Stochastic Processes - 1
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Lecture - 68
Transient Solution of Finite BDP and Finite Source Markovian Queueing Model

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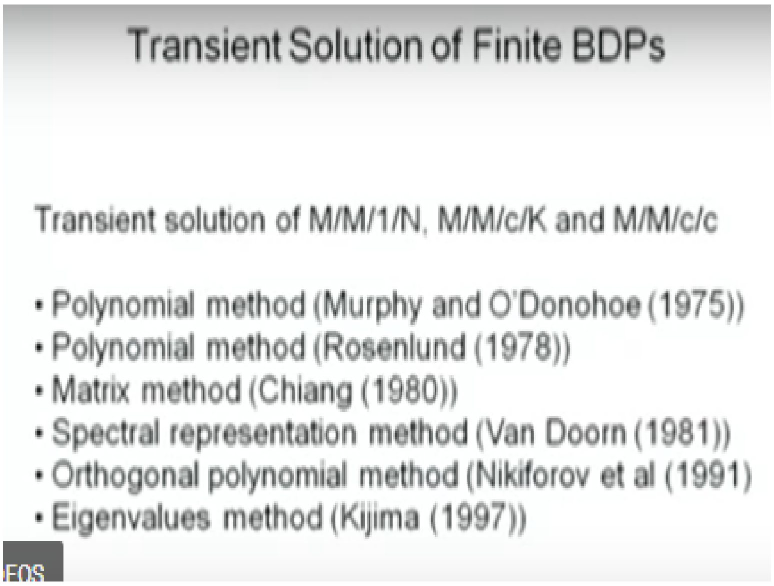

Now I am explaining the transient solution of a finite birth-death process. So using these, one can find out the transient solution of the birth-death process which I have discussed today's class MM1N, MMCK and MMCC also. So the logic is same, that means you have a birth death process with a finite state space. Therefore, the queue matrix is going to be a degree whatever be the number of states in the state space.

And it is going to be a tridiagonal matrix and you know the lambda Ns and mu Ns, birth rates as well as the death rates. And the birth rates and death rates are going to be different for these three models. There are many literatures over the transient solution of finite birth-death process started with Murphy and O'Donohoe, he uses the polynomial method. And in 1978 Rosenlund also found the transient solution for the finite BDP using again the different polynomial methods.

And Chiang in 1980, he made a matrix method to this transient solution. Then later Van Doorn gave the solution using spectral representation method. And Nikiforov et al 1991, he also gave the transient solution using orthogonal polynomial. And later Kijima also gave the solution using Eigenvalues method. So these are all the literatures for getting the transient solution of a finite birth-death process.

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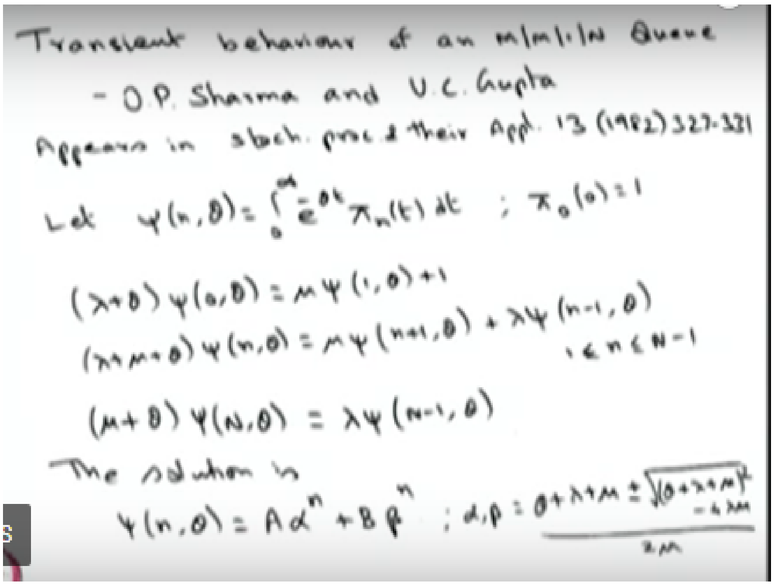

And here I am going to explain how to get the transient behaviour of MM1NQ in a very simplest form, even though there are this many literature and many more literatures for the finite birth-death process. But here I am explaining the overview of how to get the transient behaviour of MM1NQ and this is by O. P. Sharma and U. C. Gupta. It appears in Stochastic processes and their applications, volume 13-1982.

So what this method work you start with the forward Kolmogorov equation.

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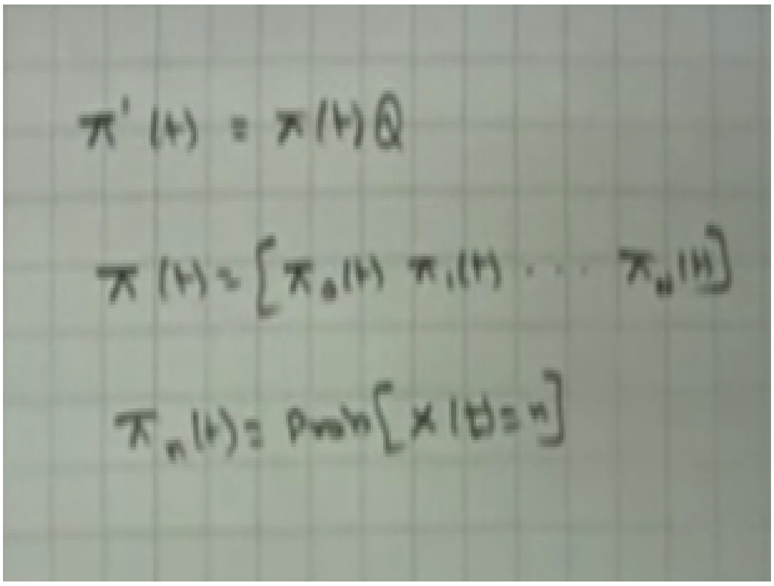

That is Pi t, Pi dash t, that is started with Pi dash of t, that is equal to Pi of t into Q matrix, where Pi is the matrix and Pi dash is the derivatives and Q is the infinite decimal matrix. Take a forward Kolmogorov equation, then use the Laplace transform for each Pi n of t you take the, sorry here the Pi dash of t is the vector, it is the distribution of a X of t.

Therefore, this is a vector and this is a vector and Q is a matrix, not the matrix which I said wrongly. So this is a vector and this is a vector and Q is a matrix. So take a Laplacian form for each probability where the Pi n of t, that is nothing but, so the Pi of t is a vector that started with Pi not of zero t Pi one of t and so on Pi n of t, where Pi n of t is nothing but what is the probability?

That the same notation that I started when I discussed the continuous Markov chain, what is the probability that n customers in the system at time t, it is a conditional probability distribution. So Pi n of t is the probability that n customers in the system at time t, and Pi n of t you get the vector and you make a forward Kolmogorov equation Pi dash of t is equal to Pi of t times Q.

And take a Laplacian form, for each Pi n of t that exist, because this is a probability and the conditions for the Laplacian sum of this function satisfies you can cross check. Therefore, you are taking a Laplacian form, so this is going to be a function of theta. Before taking a Laplacian sum, you need an initial condition also. So at time zero, you assume that no customer in the system, at time zero.

Now customer in the system, that means x of zero is equal to zero. Therefore, that probability is going to be one and all other probabilities are going to be zero. That is the initial probability vector. So use this initial probability vector and apply it over the forward Kolmogorov equation taking a Laplacian sum, you will get the system of algebraic equation.

Since you are using the Pi not of zero is equal to one, you will get the first equation with the term one and all other terms are going to be zero. And you know the Laplacian sum of derivative of the function. So you substitute, you take a Laplacian sum over the forward Kolmogorov equation with this initial condition as well as Pi n of zero is equal to zero for n not equal to zero. So you will have an algebraic equation that is n plus one algebraic equations as a function of theta.

You have to solve this algebraic equation system of algebraic equation in terms of theta. Once you are able to solve these and take an inverse Laplacian sum and that is going to be the system size at any time t. You can start saying that this is going to be of the solution A times alpha n and B times beta power n, where alpha and beta are given in this form, where alpha is equal to this plus something and beta is equal to minus something, minus square root of this expression.

So you have a alpha as well as beta and now what do you want to find out, if you find out the constant A and B you can get Laplacian sum of Pi n of t. Then you take an inverse Laplacian sum and you get the Pi n of t.

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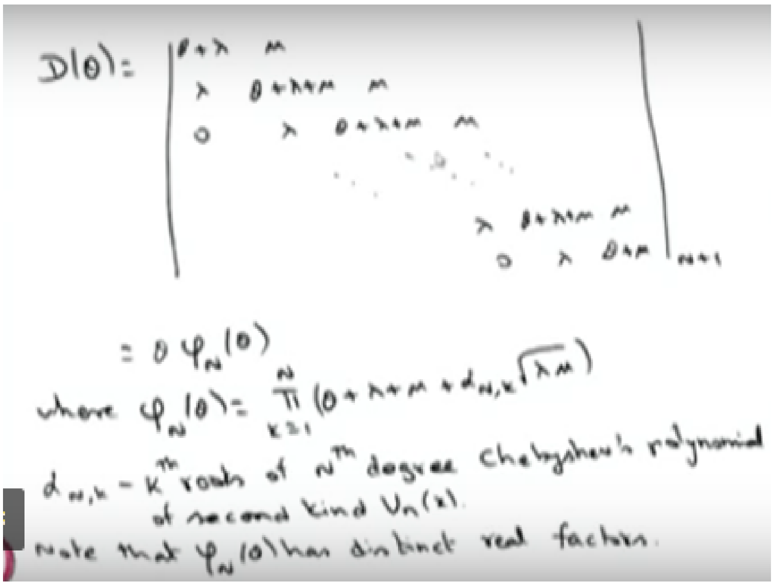

So for that you need the determinant of matrix of this form and here this is nothing but all these values are death rates, these are all the birth rates. And this is corresponding to the MM1N model. And the same logic goes for the transient solution for the MMCK as well as MMCC. So instead of this lambdas and mu's you will have a corresponding birth rates and death rates.

But ultimately you will have a n plus one matrix determinant as a function of theta. And since these three models are going to be a reducible positive recurrent the stationary probability and the limiting probabilities exist. Therefore, this determinant going to be always of the form theta times some other function as the degree, as a polynomial of degree n in the function of theta. So this theta is corresponding to the stationary probabilities or the limiting probabilities.

Therefore, always you can get the n plus one-th order matrix determine that is theta times the polynomial of degree n is a function of theta. For the MM1N model the birth rates are lambda and the death rates are mu and you can get this polynomial also in the form of product. The product of theta plus lambda plus mu times alpha of n comma k square root of lambda mu, where alpha of n, k is nothing but the k roots of n-th degree Chebyshev's polynomial of second kind.

There is a relation between the birth-death process with the orthogonal polynomial. For instant, the MM1N model the finite capacity MM1N model, the corresponding orthogonal polynomial for this birth-death process is the Chebyshev's polynomial of the second kind. Similarly, you can say the orthogonal polynomial corresponding to the MMCC model that is a Charlier polynomial, like that we can discuss the corresponding orthogonal polynomial for the finite capacity birth-death processes.

So here for the MM1N model this is related to the Chebyshev's polynomial of second kind, that is P U n of x. So once you are able to get the Chebyshev's polynomial roots and that roots is going to play a role in the product form and that is going to be the polynomial. Note that this polynomial has a distinct real factor.

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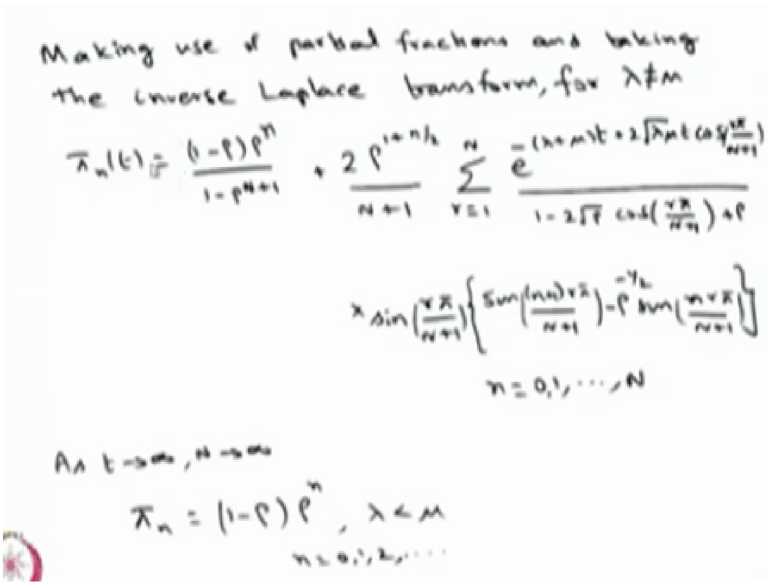

Therefore, you can use the partial fraction, then you can take the inverse Laplacian sum to finally you can get the Pi n of t. I am skipping all the simplification part and main logic is this n plus one-th order matrix determinant and that determinants has the factors. And those factors are related to the Chebyshev's polynomial roots. So once you use all those logics and use the partial fraction.

Then finally you take the inverse Laplacian sum, for lambda is not equal to mu, you will get steady state or stationary probabilities plus this expression and this is the function of t e power minus lambda plus mu times t plus two times square root of lambda mu times t cos of r by pi n plus one and denominator this expression multiplied by this.

And here this result is related to the initial condition zero, that means at time zero the system is empty. If the system is not empty, then you will have a one more expression here sin of this minus another term. So that is, you will have a little bigger expression for system size is not empty. And this theta times this, that will give the corresponding partial fraction and so on inverse Laplace it will give the terms which is independent of t and that is related to the steady state probability.

Because if you put t tends to infinity and these quantities are greater than zero. So as t tends to infinity the whole terms will tends to zero. Therefore, as the t tends to infinity, we will have Pi n of t is equal to this expression and this is valid for rho is less than one. With that condition rho is less than one, those terms will tend to zero and you will have only this term and that is going to be the steady state or limiting probabilities for MM1N model.

If you make also n tends to infinity along with the t tends to infinity, you will have Pi n's that is the steady state probability for the MM1 infinity model. So even though I have explained MM1N transient solution in a brief way. But the same logic goes for the MMCC model also, the only difference is this determinant has the lambdas and instead of mu's you will have mu2, mu3, and so on.

And instead of the Chebyshev's polynomial, you will land up with the Charlier polynomial. But there is a difference between this MM1N model and MMCC model transient solution. Since the Chebyshev's polynomial has a closed form roots, you can find out the factors. So here these are all the factors and you know the factors as well as you can get the closed form expression for the MM1N transient solution where as Charlier polynomial does not have a closed form roots.

Therefore, you will land up with the numerical result for the transient solution for MM1N, MMCC model.

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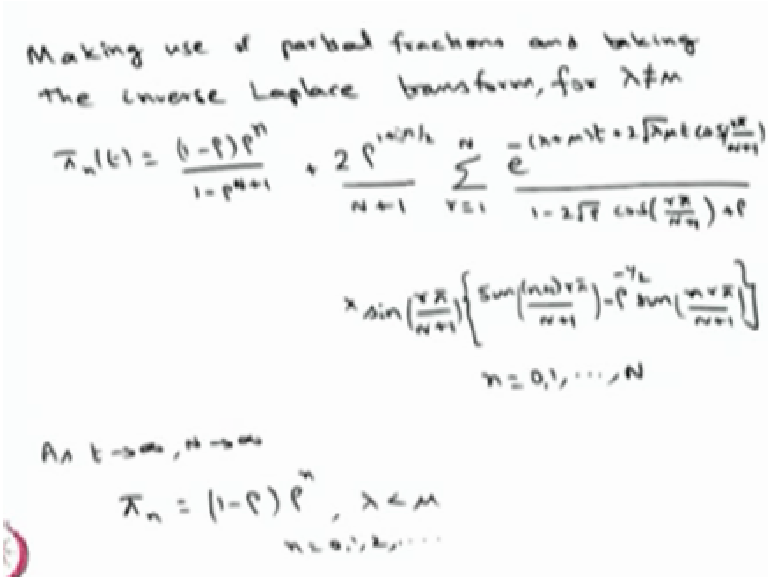

In the case of a continuous time Markov chain, that is a finite source Markovian Queuing models. This model is also known as a machine repairman model and you can think of these PCs are nothing but machines and this is nothing but the repairman. And here the scenario is we have a K PCs and each PC can give a print job and the inter arrival of print jobs that is exponentially distributed by the each PC.

Therefore, the print jobs that is follow a arrival process that is a Poisson process with the parameter lambda from each PC. And once the print jobs come into the printer, it will wait for the print. And the time taken for the each print that is also exponentially distributed with the parameter mu.

And here there is another assumption before the first print is over by the same PC, it cannot give another print command. Therefore, after the print is over by any one particular print job of any PC, then these things will go back to the same thing, then with the inter arrival of print jobs generated that is exponentially distributed, then the print job can come into the printer. So with these assumptions you can think the Stochastic process.

That means the number of print jobs at any time t in the printer that is going to form a Stochastic process and with the assumption of inter arrival of print jobs, that is exponential and actual printing job that is exponentially distributed and so on. Therefore, this is going to be a birth-death process, with the birth rates or K times lambda or K minus one times lambda and so on, where as the death rates that is mu, because we have only one repair.

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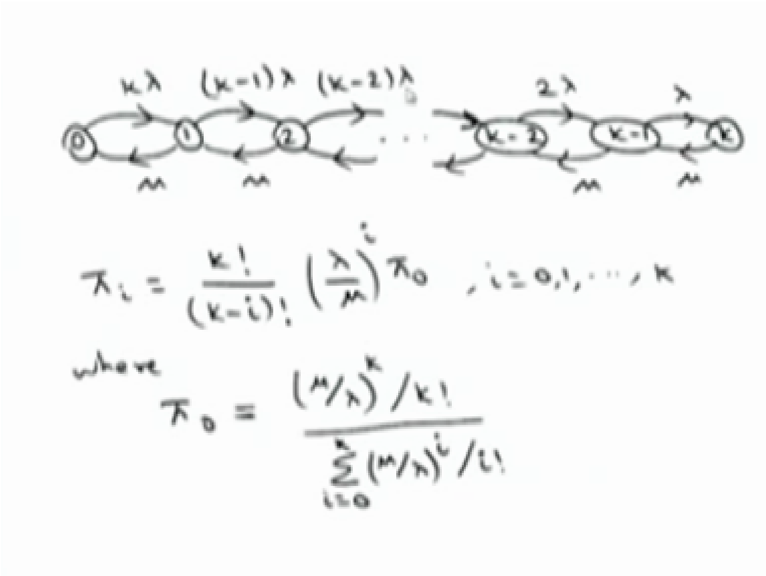

So this is nothing but system size number of jobs in the print job, printer. So therefore that varies from zero to capital K, because we are making the assumption more than one print job cannot be given by the same PC before the print is over. And from zero to one, the arrival rate will be any one of the K PCs. Therefore, the arrival rate is K times lambda and already one print job is there in the system printer.

Therefore, out of K minus one PCs one print job can come, therefore the inter arrival time that is exponentially distributed with the parameter K minus one times and lambda and so on. So this is a way you can visualise the birth rates, where as the death rates are mu. Once you know the birth rates and death rates you can apply the birth-death process concept to get the steady state probabilities.

So here we are getting the Pi A's in terms of Pi not, and using the summation of Pi A is equal to one, you are getting the Pi not also. And once you know the steady state probability, you can get the all other measures. So the difference is in this model it is a finite source, therefore the birth rates are the function of, it is the state dependent birth rates where as the death rates are mu’s only. Simulation of a Queuing model, I will do it in the next lecture.

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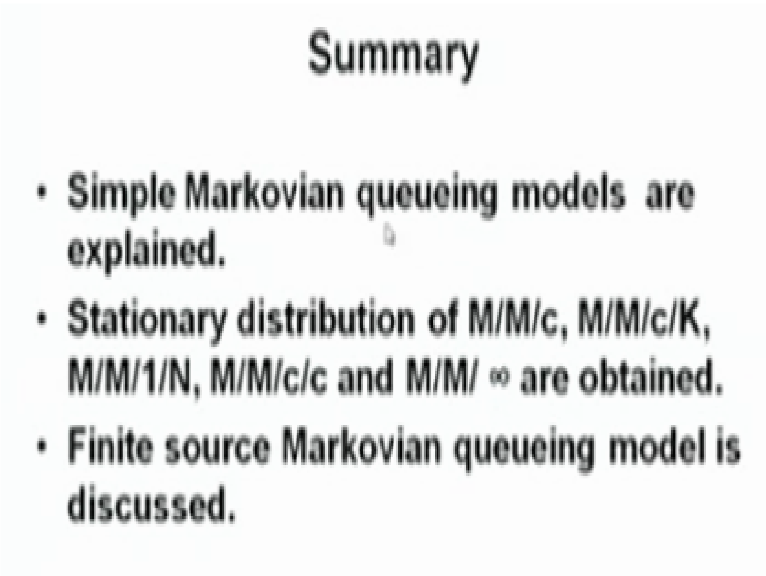

The summary of today's lecture, I have discussed the simple Markovian queueing models other than MM1 infinity, that I have discussed in the previous lecture and stationary distribution and all the other performance measures using the birth-death process we have discussed for this queueing model and finally I discussed the finite source Markovian queueing model also.

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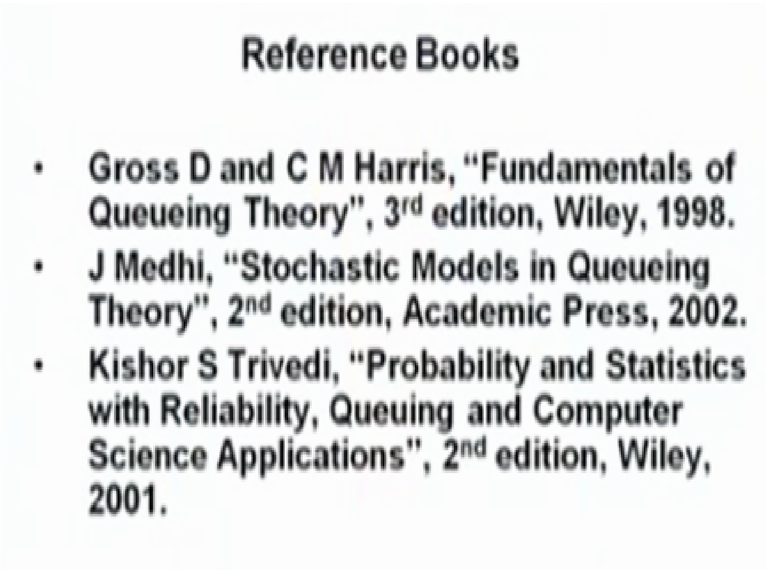

These are all the reference books. Thanks.