Module # 8
Renewal Processes

Renewal Function and Renewal Equation (contd.)

by

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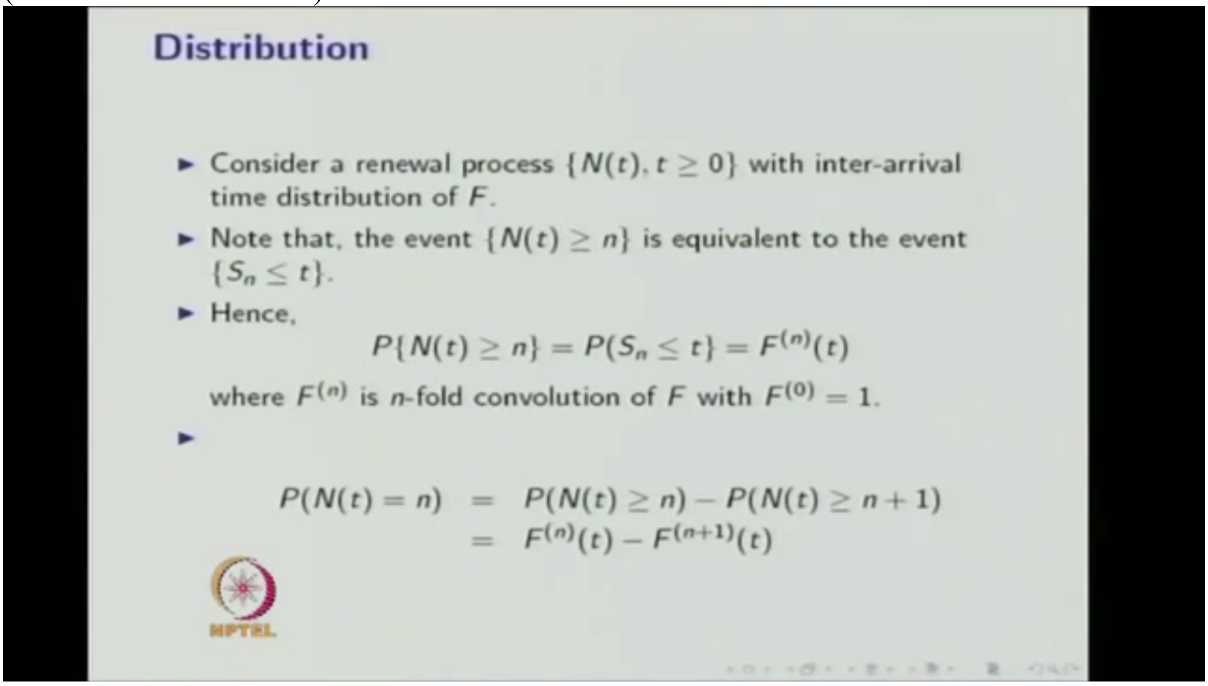

Now we are going to discuss the distribution of a renewal process.

Consider a renewal process N(t) with the inter-arrival time distribution F. We can relate the event N(t) greater than or equal to n with the S_n less than or equal to t. The event N(t) greater than or equal to n is nothing but the number of renewals that takes place till time t is more than or equal to n. The event of S_n less than or equal to t means the number of renewals -- the n number of renewals takes place on or before time t. So both the events are same.

Therefore, the probability of N(t) greater than or equal to n is same as the probability of S_n is less than or equal to t. That we are going to denote as the $F^{(n)}(t)$. That means it is a n-fold convolution of the distribution function F. So this is the distribution of the n^{th} renewal S_n, the distribution of n^{th} renewal S_n.

So using the, the distribution function of n^{th} renewal of n^{th} renewal that is S_n, you can find out the distribution of N(t). Since N(t) is the counting process, the possible values of N(t) is 0, 1, 2 and so on. So we can find the probability mass function for the random variable N(t) for fixed t.

So the probability mass function for the random variable N(t) for fixed t is probability of N(t) is equal to n is nothing but what is the probability that more than or equal to n renewals takes place on or before time t minus what is the probability that n+1 renewals takes place, more than or equal to n+1 renewals takes place on or before time t. That difference will be the probability of n renewals takes place on or before time t.

So since we know that this relation N(t) greater than or equal to n is same as the probability of -- probability of N(t) is greater than or equal to n is same as probability of S_n is less than or equal to t. That is nothing but the n-fold convolution of F. Therefore, probability of N(t) greater than or equal to n will be F superscript n-fold convolution of F minus n+1 fold convolution of F at time t.

So once you -- once you know the inter-arrival time distribution F, you can find out the n-fold convolution. Once you know the n-fold convolution of F, using this formula, you can find out the probability mass function of N(t) for fixed t.

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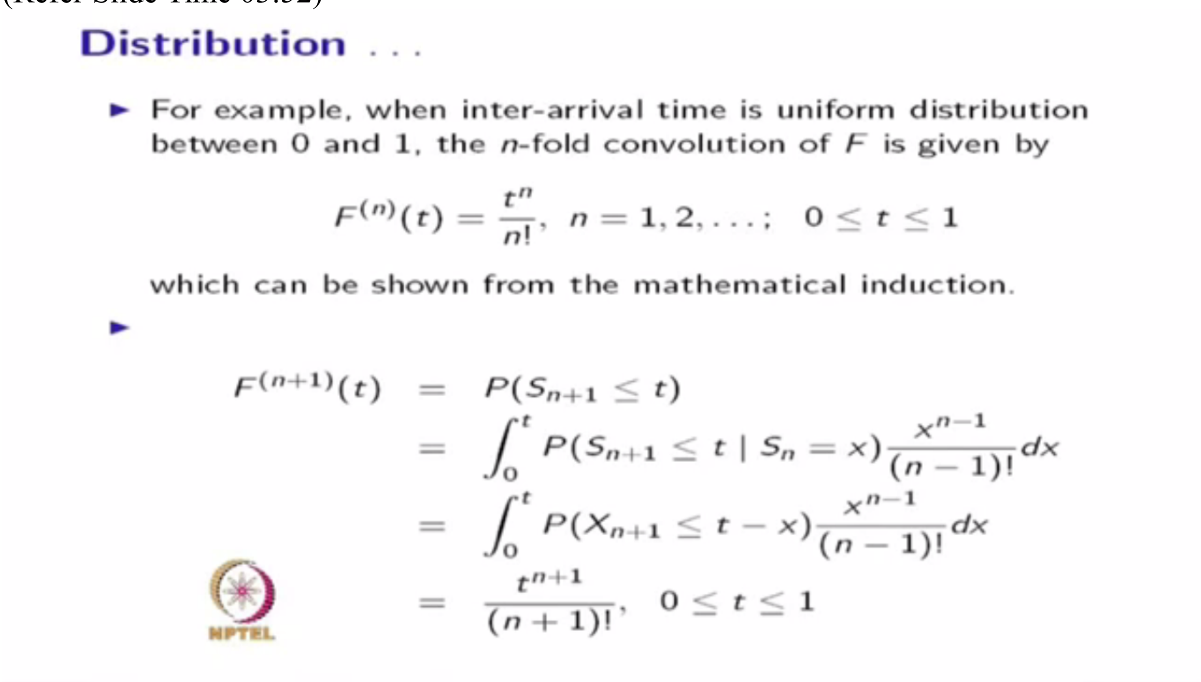

For example, when inter-arrival time is normal distribution, for example, when inter-arrival time is uniform distribution between 0 and 1, the n-fold convolution of F is given by -- given by $F^{(n)}(t)$ is nothing but t^n divided by n factorial where t is lies between the interval 0 to 1. So this can be proved by the mathematical induction.

So we can start with n+1 fold convolution of F that is nothing but the distribution function of the random variable S_{n+1}. That can be written in the conditional distribution, the probability of S_{n+1} is less than or equal to t given S_n is equal to x that is same as what is the probability that the inter-arrival time of the $(n+1)^{th}$ renewal is takes less than or equal to t - x because the given S_n is equal to x and what is the probability of S_{n+1} is less than or equal to t, that is same as what is the probability of X_{n+1} is less than or equal to t - x.

You can substitute and after simplification we can get the $(n+1)^{th}$ convolution of F that will be $t^{(n+1)}$ divided by (n+1) factorial. Hence, proved the n-fold convolution of F will be t^n by n factorial for n is equal to 1, 2 and so on and t is lies between 0 to 1.

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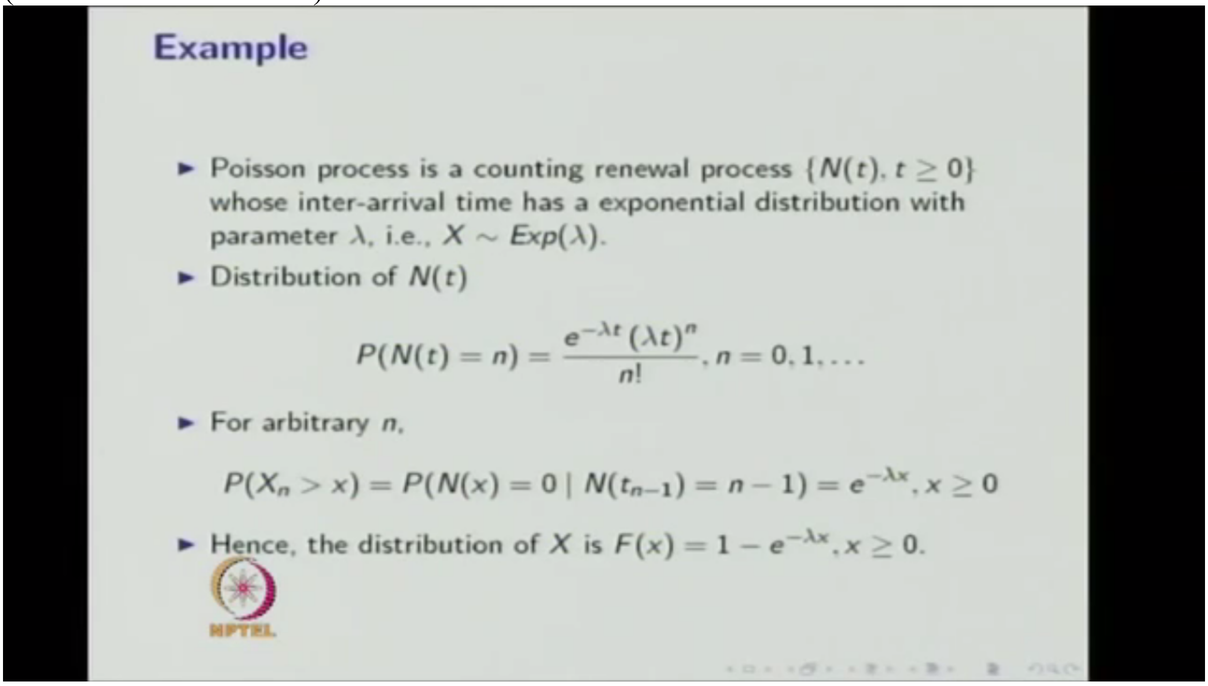

Now we will discuss the other example and find the distribution of N(t).

The Poisson process is a renewal process or a counting process whose inter-arrival time has a exponential distribution with the parameter λ. Since the inter-arrival times are IID random variables, each has the distribution exponential with the parameter λ.

Using the earlier -- earlier form of writing distribution of N(t) in terms of n-fold convolution, you can find the distribution of N(t). So here for arbitrary n, first we know the distribution of N(t) because it is a Poisson process. Therefore, it is $e^{-\lambda t}$ $(\lambda t)^n$ by n factorial. Since you know the probability of N(t) is equal to n, you can find out the inter-arrival time distribution. The inter-arrival time distribution for that, you can go for finding the complement CDF of the random variable X_n.

So the probability of X_n greater than x for arbitrary n, that is nothing but since the inter-arrival times are independent, you can find out the probability of X_n is greater than x is same as the probability of a N(x) equal to 0 given N of t minus 1 is equal to n minus 1.

That means what is the probability that n - 1 renewal takes place till the time point t_{n-1} and given this situation, what is the probability that no arrival takes place in the interval t_{n-1} to t_{n-1} plus x or since it is inter-arrival times are IID and stationary also, the N(x) will be 0, finding the probability of N(x) = 0 given that N of t minus 1 is equal to n minus 1.

Since you know the distribution of N(t), you can substitute and you can get it is $e^{-\lambda x}$ for x greater than or equal to 0 and this is valid for arbitrary n. So since the complement CDF is $e^{-\lambda x}$, the distribution of X will be exponential distribution with the parameter λ.

So to conclude, inter-arrival time is exponential distribution, we are using the property of Poisson process and the definition of the probability mass function of Poisson process also.

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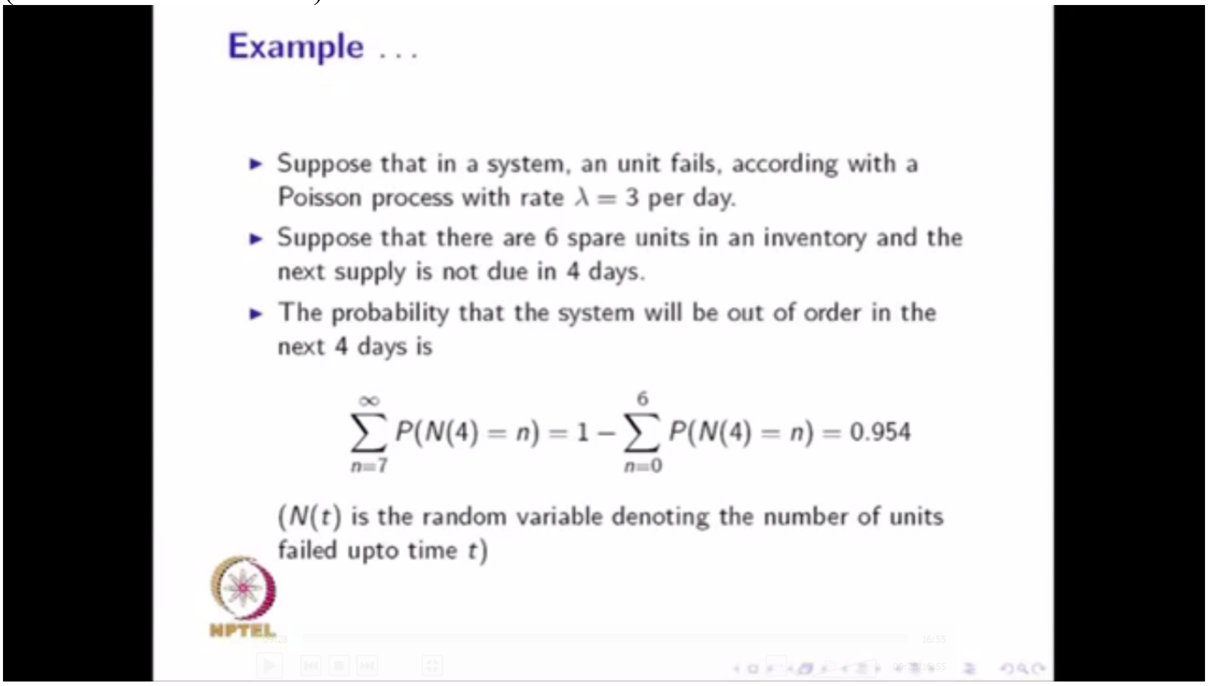

Next we are going to consider one more example for the renewal process.

Suppose that in a system, an unit fails, according with the Poisson process with the rate λ is equal to 3 per day. So this is the example for Poisson process, a special case of renewal process.

Suppose that there are 6 spare units in an inventory and the next supply is not due in 4 days. Our interest is to find out what is the probability that the system will be out of order in the next 4 days. That is nothing but what is the probability that at the time point 4, the number of fail units in the system is greater than or equal to 7? That is the probability of system will be out of order in the next 4 days.

So this probability is same as 1 minus the summation n is equal to 0 to 6 with the probability that N(4) = n. You substitute the probability mass function, the λ value and a t equal to 4. You will get the required probability. That will be 0.954. So this is the example of a Poisson process.

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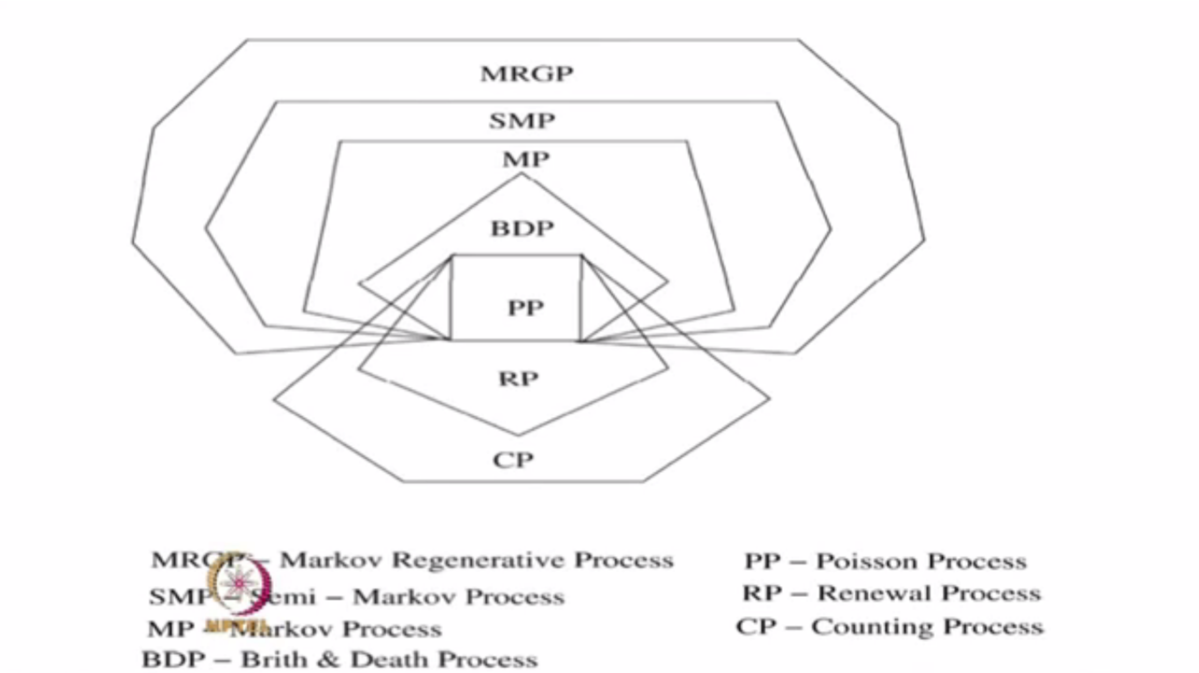

This is the pictorial representation of a Poisson process and all other process in one graph. The Poisson process is in the PP. Poisson process is a special case of pure birth process and pure birth process is a special case of Birth Death Process and the Birth Death Process is a special case of continuous-time Markov chain and the continuous-time Markov chain is a special case of Markov process and the Markov process is a special case of a semi-Markov process and the semi-Markov process is a special case of Markov regenerative process.

The other side, the other side, Poisson process is a special case of renewal process and the renewal process is a special case of semi-renewal process and the semi-renewal process is a special case of counting process. So this is the way one can relate the Poisson process with all other stochastic processes.

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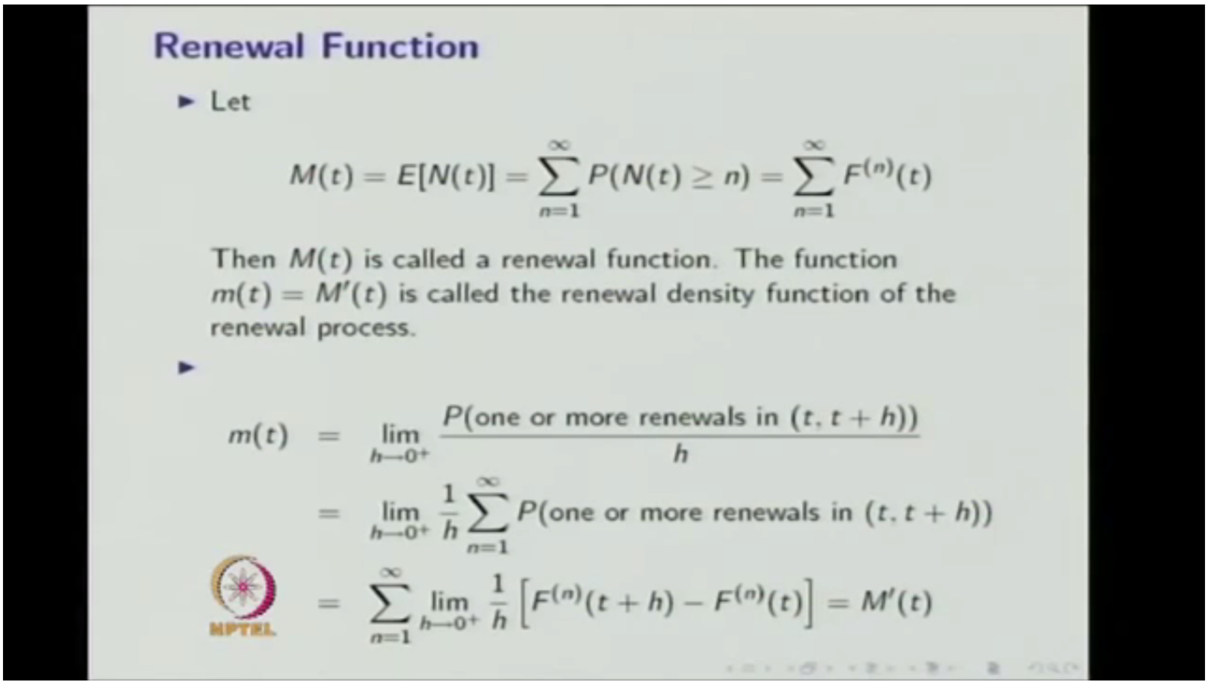

Now we are moving into the concept called renewal function. Let M(t) is equal to expectation of N(t) for fixed t. That is called the renewal function. Since N(t) is a counting process and the possible values are 0, 1, 2 and so on, you can find out the expectation of N(t) with the summation n is equal to 1 to infinity of probability N(t) greater than or equal to n since it is a discrete random variable with the possible values 0, 1, 2 and so on.

Since for fixed t, N(t) is a discrete random variable with the non-negative integer values 0, 1, 2 and so on, the expectation is the sum of its complementary cumulative distribution function. Alternatively, the expectation can be computed as a sum of n times P(N(t)) that is same as sum of n times capital $F^{(n)}(t)$ minus capital $F^{(n+1)}(t)$, which gives sum of $F^{(n)}(t)$. The expected number of renewals up to time t first moment is referred to as a renewal function that is capital M(t) that is equal to expectation of N(t).

We know that the probability of N(t) greater than or equal to n, that is same as the probability of S_n is less than or equal to t and that is nothing but the CDF of S_n, that is nothing but the n-fold convolution of F. Therefore, summation of probability of N(t) greater than or equal to n is same as the summation of n-fold convolution of F where n is running from 1 to infinity.

So using this one can find the renewal function and the derivative will be the renewal density function of the renewal process. So the capital M(t) is a renewal function and the small m(t), that is nothing but the derivative of renewal function is called a renewal density function.

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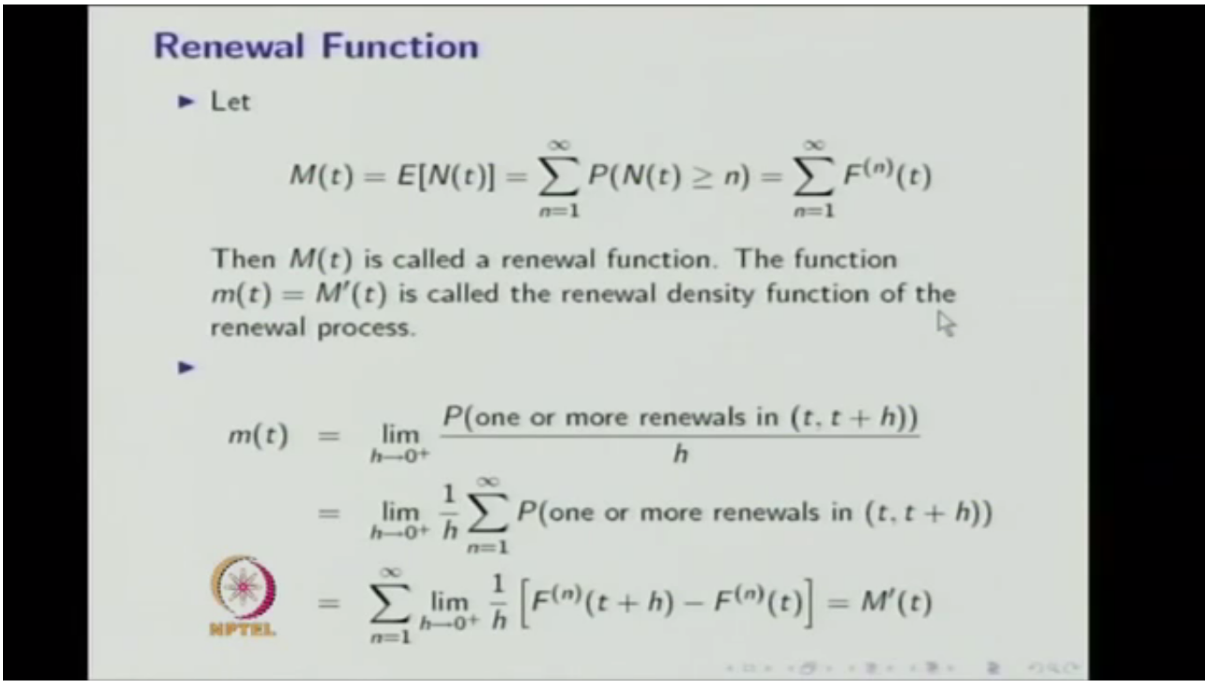

The function small m(t) specifies the mean number of renewals to be expected in a narrow interval near t.

One can find the renewal density function using limit h tends to 0, the probability of one or more renewals occurs in the interval t to t + h divided by h. That is nothing but limit h tends to 0, 1/h the probability of zero or more renewals. That is nothing but the summation of n is equal to 1 to infinity of probability of n renewals in the interval t to t + h.

The limit and the summation can be interchanged and the probability of n renewals in the interval t to t + h where n is running from 1 to infinity, that can be simplified in the form of n-fold convolution of F between the interval t to t + h.

By using the M(t) is nothing but the summation of $F^{(n)}(t)$, therefore, the limit h tends to 0, 1/h of $F^{(n)}(t+h)$ - $F^{(n)}(t)$ is nothing but the derivative. Therefore, it is a capital $M'(t)$. So the derivative of renewal function is nothing but the renewal density function.