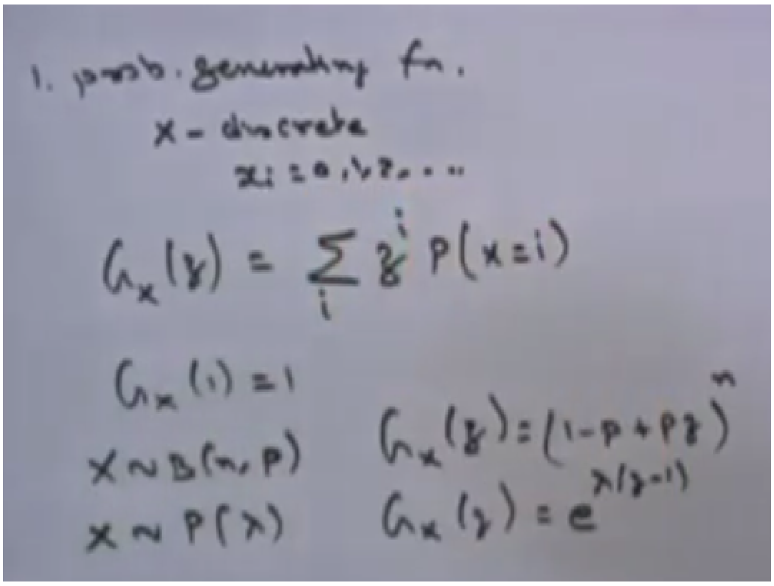
Stochastic Processes - 1
Dr. S. Dharmaraja
Department of Mathematics
Indian Institute of Technology – Delhi

Lecture - 08
Generating Functions, Law of Large Numbers and Central Limit Theorem

Now we are going to discuss few generating functions.

(Refer Slide Time: 00:11)



So the first one is called probability generating function. So this is possible only with a random variable is a discrete random variable and the possible values of xi has to take zero or 1 or 2 like that, that means if the possible values of the random variable x takes value only zero, 1, 2 and so on, then you can be able to define what is the probability generating functions for the random variable x as with the notation Gxz.

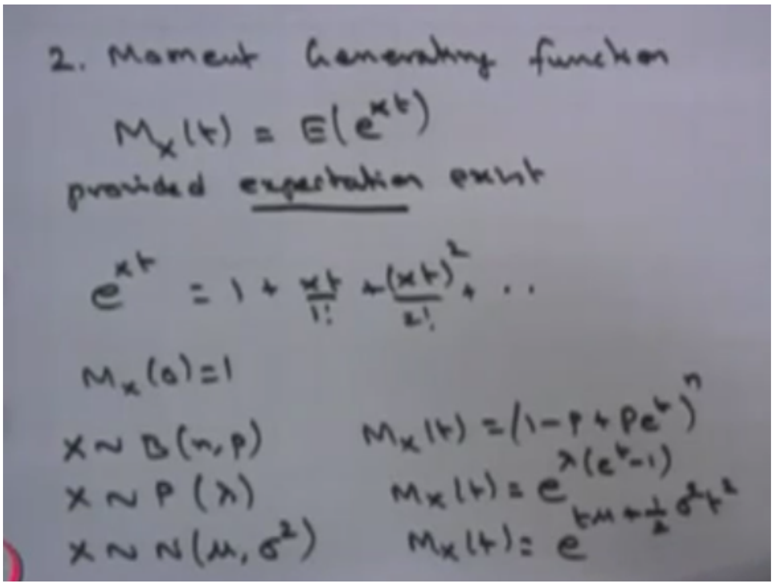
That is probability generating function for the random variable x as a function of z, that is nothing but summation z power i, and what is the probability x takes the value i for all the possible values of i. That means if the discrete random variable takes only countably finite value, then the probability generating function is a polynomial. If the discrete random variable takes a count ably infinite values, then it is going to be the series.

So this series is going to be always converges and you can able to find out what is the value at 1 that is going to be 1. And since it is going to be z power i by differentiating you can get, there is easy formula or there is a relation between the moment of order n with the probability function in the derivative of n-th derivative and substituting z is equal to one. And if suppose x is going to be a binomial distribution with the parameters n and p, then you can find out what is the probability generating function for the random variable x.

That is going to be one minus p plus p times z power n, because the binomial distribution has the possible values are going to be zero to n. Therefore, you will get the polynomial of degree n. Suppose x is going to be a Poisson distribution with the parameter lambda, because this is also a discrete random variable and the possible values are going to be countably infinite, whereas here the possible values are going to be countably finite.

So here also you can find out what is the probability mass function, sorry what is the probability generating function for random variable x and that is going to be e power lambda times z minus one. So like that you can find out a probability generating function for only of a discrete type random variable with a possible values to be countably finite or countably infinite with zero, 1, 2, and so on.

(Refer Slide Time: 02:55)



The next generating function, which I am going to explain, that is moment generating function. The way we use the word moment generating function, it will use the moments of all order n. That means it uses the first order moment, second order moment, and third order moment. And you can define the moment generating function for the random variable x as a function of t, that is nothing but expectation of e power x times t provided the expectation exist, that is very important.

That means since I am using the expectation of a function of a random variable and that too this function is e power xt you can expand e power xt as 1 plus xt by factorial 1 plus xt power 2 by factorial 2 and so on. Therefore, that is nothing but the moment generating function for the x is nothing but expectation of this expansion. That means expectation of one plus expectation of this plus expectation of this plus one.

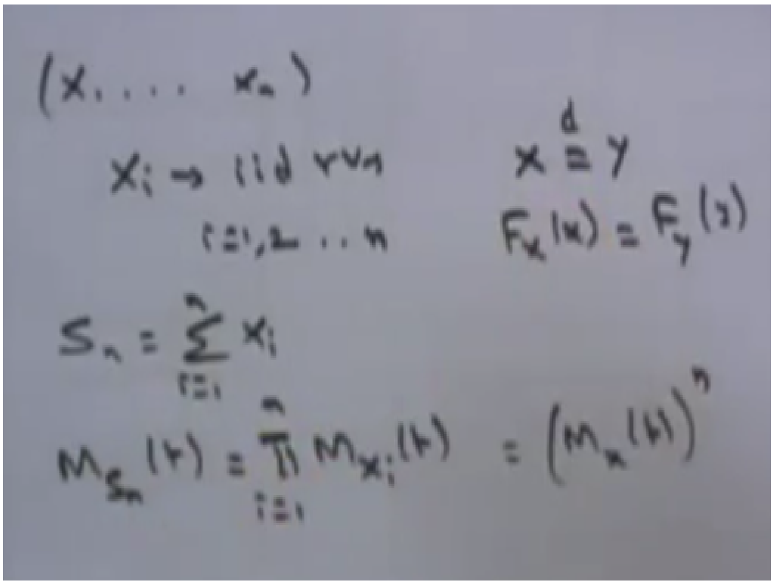
That means if the moment of all order n exists, then you can able to get what is the moment generating function for the random variable x. That is the provided condition is important as long as the right hand side expectation exists, you can give the moment generating function for the random variable x. So here also many properties are there, I am just giving one property Mx of zero going to be one.

And there are some property which relate with the moment of order n with the derivative of moment generating function. And I can give one simple example, if x is going to be binomial distribution with the parameters n and p, then the moment generating function for the random variable x that is going to be one minus p plus p times e power t power n.

Similarly, if x is going to be Poisson with the parameter lambda, then you may get the moment generating function is going to be e power lambda times e power t minus one. And you can go for continuous random variable also. If x is going to be a normal distribution with the parameters mu and sigma square, then the moment generating function is going to be e power t times mu plus half sigma square t square.

So this is very important moment generating function, because we are going to use this moment generating function of normal distribution in the stochastic process part also. There is some important property over the moment generating function.

(Refer Slide Time: 05:50)



Suppose you have n random variables and all xis random variables are iid, that means independent identically distributed random variable. That means the distribution of, when you say the random variable x and y are identically distributed. That means the CDF of x and the CDF of y are same. For all x and y both the values are going to be same, then you can conclude both the random variables are going to be identically distributed.

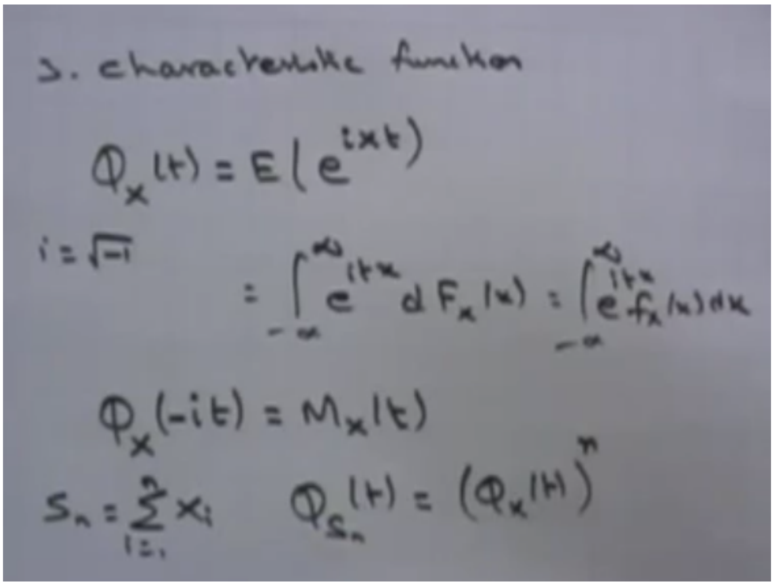
So here I am saying the n random variables are iid random variables. That means not only identical they are mutually independent also. If this is the situation, my interest is to find out what is the MGF of sum of n random variables, that is Sn. So the moment generating function for the random variable Sn is going to be the product of the MGF of individual random variable.

Since they are identical the MGFs is also going to be identical. Therefore, this is same as you find out MGF of any one random variable then make the power. So this independent random variables having the property when you are trying to find out the MGF of sum of random variable, that is same as the product of MGF of individual random variables.

Here, there is a one more property over the MGF, suppose you find out the MGF of some unknown random variable and that matches with the MGF of any standard random variables, then you can conclude the particular unknown random variable is also distributed in the same way. That means the way you are able to use the CDFs are same, then the corresponding random variables are identical.

Same way if the MGF of two different random variables are same, then you can conclude the random variables also identically distributed.

(Refer Slide Time: 08:00)



Third we are going to consider the another generating function that is called characteristic function. This is important than the other two generating function, because the probability generating function will exist only for the discrete random variable. And the moment generating function will exist only if the moments of all order n exist, whereas the characteristic function exist for any random variable.

Whether the random variable is a discrete or the moments of all order n exist or not, immaterial of that the characteristic function exist for all the random variable. That I am using the notation phi suffix x as a function of t, that is going to be expectation of e power i times xt. Here the i is the complex number, that is the square root of minus one. So this play a very important role such that this expectation is going to be always exist, whether the moment exist or not.

Therefore, the characteristic function always exists. You can able to give the interpretation of e power this is same as minus infinity to infinity e power i times tx d of CDF of that random variable. So that means whether the random variable is a discrete or continuous or mixed you are integrating with respect to the CDF of the integrant function is e power i times tx, where i is a complex quantity and if you find out the absolute.

This absolute, this is going to be using the usual complex functions you can make out this is going to always less than or equal to one in the absolute sense. Therefore, this integration is exist and this integration is nothing but the Riemann–Stieltjes integration. And if the function is going to be the if the random variable is continuous, then you can able to write this is same as minus infinity to infinity e power i times tx of the density function integration with respect to x. That means, this is nothing but the four way transform of f.

And here we have this f is going to be the probability density function and you are integrating the probability density function along with e power i times tx and this quantity is going to be always converges whereas the moment generating function without the term i, the expectation may exist or may not exist. Therefore, the MGF may exist or may not exist for some random variable.

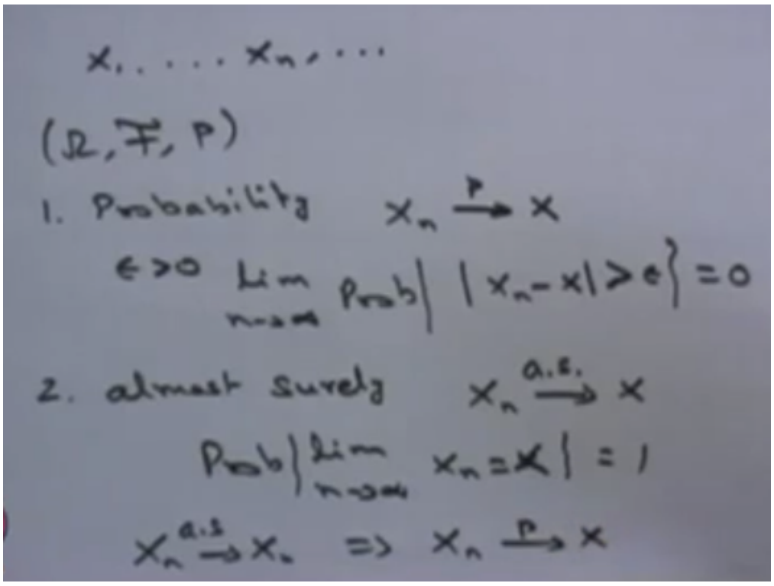
And I can relate with the characteristic function with the MGF with the form phi x of minus i times t, that is same as MGF of the random variable t. That means I can able to say what is a MGF of the random variable x that is same as the characteristic function of minus i times t, where i is the complex quantity. And here also the property of the summation of, suppose I am trying to find out what is the characteristic function of sum of n random variables.

And each all the random variables are iid random variable, then the characteristic function of Sn is same as, when xis' are iid random variable, then the characteristic function each random variable power n. And this also has the property of uniqueness, that means if two random variables characteristic functions are same, then you can conclude both the random variables are identically distributed.

So as a conclusion, we have discussed three different function, first one is probability generating function, and the second one is moment generating function and the third one is characteristic function. And we are going to use all those functions and all other properties of joint probability density function and distribution and everything, we are going to use it in the at the time of stochastic process discussion.

Next we are going to discuss what is the, how to define or how we can explain the sequence of random variable converges to one random variable.

(Refer Slide Time: 12:15)



Till now, we started with one random variable then using the function of the random variable you can land up another random variable or from the scratch you can create another random variable, because random variable is nothing but a real valued function satisfying that one particular property inverse images also belonging to f. Therefore, you can create many more or countably infinite random variables are uncountably many random variables also over the same probability space.

That means you have a one probability space and in the single probability space you can always create either countably infinite or uncountably many random variables and once you are able to create many random variables. Now our issue is what could be the convergence of sequence of random variable. That means if you know the distribution of each random variable and what could be the distribution of the random variable Xn as n tends to infinity.

So in this, we are going to discuss different modes of convergence, that is the first one is called convergence in probability. That means, if I say a sequence of random variable Xn converges to the random variable some x in probability. That means if I take any epsilon greater than zero, then limit n tends to infinity of probability of absolute of Xn minus X which is greater than epsilon is zero.

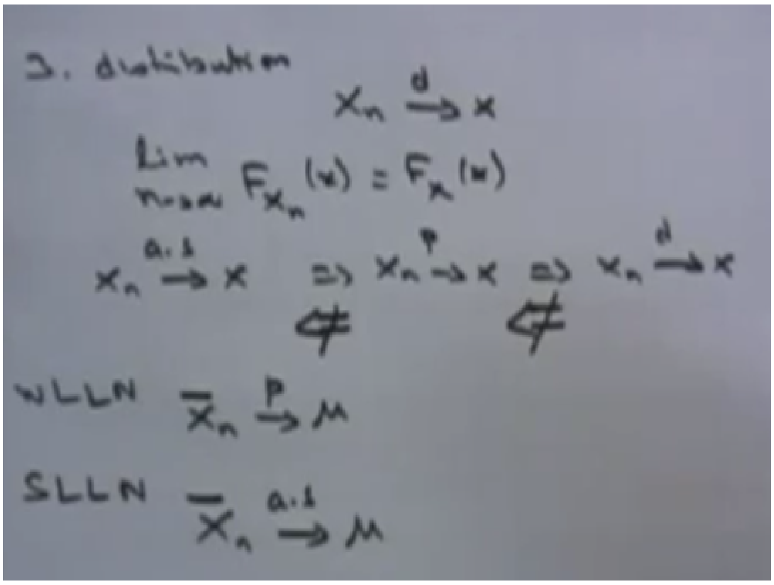
If this property is satisfied for any epsilon greater than zero, then I can conclude the sequence of random variable converges to one particular random variable x in probability. That means this is a convergence in probability sense. That means you collected possible outcomes that find out the different of Xn minus X, which is in the absolute greater than epsilon.

That means you find out what are all the possible event which is away from the length of two epsilon, you collect all possible outcomes and that possible outcomes is that probability is going to be zero, then it is convergence in probability. That means you are not doing the convergence in the real analysis the way you do, you are trying to find out the event, then you are finding out the probability. Therefore, this is called the convergence in probability.

The second one it is convergence almost surely. So this is the second mode of convergence, this notation is Xn converges to X a dot s dot. That means the sequence of random variable has n tends to infinity converges to the random variable x has n tends to infinity that is almost surely, provided the probability of limit n tends to infinity of Xn = x, or Xn is equal to capital X, that is going to be one.

That means first you are trying to find out what is the event for n tends to infinity the Xn takes the value X. That means you are collecting the few possible outcomes that as n tends to infinity what is the event which will Xn same as the X. Then that event probability is going to be one if this condition is satisfied, then we say it is going to be almost surely. I can relate with the almost surely with if any sequence of random variable converges almost surely that implies Xn converges to X in probability also.

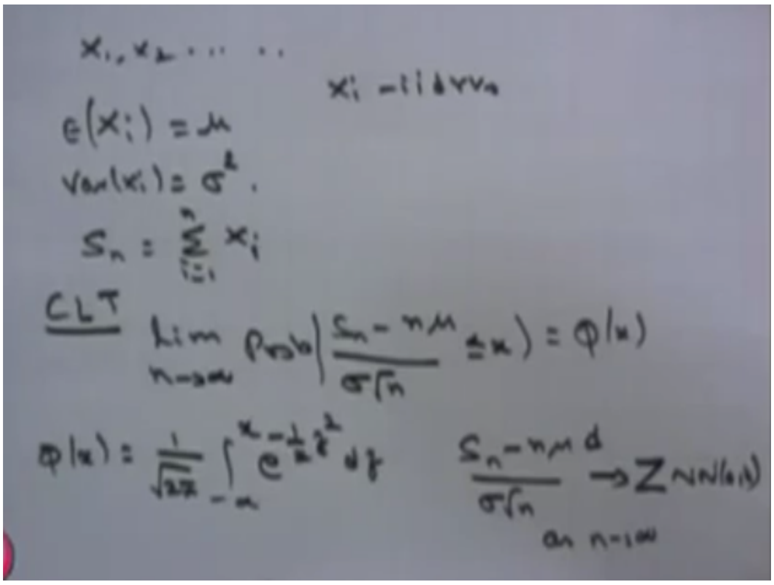
(Refer Slide Time: 16:15)



This is a third mode of convergence, that means if the sequence of random variable CDF converges to the CDF of the random variable X, then you can say that the sequence of random variable is converges to the random variable in distribution and I can conclude the sequence of random variable converges in almost surely implies in probability, that implies in distribution, where as the converse is not true.

And when I categorise this into the law of large numbers as a weak law of large numbers and strong law of large numbers, if the mean of Xn that converges to mu in probability, then we say it is weak law of large numbers. Similarly, if the convergence in the almost surely, then we conclude this is going to be satisfies the strong law of large numbers.

(Refer Slide Time: 17:02)



The final one that is the central limit theorem. You have a sequence of random variable with each are iid random variables and you know the mean and variance and if you define the Sn is in the form, then Sn minus n mu divided by sigma times square root of n, that converges to standard normal distribution in convergence in distribution.

That means whatever be the random variable you have, as long as they are iid random variable and even these things can be relaxed, the sequence of random variable the summation will converges to the normal distribution or their mean divided by the standard deviation will converges to the standard normal distribution. With this, I complete the review of theory of probability in the two lectures. Then, the next lecture onwards I will start stochastic process. Thank you.