Module # 8
Renewal Processes

Renewal Function and Renewal Equation contd.

by

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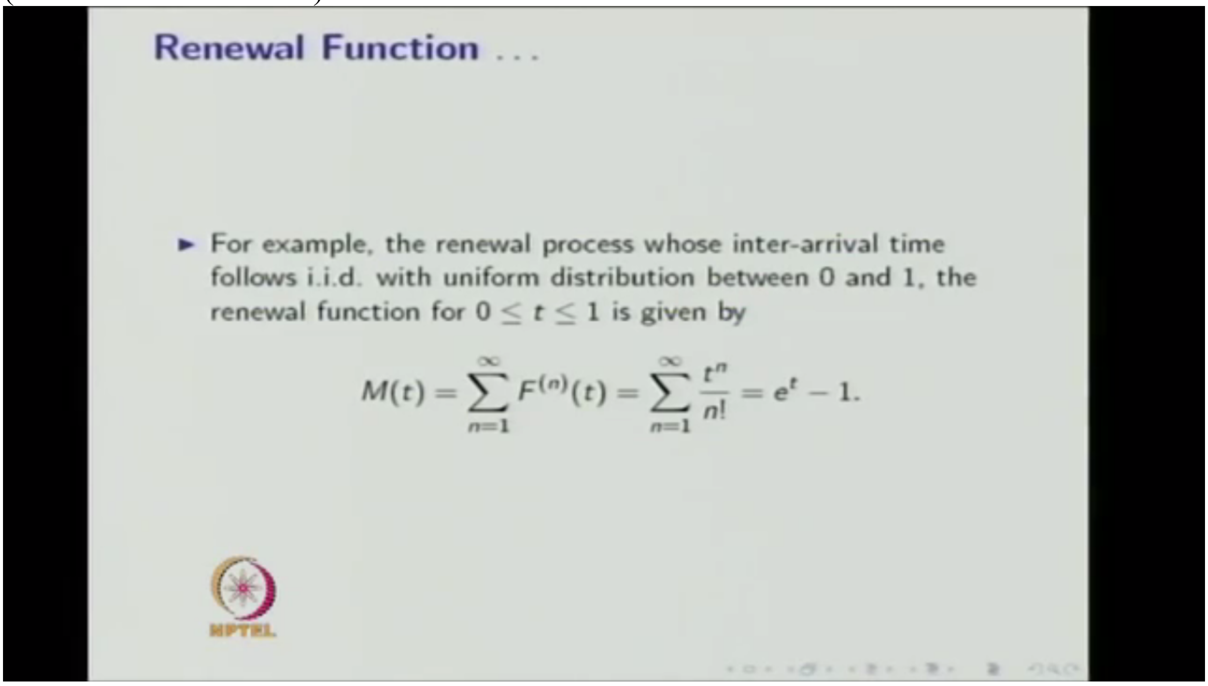

For example, the renewal process whose -- whose inter-arrival time follows IID with the uniform distribution between the interval 0 to 1, the renewal function is e^t - 1 because the inter-arrival time is uniformly distributed between the interval 0 to 1. Therefore, you know what is the $F^{(n)}(t)$, n-fold convolution of F. Substitute that. That is t^n divided by n factorial. After simplification, you will get e^t - 1. That is the renewal function for a renewal process whose inter-arrival times are IID random variable with uniform distribution between the interval 0 and 1.

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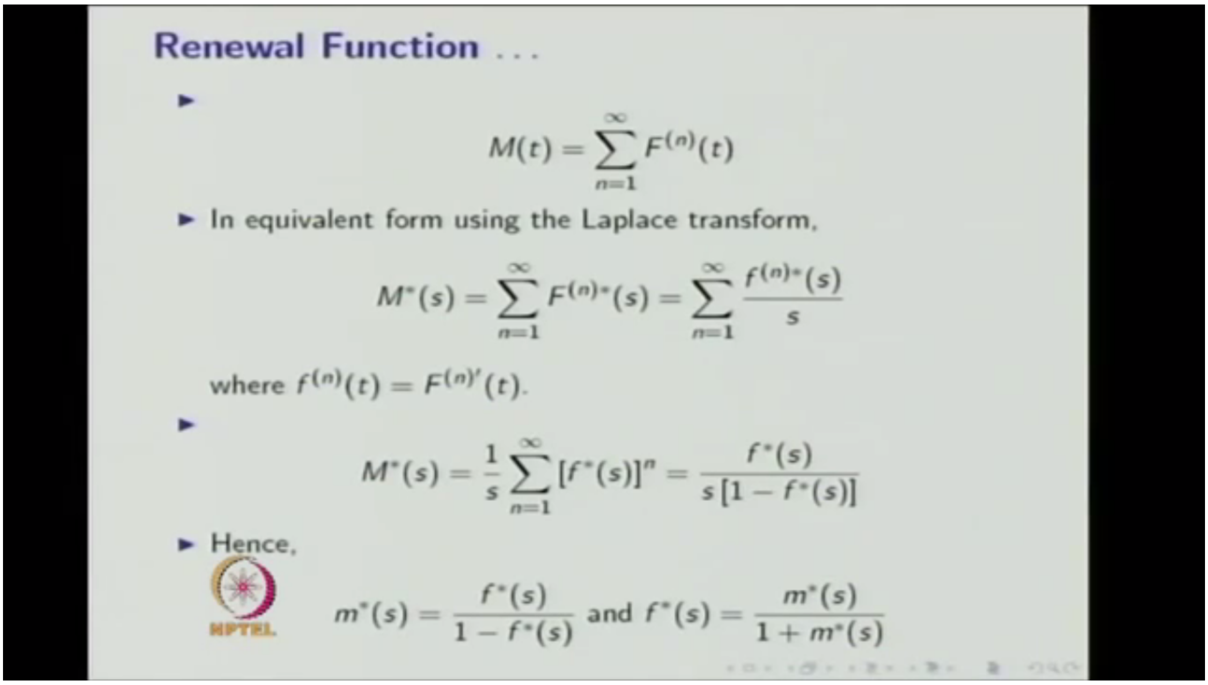

Now we are going to relate the probability density function of inter-arrival time -- inter arrival time with the renewal density function and renewal function in Laplace form. We know that the renewal function is nothing but the summation n is equal to 1 to infinity $F^{(n)}(t)$, n-fold convolution of F where F is the CDF of inter-arrival times.

In equivalent form using the Laplace transform, the $M^*(s)$ that is nothing but the Laplace transform of M(t) renewal function. That is nothing but summation n is equal to 1 to infinity, the n-fold convolution of F in Laplace, in Laplace transform as a function of s. That is same as the small f of n-fold convolution in Laplace form divided by s.

We know that the derivative of n-fold convolution of CDF is nothing but the n-fold convolution of the probability density function. So this can be written as a 1/s summation n is equal to 1 to infinity the Laplace transform of probability density function over power n. You can simplify. After that you can find the renewal density function in Laplace. That is nothing but f of s divided by 1 minus f of s where f of s is nothing but the probability density function in Laplace transform. So from the same equation you can get the f of s in terms of m of s also where m of s is the renewal density function in Laplace.

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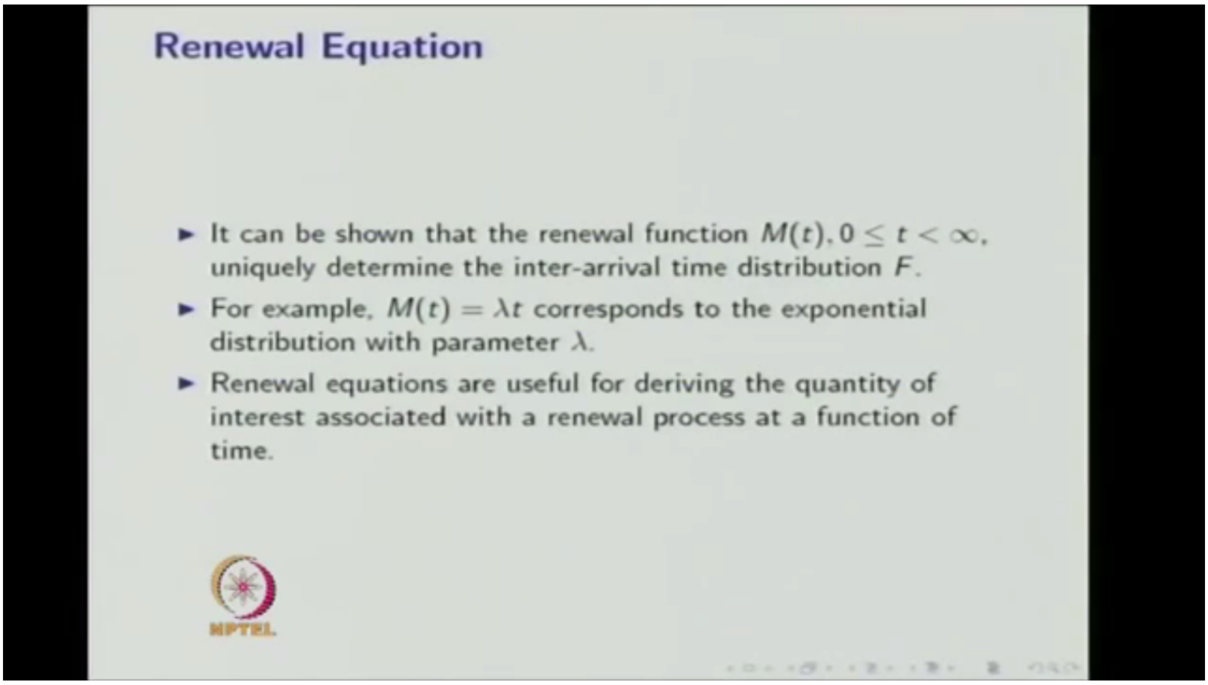

Now we are moving into the next concept that is called renewal equation. It can be shown that the renewal function M(t) uniquely determine the inter-arrival time distribution F.

For example, M(t) is equal to λT corresponds to the exponential distribution.

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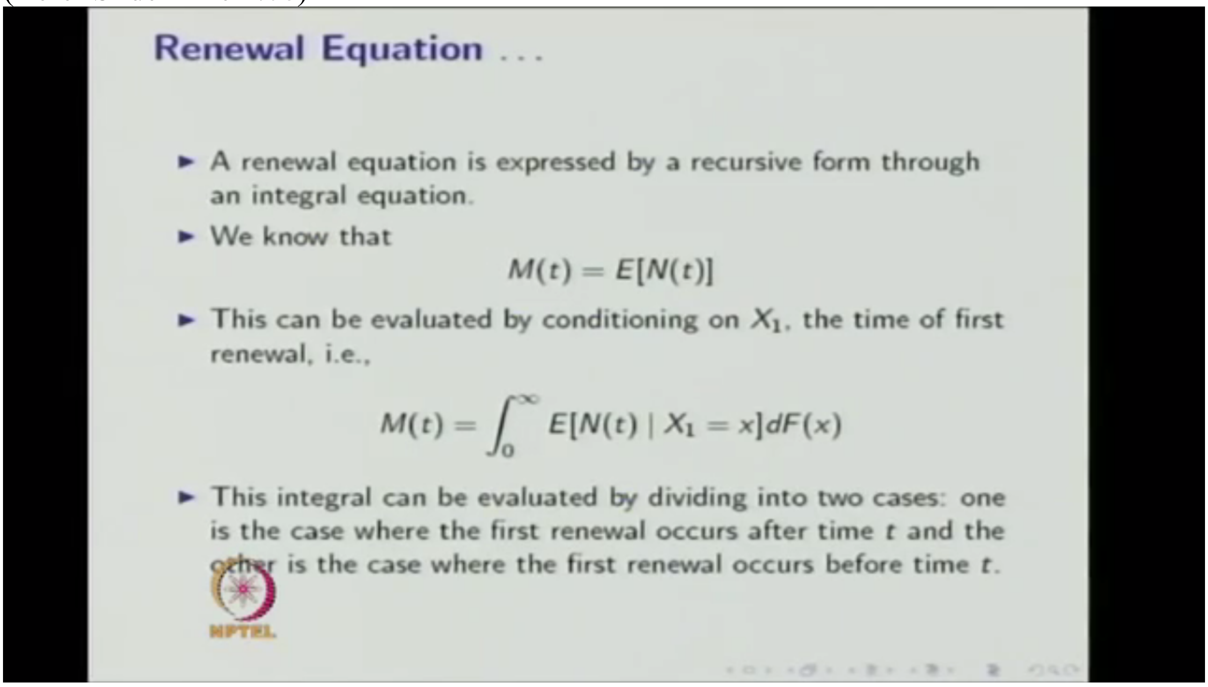

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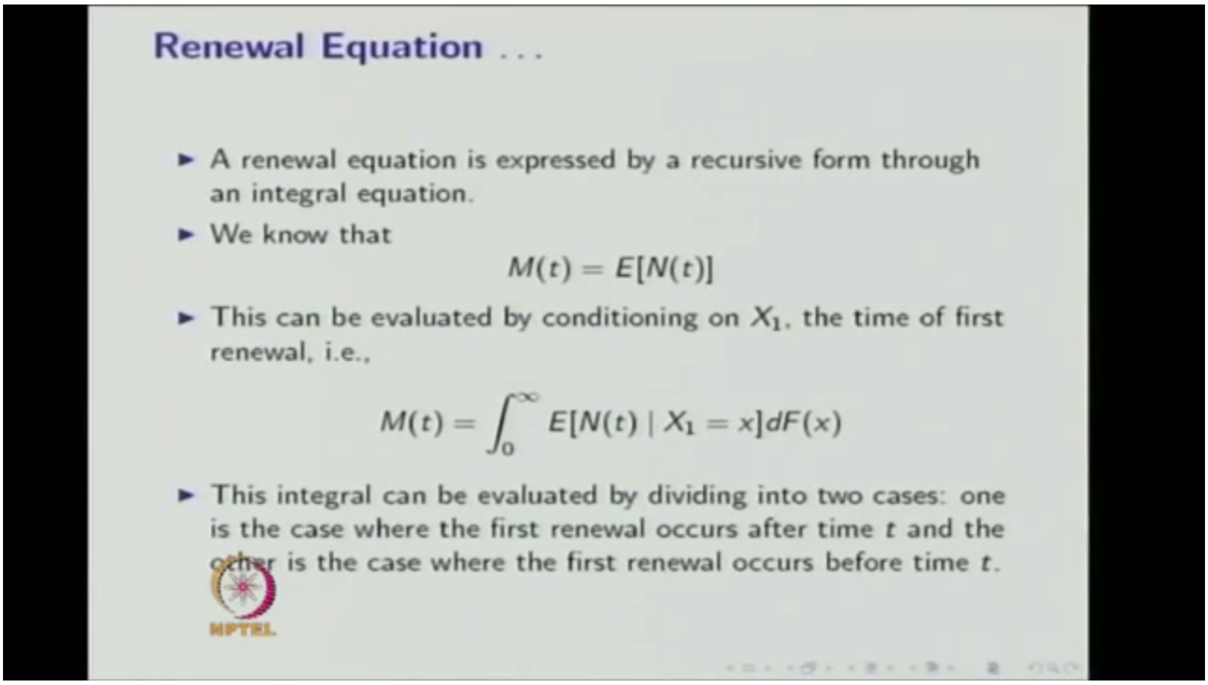

Now we are moving into the next concept called renewal equation. A renewal equation is expressed by a recursive form through an integral equation.

We know that the renewal function M(t) is expectation of N(t). This can be evaluated by conditioning on X_1. That X_1 is nothing but the time of first renewal. Hence, the renewal function M(t) is equal to integration 0 to infinity expectation of N(t) given X_1 is equal to x integration with respect to the CDF of inter-arrival time distribution.

This integral can be evaluated by dividing into two cases. One is the case where the first renewal occurs after time t and the other is the case where the first renewal occurs before time t.

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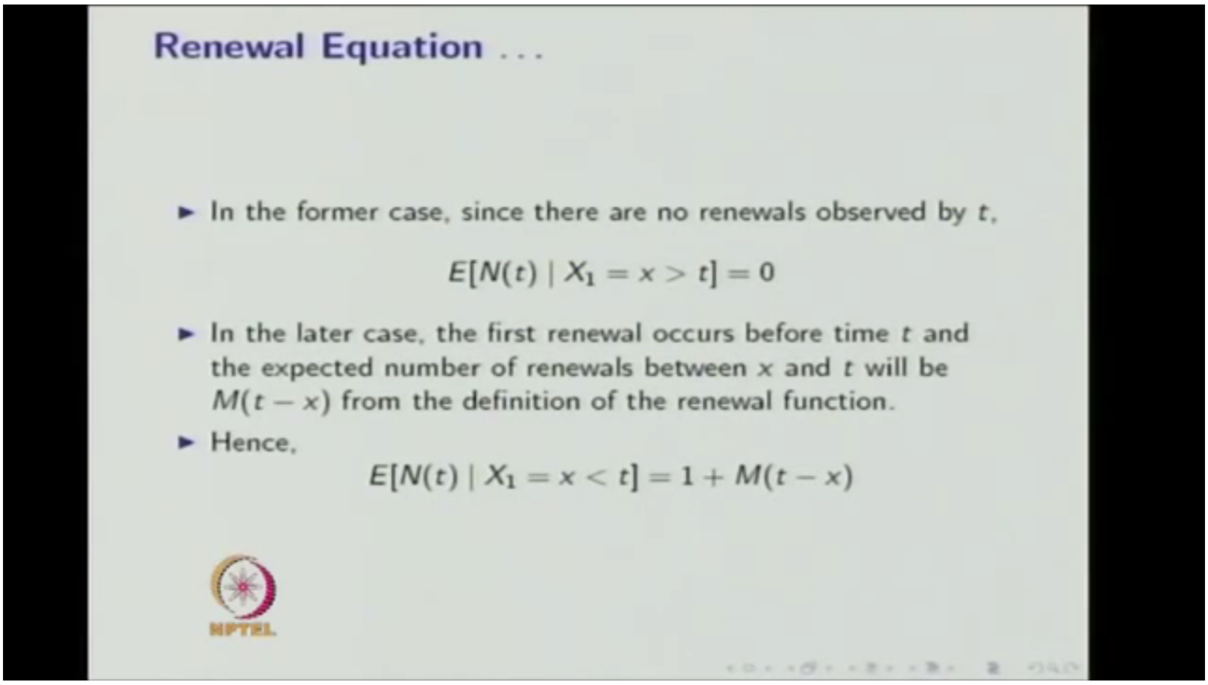

In the former case, since there are no renewals observed by time t, the conditional expectation will be 0. In the later case, the first renewal occurs before time t and the expected number of renewals between x and t will be the renewal function M(t - x) from the definition of renewal function.

Hence, the conditional expectation is one renewal already takes place before time t, therefore, 1 plus the expected number of renewals between the interval x to t will be M(t - x). Therefore, the conditional expectation is 1 + M(t - x).

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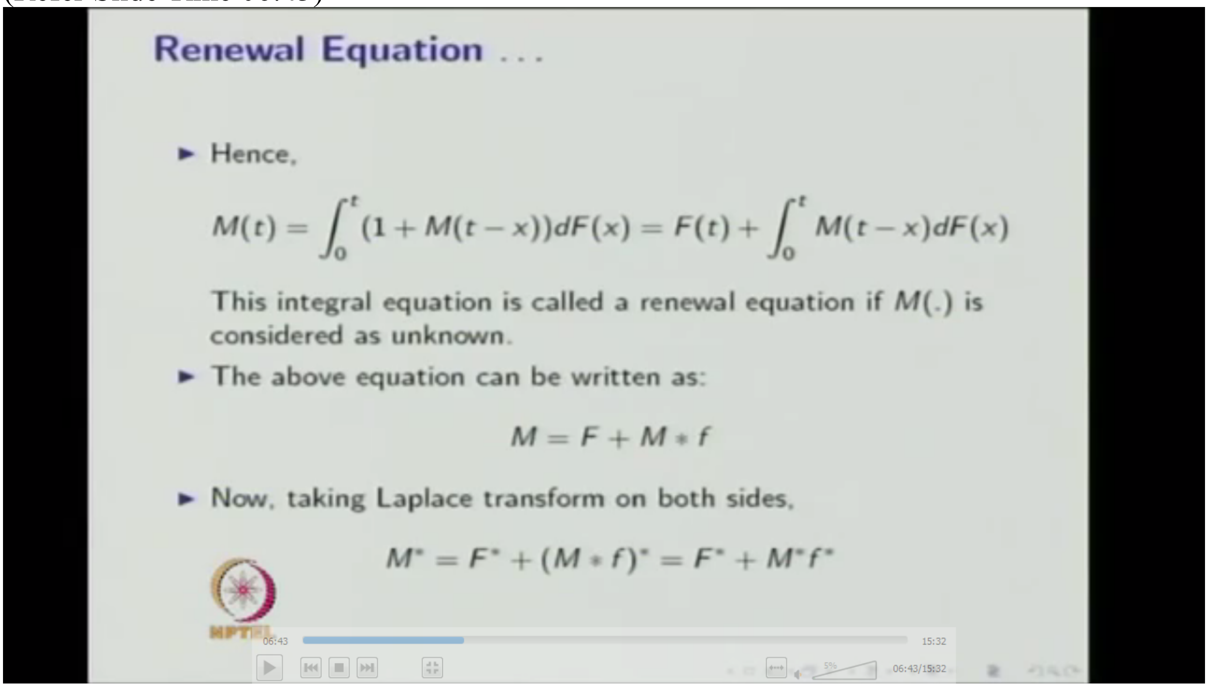

Therefore, the M(t), that conditional expectation integration will be splitted into two. Hence, the one integration will be 0 because the conditional expectation is 0. The other integration will be 0 to t 1 + M(t - x) dF(x). That is same as capital F(t) plus 0 to t M(t - x) dF(x).

So this integral equation is called the renewal equation if M is the -- M is considered as unknown because the left hand side is the M(t), that's a renewal function. If M(t) is unknown, unknown, then M(t) is written in the form of M(t) is equal to F(t) plus integration 0 to t. The integrant is M(t - x). Therefore, this is called the integral equation and this integral equation is called the renewal equation.

Later we are going to give the renewal limiting theorems as t tends to infinity for the integral also. M(t - x)dF(x) can be evaluated as t tends to infinity if certain conditions are satisfied, but here we are presenting the renewal equation where the renewal function is the consider as unknown.

The above equation can be written as M = F + M * f where small f is nothing but the probability density function of inter-arrival time.

Now, taking Laplace transform on both sides, we will get M^* is equal to F^* + M^* f^*.

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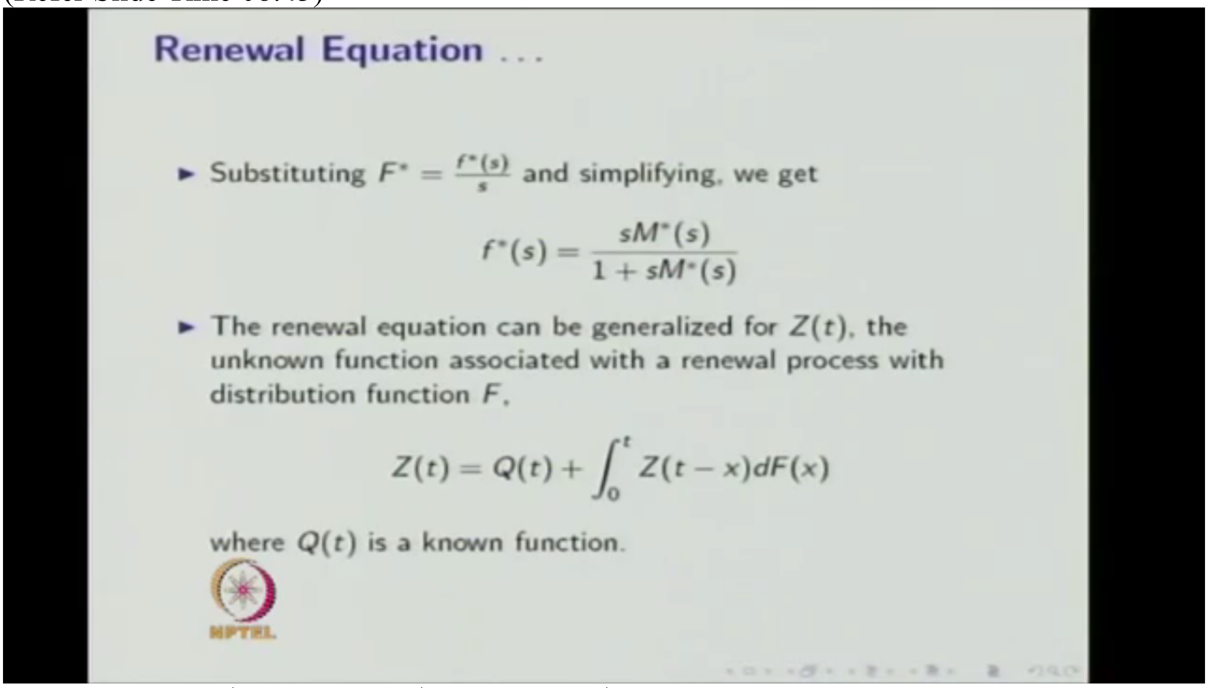

You can relate F^* is equal to f^* by s where * is nothing -- denoted for Laplace form and simplifying we get F(x) is equal to s times M^* divided by 1 + s times M^* where capital M^* is nothing but the renewal function in Laplace.

The renewal equation can be generalized for some unknown Z(t), unknown function associated with the renewal process with the distribution function capital F as Z(t) is equal to Q(t) plus integration 0 to t Z(t - x)dF(x). So the renewal function can be extended, can be generalized for any unknown function associated with the renewal process that is Z(t) as Q(t) plus integral 0 to t Z(t - x)dF(x) where Q(t) is a known function.

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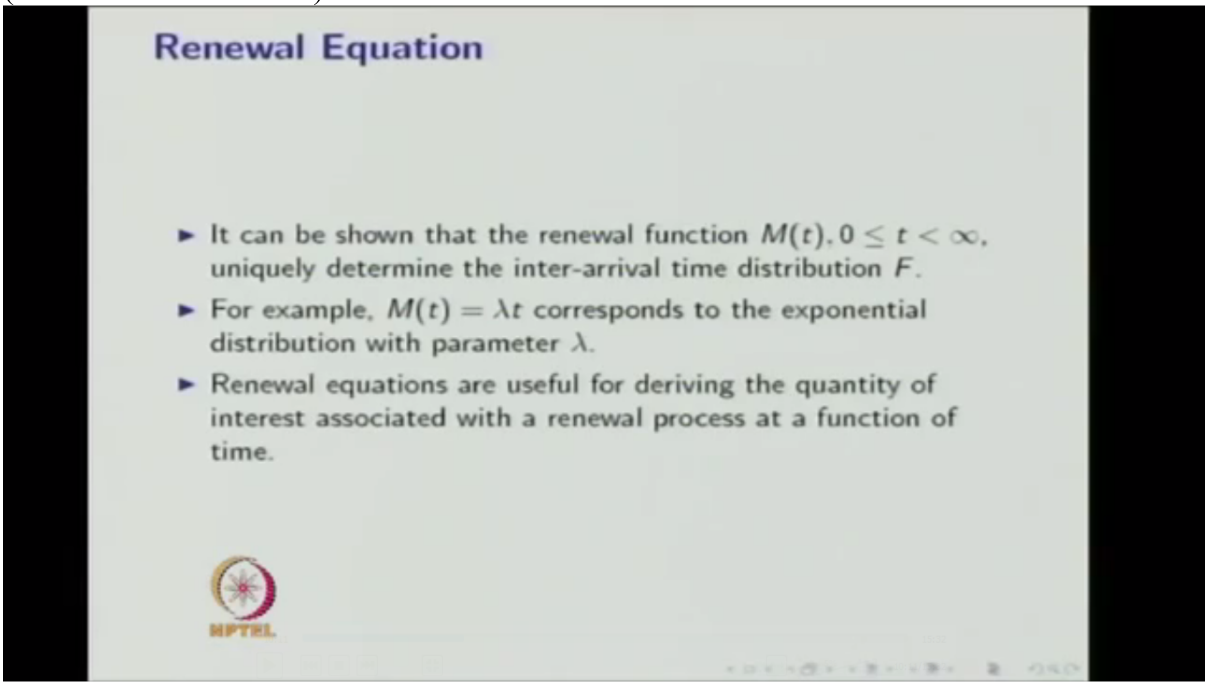

It can be shown that the renewal function uniquely determines the inter-arrival time distribution. Once you know the renewal function, you can uniquely determine the distribution function.

For example, if I know the renewal function is λt, then you can say that the inter-arrival time distribution is exponential with the parameter λ.

Hence, the renewal equations are useful for deriving the quantity of interest associated with the renewal process at a function of time.

So first we discuss the renewal equation in terms of renewal function and this can be generalized to any unknown function of time which is associated with the renewal process also.

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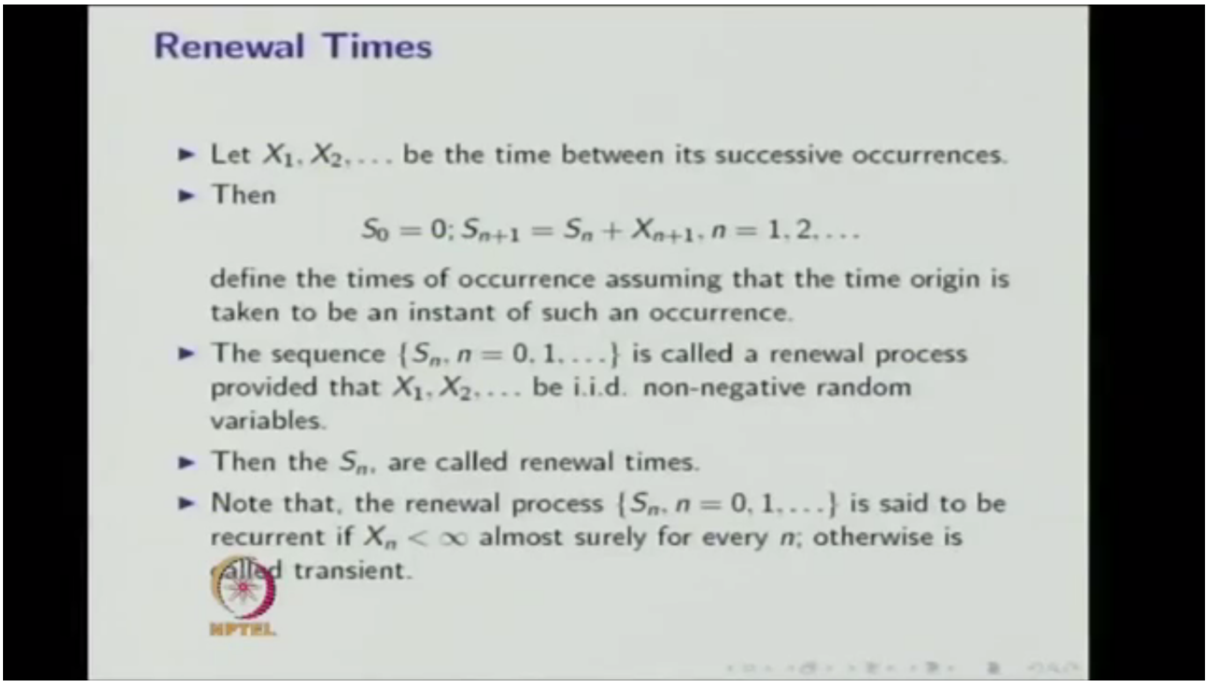

Now we are moving into the next concept, renewal times. Let X_1, X_2 be the time between its successive occurrence. Then S_0 is equal to 0 and S_{n+1} is equal to $S_n + X_{n+1}$ defines the time of occurrence assuming that the time origin is taken to be an instant of such occurrence. The sequence S_n is called the renewal process provided the inter occurrence times are i.i.d. non-negative random variables. Then each S_n is called the renewal times.

Note that the renewal process is said to be recurrent if X_n is finite almost surely for every n. Otherwise, the renewal process is called transient.

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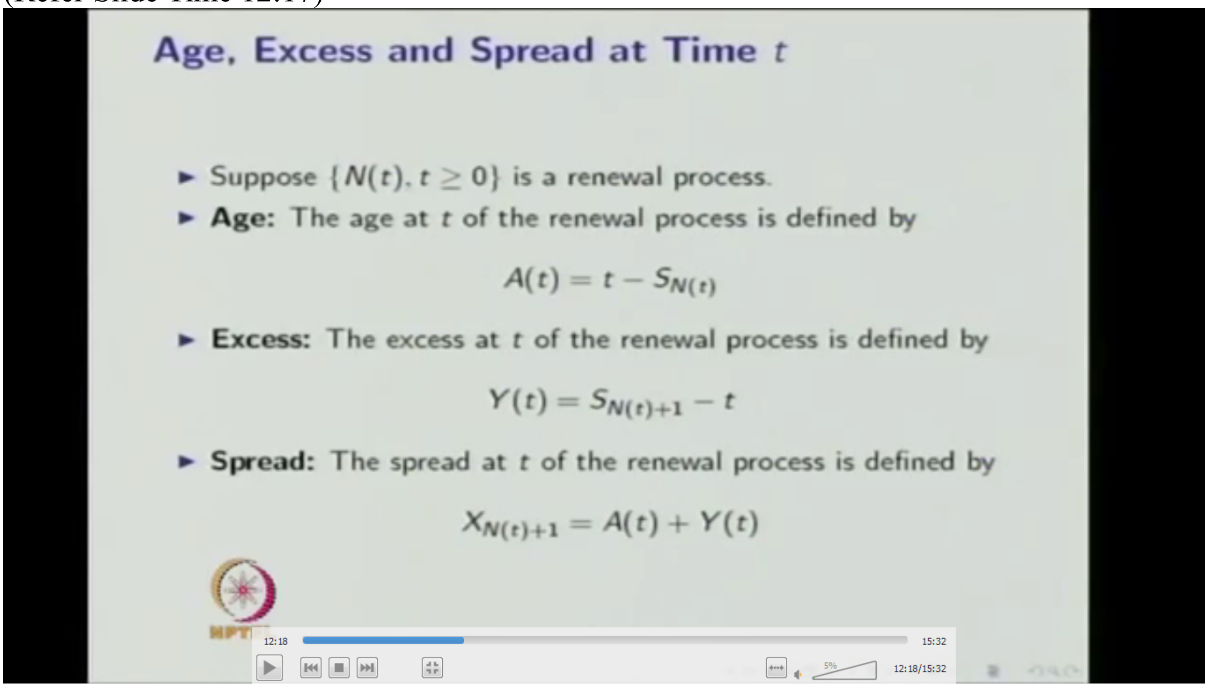

Now we are moving into the last concept in the -- in this lecture that is age, excess and spread.

Suppose N(t) is a renewal process and S_n forms the renewal time. You can define age at time t of the renewal process as A(t) is equal to t - $S_{N(t)}$. The excess at time t of the renewal process is defined as Y(t) is equal to $S_{N(t)+1)}$ - t. That is the excess at time t whereas the spread is the addition of age and excess. That is the spread of -- spread at time t of the renewal process.

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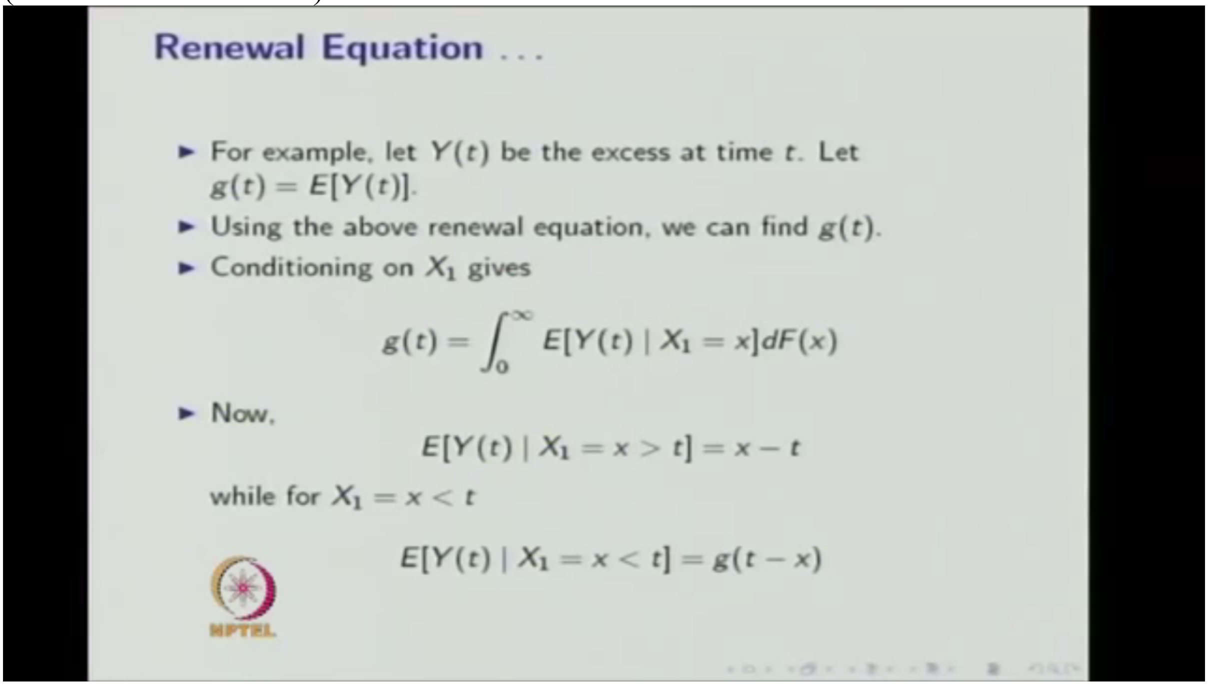

Here we are going to calculate what is the average excess or expectation of expectation of excess. Let Y(t) be the excess at time t and let G(t) be the expectation of excess at time t. Using the renewal equation which we have discussed now, we are going to find the expectation of excess at time t.

First, we are making condition on X_1 and the conditional expectation will be expectation of Y(t) given X_1 is equal to x, which is greater than t, will be x - t while X_1 is equal to x, which is less than t, then the conditional expectation will be unknown still g(t - x).

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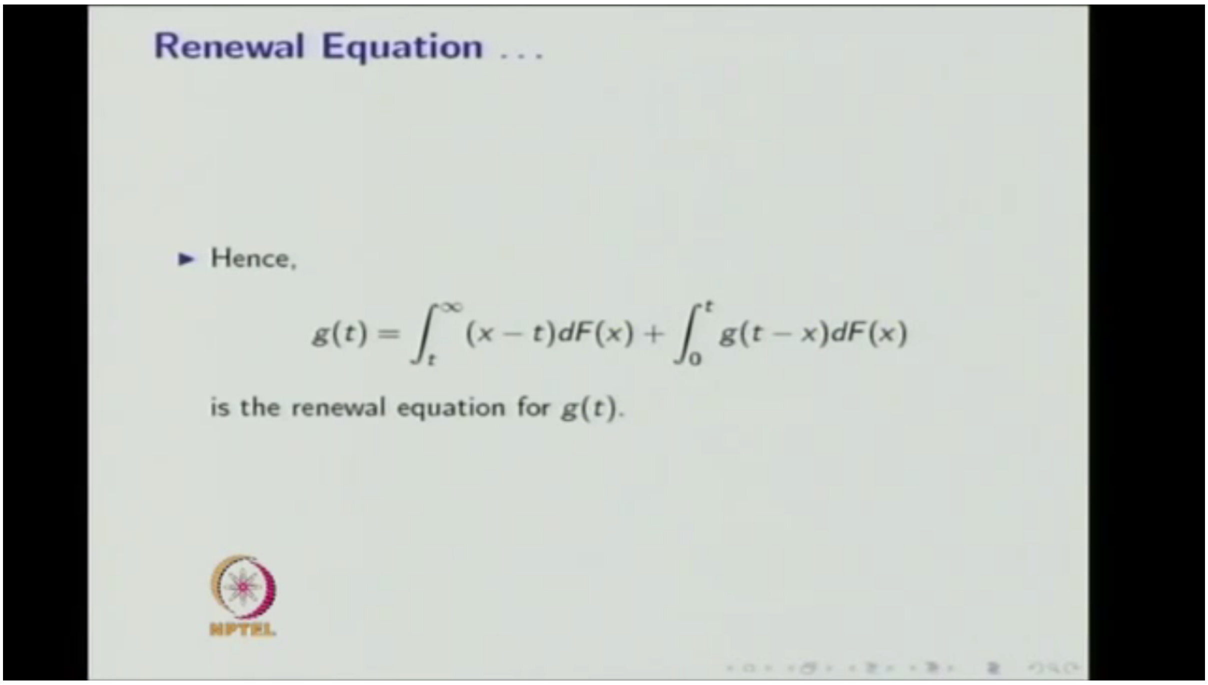

Therefore, the g(t) will be two integration. One integration is from t to infinity and the integrand is x - t and the second integration 0 to t and the unknown is g(t - x).

Since g(t) is unknown function of time t, this is an integral equation. One can solve the integral equation. This is nothing but the renewal equation. One can solve the renewal equation and obtain g(t).

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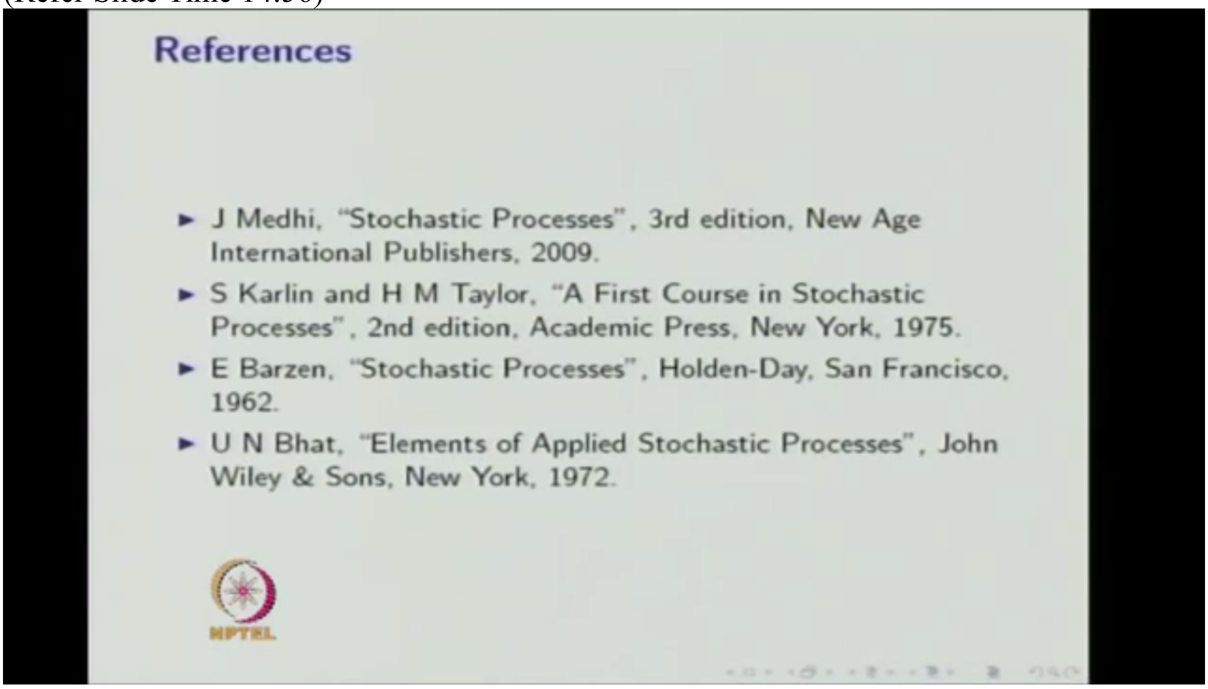

Here is the list of reference for Lecture 1.

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