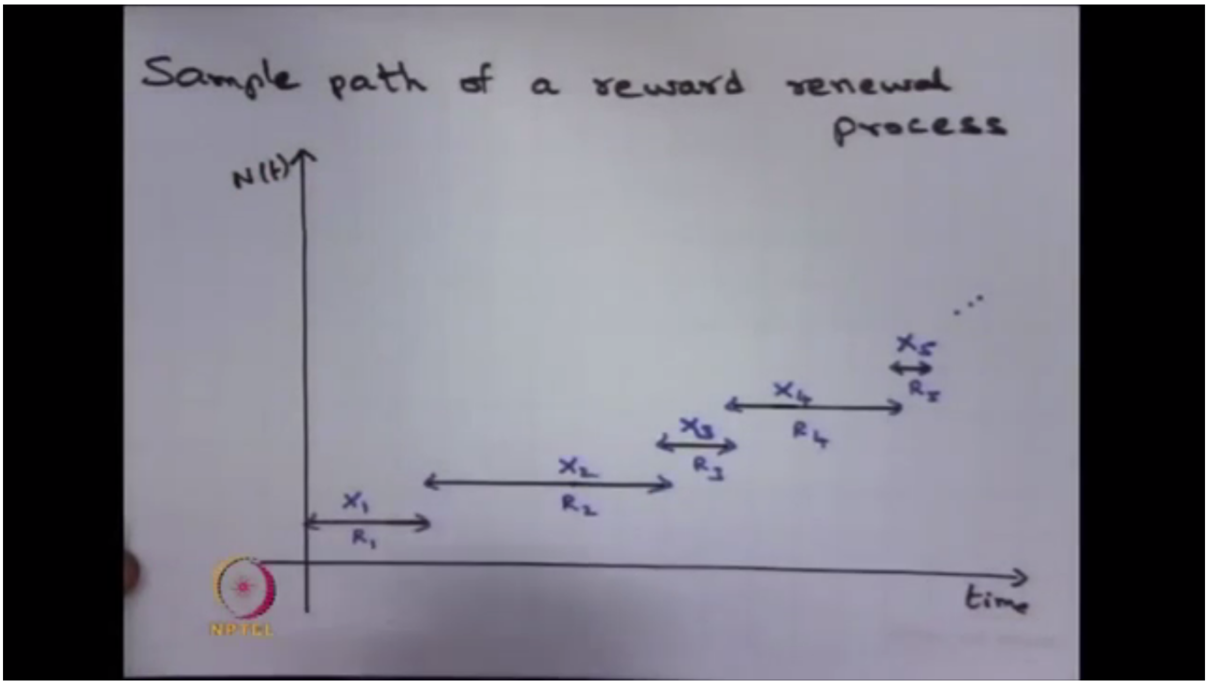
Note that unlike the X_n, each R_n may take a negative values as well as the positive values. The X_i's are nothing but the inter-arrival time of the renewals whereas the rewards may take negative values as well as positive values.

(Refer Slide Time 04:27)



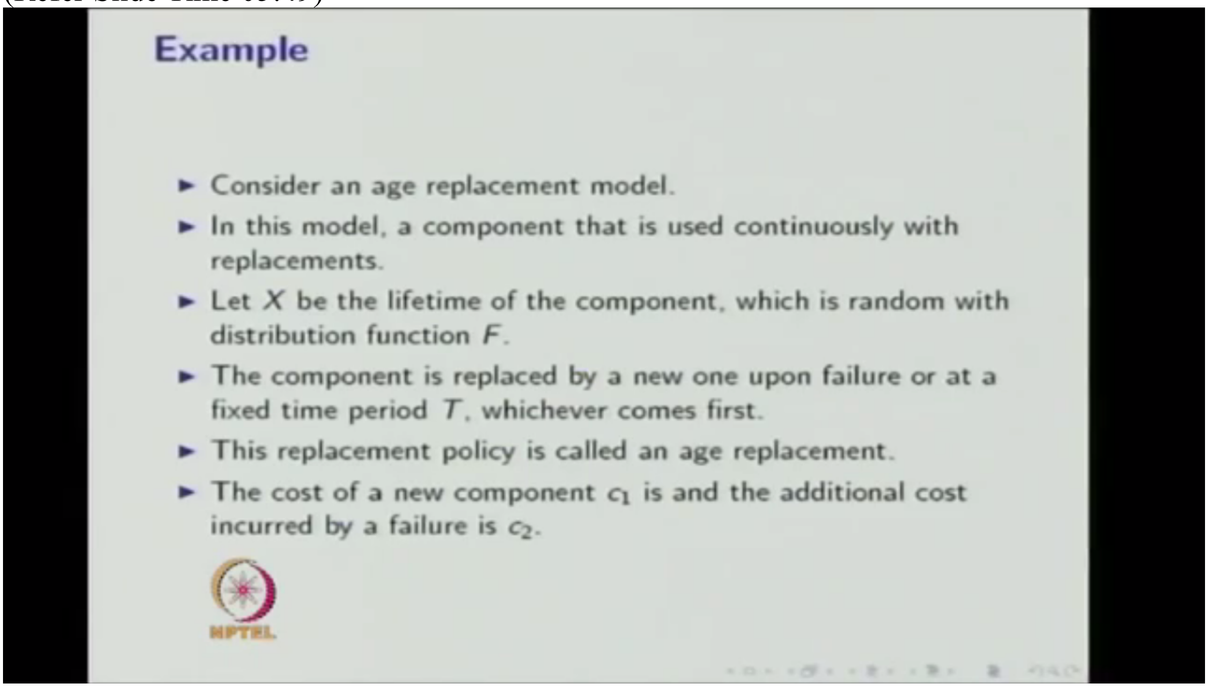
See the sample path of a reward renewal process. So the x-axis is the time. The y-axis is N(t).

(Refer Slide Time 04:47)



And these are all the time points in which the renewals takes place. The first renewal takes place, second renewal and so on. So this is the inter-arrival time and attaching the reward R_i to the each X_n and these R_i's may be negative or positive, then the collection of R_i's with this form R_t is equal to summation n is equal to 1 to N(t) R_n will be the reward renewal process. So it is very difficult to show the sample path of R(t). So here we are showing the sample path of renewal process with the rewards attached with the each X_i's.

(Refer Slide Time 05:49)



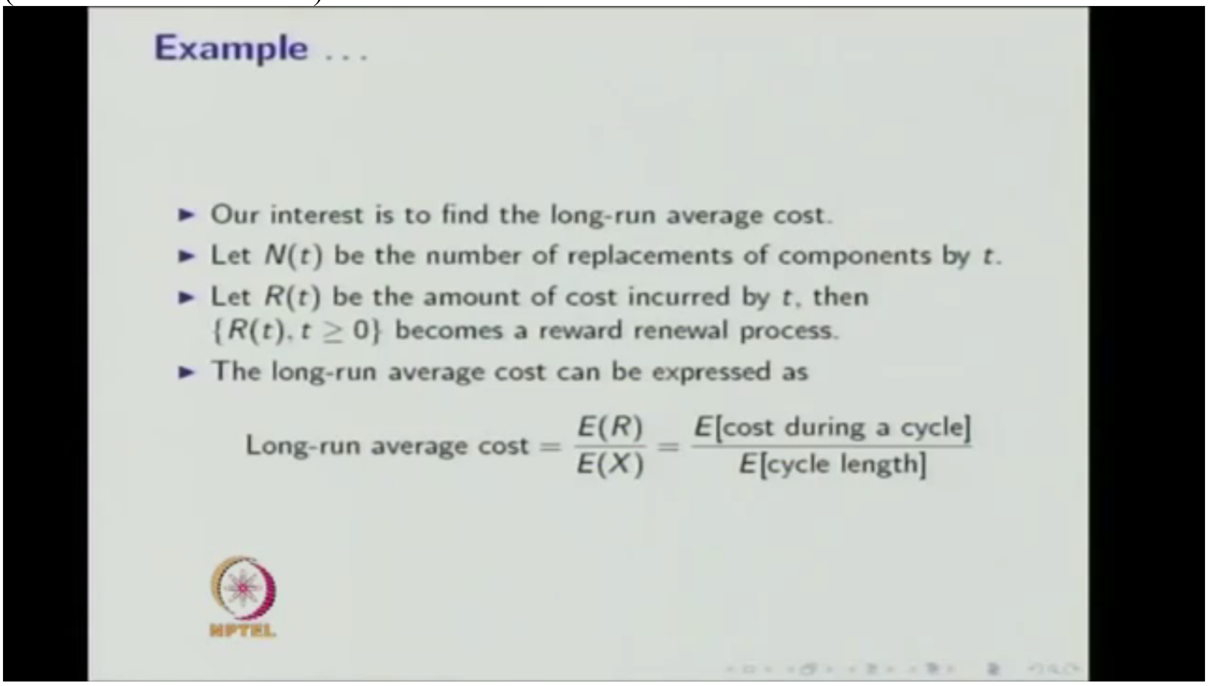
Now we are moving into the simple example of reward renewal process. Consider an age replacement model. In this model, a component that is used continuously with replacement -- replacements.

Let X be the lifetime of the component, which is random with the distribution function F. The component is replaced by a new one upon failure or at a fixed time period capital T, whichever comes first.

The replacement policy is called an age replacement because the component is replaced by a new one upon failure or at a fixed time period T.

The cost of a new component is c_1 and the additional cost is incurred by a failure is c_2.

(Refer Slide Time 07:06)



Our interest is to find the long-run average cost.

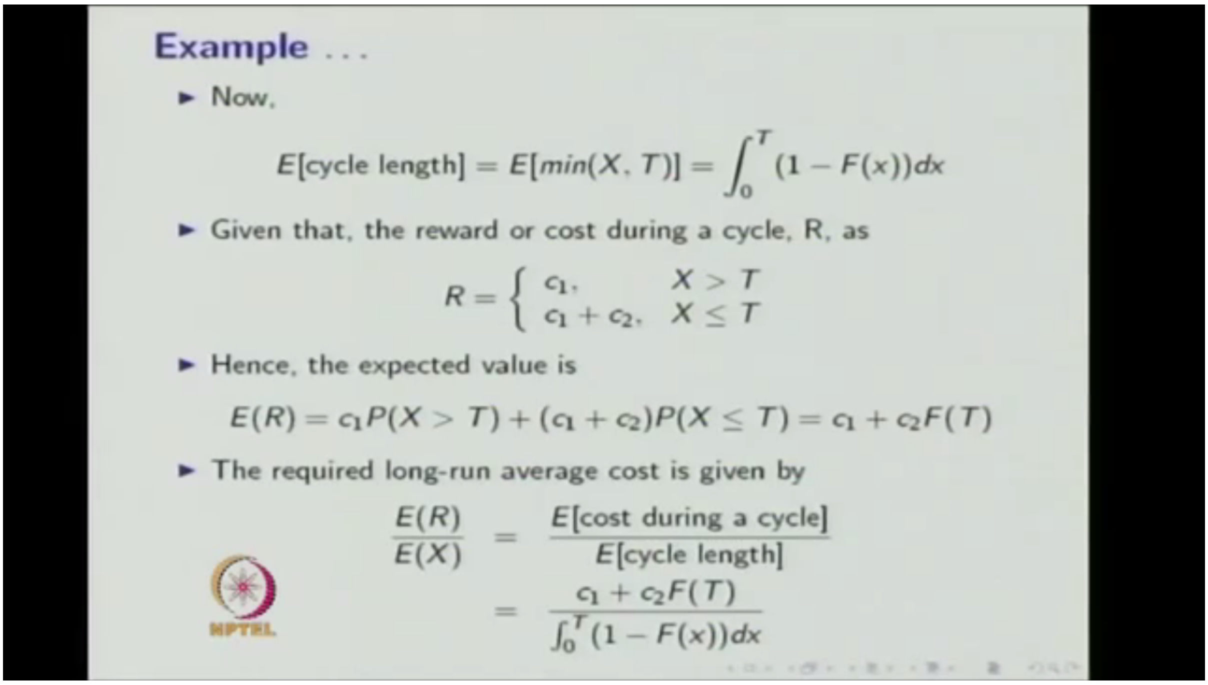
Let N(t) be the number of replacements of components by time. N(t) is a renewal process.

Let R(t) be the amount of cost incurred by time t. So this is a reward renewal process.

So the long-run average cost can be expressed as the ratio of expectation of R by expectation of X that we are going to conclude later. Now we are using the long-run average cost is the expectation of R divided by expectation of X.

Expectation of R is nothing but expectation of cost during a cycle and expectation of X is nothing but expectation of the cycle length.

(Refer Slide Time 08:08)



Now one can find the expectation of cycle length, that is nothing but the expectation of the cycle length is a random variable, which is nothing but the minimum of X or T.

Here the component is replaces either upon a failure or at each capital T. Time between replacement is called a cycle.

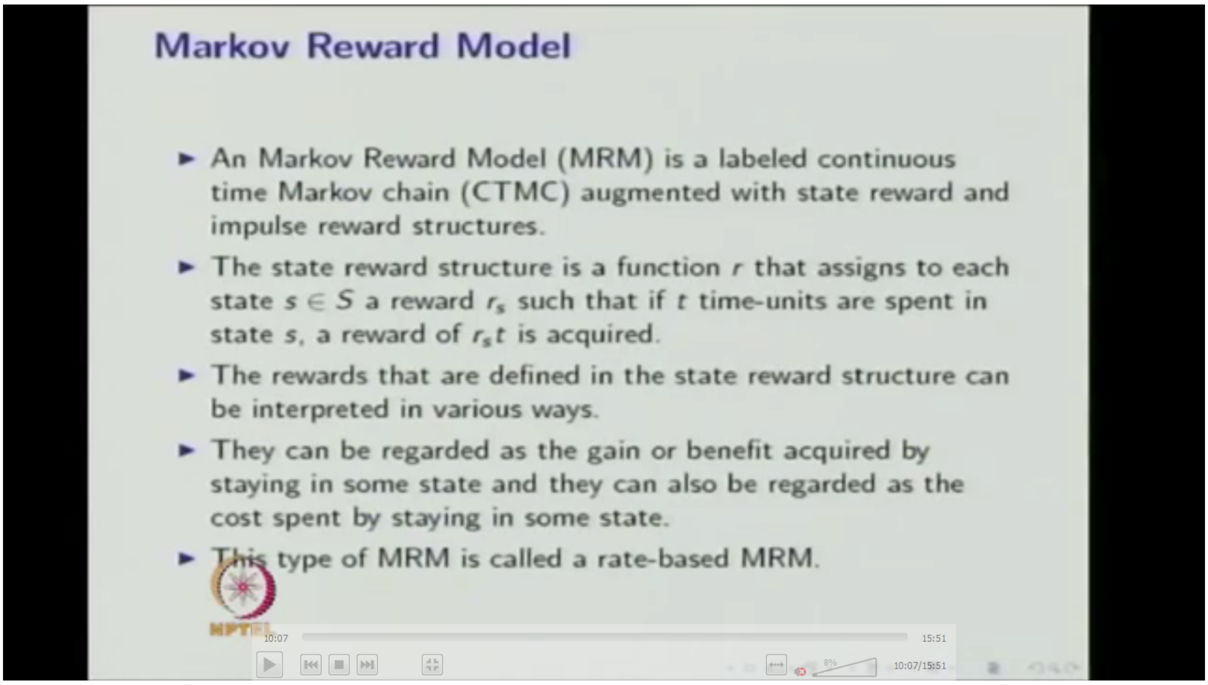
That is nothing but the integration 0 to t (1 - F(x))dx.

Given that, the reward or cost during the cycle R, therefore, R will be c_1 if X is greater than T, if it fails after T. If it fails before T, then there is an additional cost c_2. Therefore, the cost will be $c_1 + c_2$ if failure occurs before the fixed time T.

Hence, the expected value that is the expected cost is either c_1 with the probability X is greater than T or c_2 plus -- $c_1 + c_2$ with the probability X is less than or equal to T. So that is nothing but $c_1 + c_2$ times F(T).

Hence, the long-run average cost is nothing but expectation of R divided by expectation of X. That is nothing but expectation of cost during a cycle divided by expectation of cycle length. Substitute the values. You will get the long-run average cost.

(Refer Slide Time 10:07)

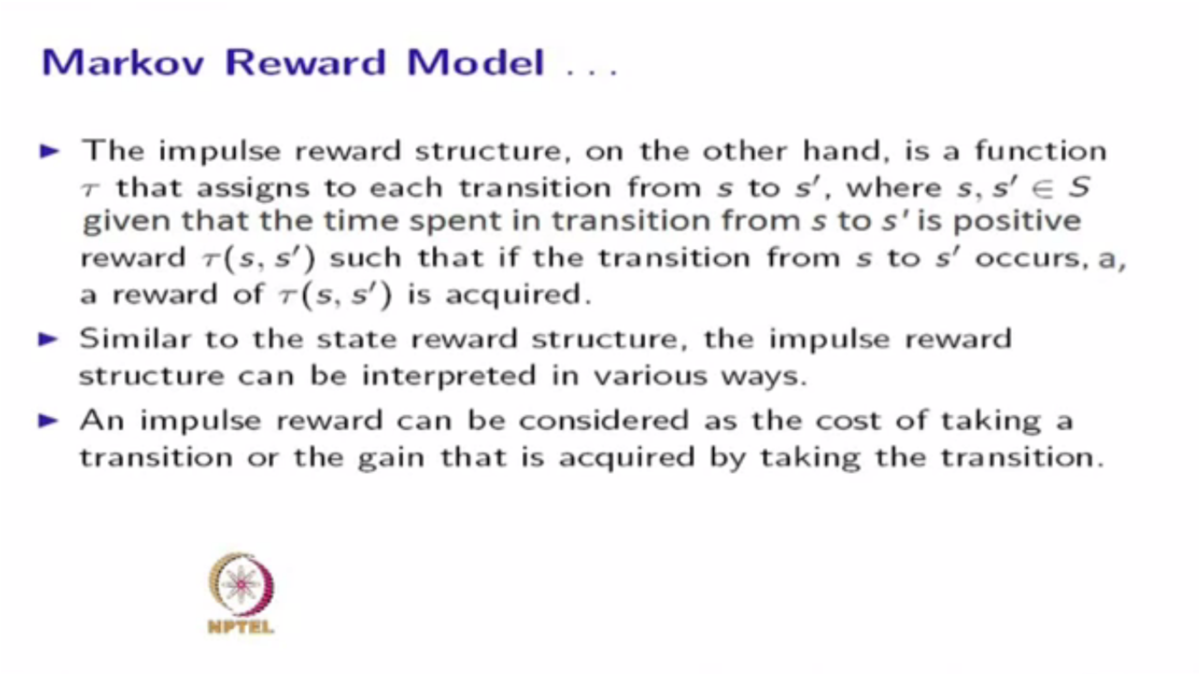


Now we are moving into the special case of a reward renewal process that is a Markov Reward Model. A Markov Reward Model is a labeled continuous-time Markov chain augmented with a state reward and impulse reward structures.

The state reward structure is a function r that assigns to each state a reward r_s such that if t time units are spent in the state s, a reward of r_s times t is acquired.

The rewards that are defined in the state reward structure can be interpreted in various ways. They can be recorded as the gain or benefit acquired by staying in some state and they can also be regarded as the cost spent by staying in some state, and this type of Markov reward model is called a rate-based Markov reward model.

(Refer Slide Time 11:33)

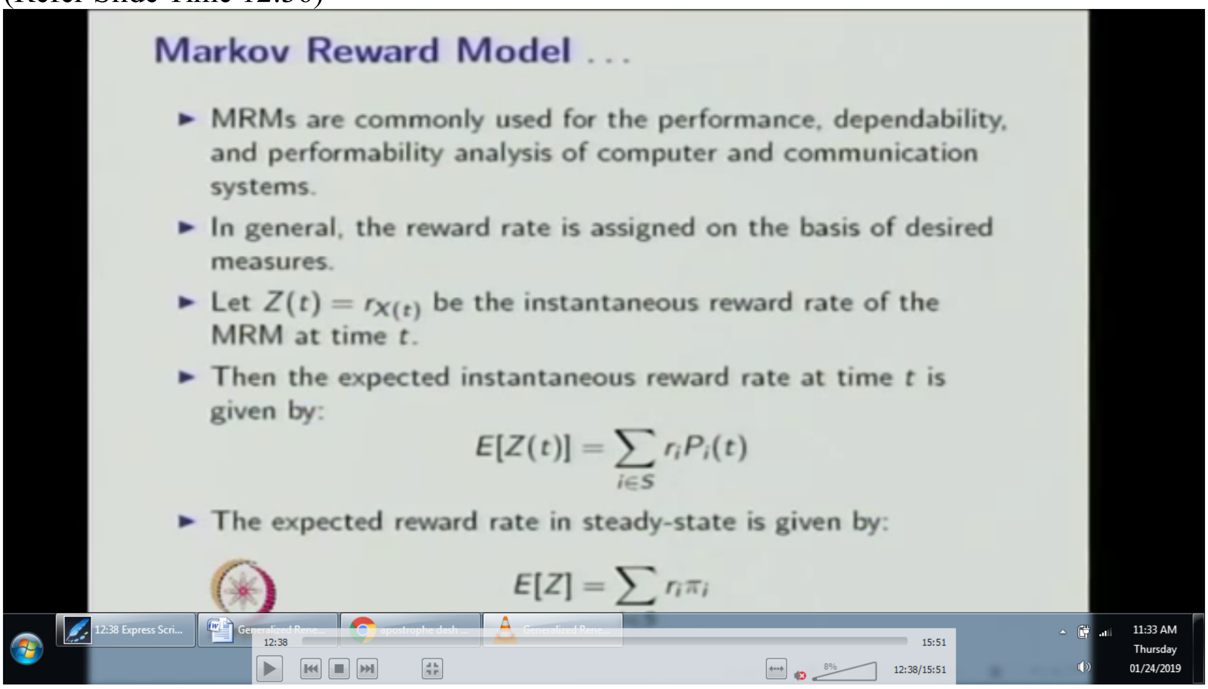


The impulse reward structure, on the other hand, is a function of -- function τ that assigns to each transition from s to s' where s and s' are belonging to the state space S, and the spending time in transition of s to s' is positive, and the reward τ of s to s' such that if the transition from s to s' occurs, a reward of τ(s, s') is acquired.

Similar to the state reward structure, the impulse reward structure can be interpreted in various ways.

An impulse reward can be considered as the cost of taking a transition or the gain that is acquired by taking the transition.

(Refer Slide Time 12:36)



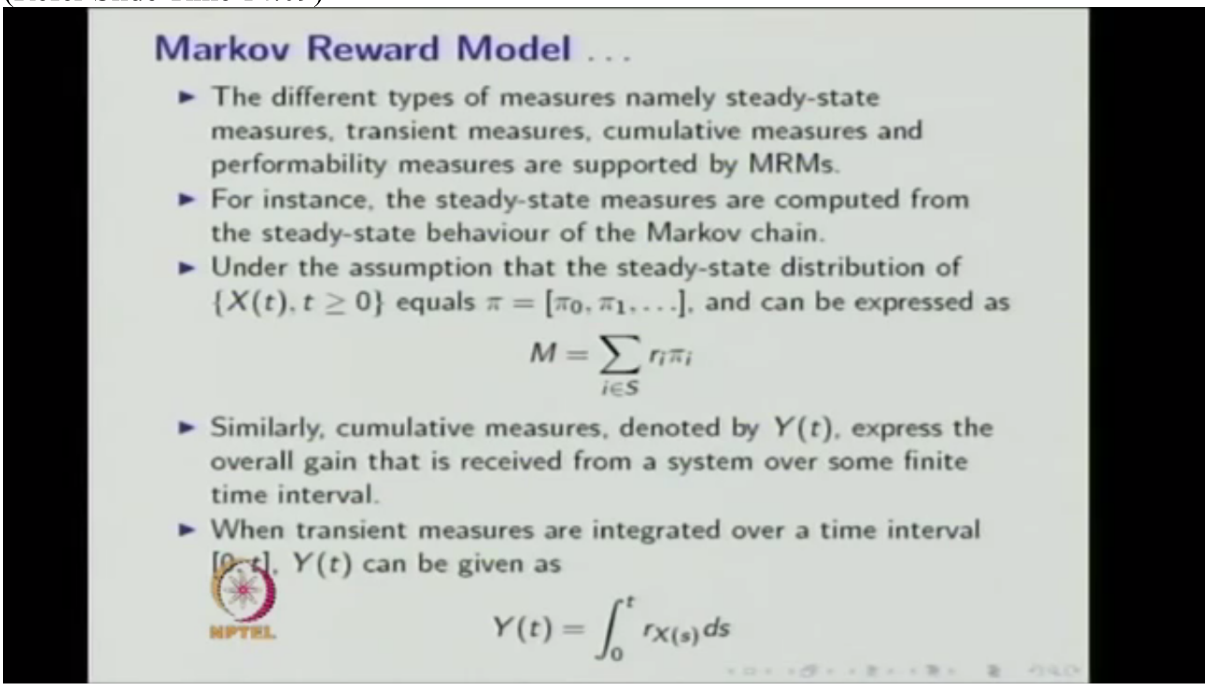
Markov reward models are commonly used for the performance, dependability and performability analysis of computer and communication systems.

In general, the reward rate is assigned on the basis of desired measures.

Let Z(t) is nothing but $r_{X(t)}$ be the instantaneous reward rate of the Markov reward model at time t. Then the expected instantaneous reward rate at time t is given by: expectation of Z(t), that is nothing but summation of r_i $P_i(t)$.

The expected reward rate in steady state is nothing but expectation of Z is summation of r_i π_i. Suppose in the perform -- availability model, our interest is to find out the availability of the system, then one can assign the rewards for the up states are 1 and rewards for the down states are zeros. Then the expected reward rate will be the availability for the system by making a summation of r_i's and probability of being in those states.

(Refer Slide Time 14:09)



The different types of measures namely steady-state measures, transient measures, cumulative measures and performability measures are supported by Markov reward models.

For instant, steady-state measures are computed from the steady-state behavior of the Markov chain. So for the availability model, if you know the steady-state behavior of the Markov chain such as a steady-state probability of being in this system, then by assigning the rewards 1 to the up states and 0s to the down states, one can get the steady-state availability of the system.

Similarly, reverting the rewards, you can get the steady-state unavailability of the system also.

Under the assumption that the steady-state distribution of X(t), one can find the steady-state measures by multiplying r_i's with the π_i's where i is -- i belonging to the state space S.

Similarly, the cumulative measures denoted by Y(t) express the overall gain that is received from a system over some finite time interval. When transient measures are integrated over the time interval 0 to t, then Y(t) can be given as integration 0 to t $r_{X(s)}$ ds.