Stochastic Processes - 1
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Lecture - 71
Stationary Distribution and Open Queueing Network

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Stationary Distribution

Solve

$$\pi Q = 0; \sum_{(n_1, n_2) \in \Omega} \pi_{(n_1, n_2)} = 1$$

$$P(X = n_1, Y = n_2) = (1 - \rho_1)\rho_1^{n_1}(1 - \rho_2)\rho_2^{n_2}$$

$$\rho_1 = \frac{\lambda}{\mu_1} < 1; \rho_2 = \frac{\lambda}{\mu_2} < 1; n_1 \geq 0; n_2 \geq 0$$

Since we land up the underlying stochastic process is a continuous time Markov chain, now we can find out the stationary distribution that is our interest. Solve pi Q is equal to zero and the summation of pi a is equal to 1. Now the pi a is not just one pi a, it is a two index term. So by solving this equation, that means you write the balance equation, then use the summation of probability is equal to 1, you need a condition.

So as long as the rho 1 is lambda divided by mu 1 that is less than 1 as well as rho 2 that is lambda divided by mu 2, that is also less than 1, then the stationary distribution exists and confined on the stationary distribution probability as a probability that n1 customer in the first node and n2 customer in the second node, that is nothing but n1 customers in the first node, that probability distribution is 1 minus rho 1 times rho 1 power n 1.

And n2 customers in the second, that is nothing but 1 minus rho 1 times rho 2 power n2. So this is sort of a product. The probability of n1 customer in the first node and the probability of n2 customer in second node together that is same as what is the probability the n1 customer in the first node multiplied by what is the probability that n2 customers in the second node and this form is called the product form solution.

So for a tandem queueing network, the stationary distributions are of the product form. This together probability is the product of individual MM1 infinity stationary distribution probabilities. Therefore, this is called the product form. So this exists as long as rho 1 is less than 1 and rho 2 is less than 1. If it is greater than or equal to 1, then this system is not stable, equivalently stationary distribution does not exist.

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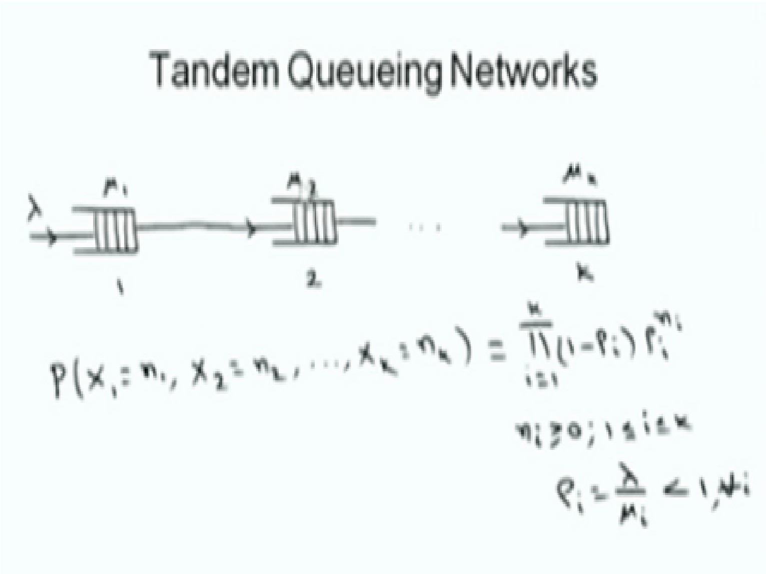

Now I can extend that these two queues queueing network into K queues or K nodes. So there also my interest is to find out the stationary distribution and the probability of n1 customer in the first node, n2 customer in the second node and nK customer and so on till nK in the Kth node that joint distribution is same as product of individual distribution of ni customers in ith node.

So this is the product form, product of i is equal to 1 to K and this result is nothing but stationary distribution of MM1 infinity que. So this exists as long as all the rho i is less than 1 and since the mean arrival is lambda, therefore all the rho is lambda divided by the service rates or the corresponding que.

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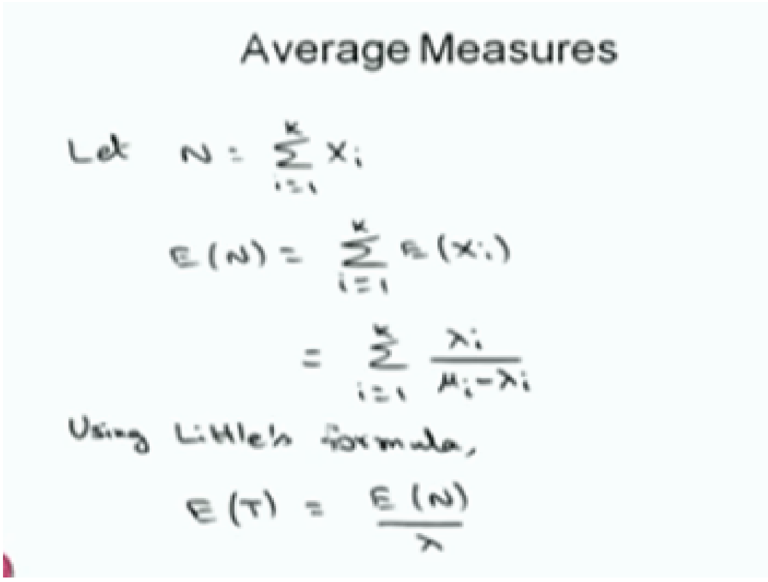

So once we know the number of customers in what is the distribution of number of customers in each node for the K node tandem queueing network, you can find out what is the total number of customers in the whole queueing networks. That means you have to sum it up all the average number of customers in each node. If you sum it up that will be the total number of customers in the whole queueing network.

And since each node is going to behave like MM1 queueing system, therefore, you can use the average number of customers in MM1 queueing that result, that is lambda a divided by mu a minus lambda e, whereas here all the lambda e are lambda, because it is tandem queue and if you sum it up, that is going to be the number of customers in the system. Once you know the total number of customers in the system.

System is the whole queueing network, using Little's formula, you can find out the average time spent in the system. Some books, they say average response time, average sojourn time, so all those things are the same, so the average sojourn time, using Little's formula, you can get it after substituting expected number in the system divided by the arrival rate. Because Little's formula is applied to the queueing system in the sense.

The arrival rate to the system is lambda and total number of customers in the system, so that the all that series queuing, all the queues that hold tandem queue that you treat it as one system. And it satisfies all the Little's law conditions, that is first come, first served, and the arrival rate is lambda and after the service is over, the customer leave the system, and so on.

So you cannot play the Little's formula and get the average time spent in the system, for the whole system, not for the individual ques.

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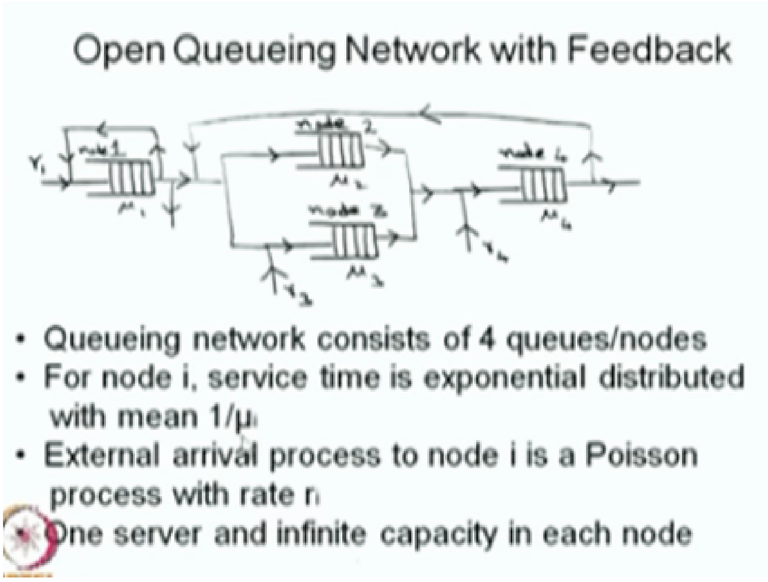

Now, I am moving into the open queueing network with feedback. So I am just giving one simple scenario of four nodes queueing network with the feedback also. So this is the open queueing network and here the assumptions are for each node, the service time is exponentially distributed with the mean 1 divided by mu i or the parameter mu i. Only one server in each node and infinite capacity for each node also.

The external arrival process to the node i that is Poisson process with the rate lambda i. That means for the node 1, there is external arrival that is r1. There is no external arrival to the node 2 whereas there is external arrival for the node 3, that is r3, that arrival is a Poisson process and this arrival is also Poisson process with the parameter r4. And I have not supplied the routing probabilities.

So after the service is over in the first node with some probability, it moved into the again node 1. With some probability, it moved into the node second as well as node 3 in this quotation. You can multiply this and this, then with some probability it goes away from the system, so the summation of these plus these plus these and this probability has to be 1. Sometimes we would not draw this outgoing r.

So there are two possibilities, either this r probability is 0, or non-zero whenever the summation of other than going out, if that probability is not equal to 1. That means 1 minus of summation of probability that is 1 minus of the summation of probability is greater than 0, then that is the probability that the system, after the service is over, the customers who finishes the service in the node 1, leave the system.

Whenever the summation of arcs from the node i to all other nodes, if that summation of probability is less than 1, then 1 minus of that summation of probability, that is the probability in which after the service is over, the customers need the system. So that is the way, you can make out the probability for this, otherwise we need to supply the routing probability matrix.

So here I am concluding with the open queueing network with the feedback and the assumptions are infinite capacity in each node and single server and the external arrival process is a Poisson process with the rate ri, whereas the actual or the total arrival rate that is different from r, because the external arrival rate is a Poisson process with r1. But there are some customers after finishing the service in the first node.

They are again coming back, therefore the total or actual arrival stream into the node 1 that is different from r1, so that we are going to calculate later.

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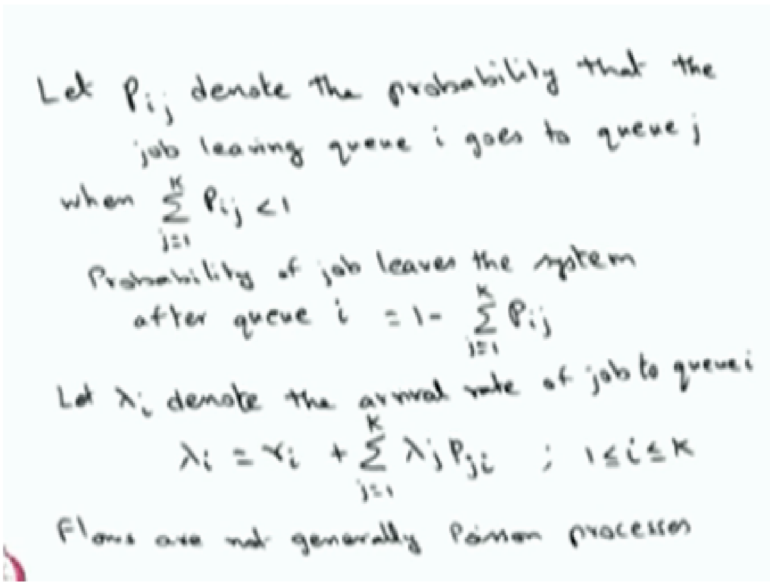

So I am going for the routing probability with notation P of i, j. P i, j denotes the probability that the job leaving queue i goes to the queue j. Whenever this probability is the summation of probability that rho sum is less than 1, then the probability of job leaves the system after the queue i that is going to be 1 minus of summation j is equal to 1 to K. We have K, K is the total number of queues in the open queueing network with the feedback.

And Paj is the routing probability of jobs going from the node i to node j. Either you can say node or que. The actual or total arrival rate that lambda i denotes the arrival rate of a job to the queue i, that can be computed using these formula. So the lambda i is equal to ri plus what are all the different rates lambda j and the packets or the jobs are moving from the queue j to queue i.

So if you multiply these routing probability and the arrival rate, that summation plus the external arrival that rate ri, that will give the arrival rate of job to the node que, where i is running from 1 to K.

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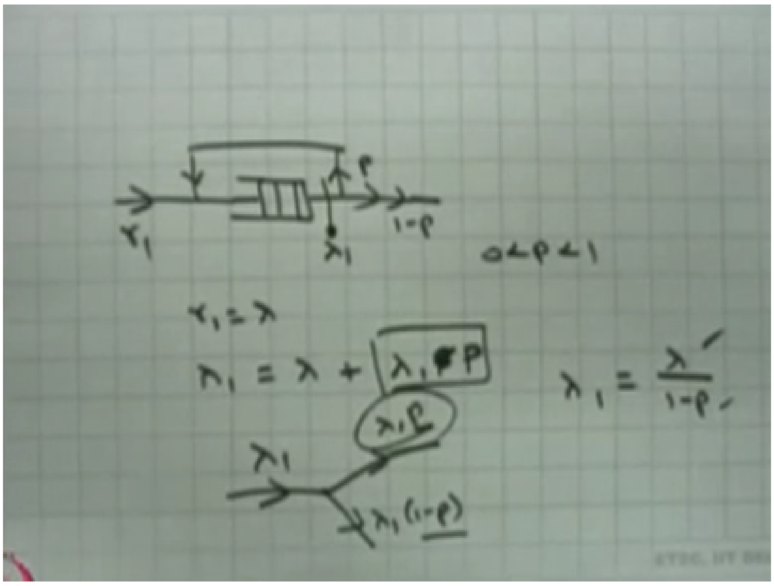

Let me give one simple situation, how one can calculate the arrival rate for any que. Suppose you think of the external arrival rate is r1 and after the service is over, with the probability P, the customer who finishes the service in the first node and come back. With the probability 1 minus P, it leaves the first node. Therefore, our interest is to find out what is the arrival rate for the node 1, with a simple only one feedback.

So here, suppose I make it r1 is lambda, that is the external arrival rate. So the arrival rate to the node 1 that is lambda 1, that is same as the external arrival rate plus what are all the possibilities in which the node 1 build up. So that with the probability P, so this P times, suppose lambda 1 is the arrival rate and lambda 1 times P that is going to be the proportion, the P is the proportion in which it is coming back, we can use the Buch's theorem.

The departure process is the Poisson process. If the arrival rate is lambda 1, then the lambda 1 times P using the Poisson process split, one Poisson process can be split into. Suppose this is departure process, the process is split into two Poisson process lambda 1 P and lambda 1 times 1 minus P. So the lambda 1 times P that is fed again into the node 1. Therefore, that will give lambda 1 times P.

So the lambda 1 is equal to lambda plus lambda 1 times P, that is the way. So here, the arrival rate lambda i is equal to ri plus the summation of lambda j, P j, i that is the same thing applying for i is equal to 1. So the lambda 1 is equal to the lambda that is the external arrival rate plus what are all the ways you have an input arc to the node 1. So this point the departure process is Poisson process using the Buch's theorem.

Therefore, suppose you make it arrival rate is lambda 1, therefore the departure process is the Poisson process with the parameter lambda 1 and the Poisson stream is split into two with Poisson stream with the proportion P and the another Poisson stream with the proportion 1 minus P. Therefore, you have two Poisson stream with the parameters lambda 1 times P and lambda 1 times 1 minus P.

These two are the two independent Poisson streams and this Poisson stream is fed again into the node 1. Therefore, this is the only one input for the node 1, therefore you will have a lambda 1 is equal to lambda 1 plus lambda 1 times P. Like that, if you have many k nodes, you have to write the equation for all the k nodes, then you have to solve for lambda i, then you will get what is the arrival rate for the node i.

So here, I have only one equation, so I can get lambda 1 is going to be lambda divided by 1 minus P by solving this equation, lambda 1 is equal to lambda plus lambda 1 times P, therefore lambda 1 is going to be lambda divided by 1 minus P. Lambda is known to me and P is known to me, therefore using these two I can get lambda 1 that is the arrival rate for the node 1.

So this is the very simplest example, but for any open queueing network with the feedback by framing this K equations, by solving you can get the lambda i. The product form solution is valid for the stationary distribution, that is what given as a Jackson theorem.