Stochastic Processes - 1
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Lecture - 67
M/M/c/K Model, M/M/c/c Loss System, M/M/? Self Service System

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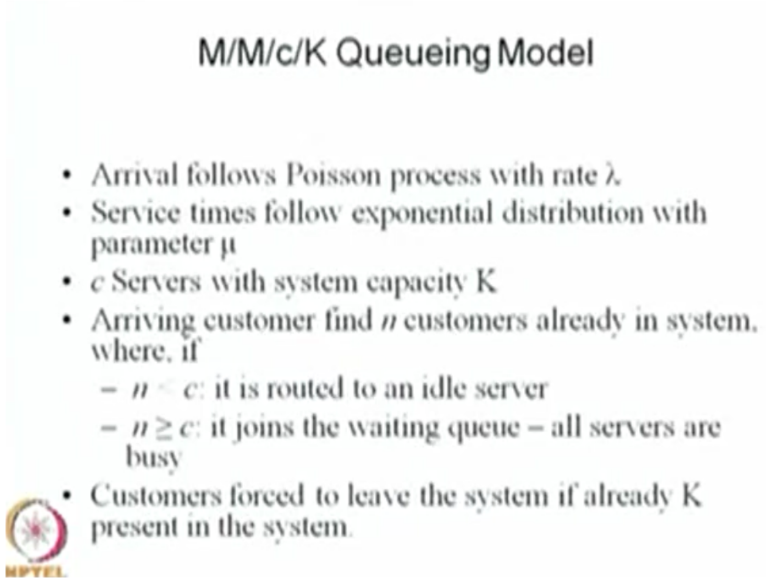

Now I am moving into MMcK model. So here the change is instead of one server in the MM1 model you have more than one servers c and you have a finite capacity that is capital K, capacity of the system. So the arrival follows the Poisson process, service is exponential, we have c identical servers. The capacity is capital K and this is the scenario in which whenever the system size is less than c it will be routed into the ideal server.

And if it is greater than or equal to c, that means all the servers are busy that means the customer has to wait. But if the system size is full, that means c customers are under service and k minus c customers are waiting in the queue for the service and then whoever comes it will be rejected for us to leave the system.

And therefore you have a waiting as well as blocking because it is a finite capacity that is blocking and since you have, always we choose k such that it is k is always greater than or equal to c. If k is equal to c, then it is a loss system and then if k is greater than c, then k minus c customers maximum can wait in the system, in the queue.

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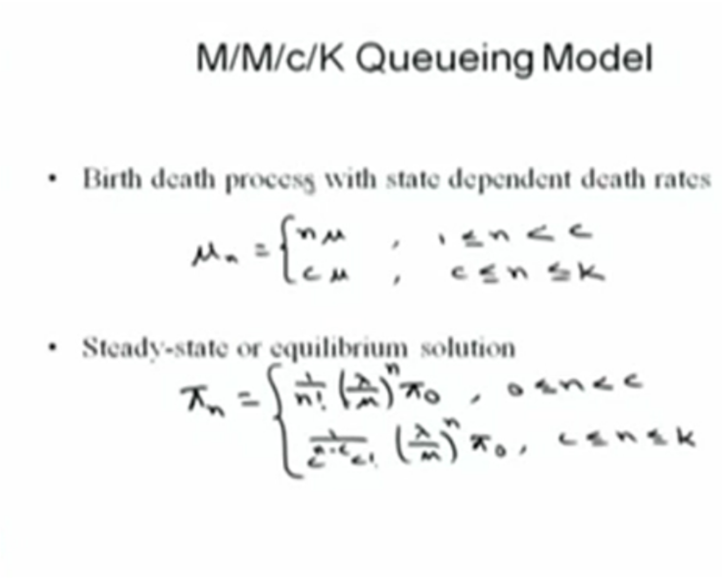

Therefore, the underlying stochastic process, here the stochastic process is again number of customers in the system at any time t. Therefore, this stochastic process is also going to be a continuous time Markov chain because of these assumptions, inter arrivals or exponential distribution service, each service by each server is exponentially distributed and all are independent and so on.

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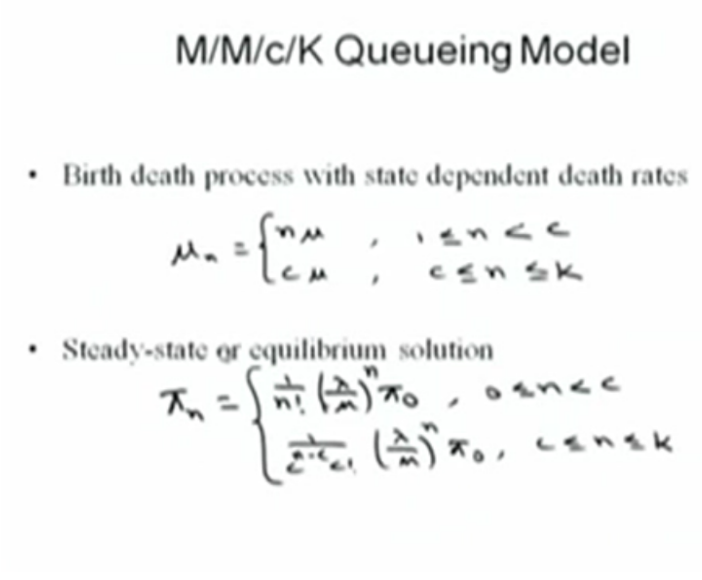

So with these assumptions these stochastic processes continuous time Markov chain and at any time only, only one forward or only one backward the system can move. Therefore, it is going to be a birth death process also and the birth rates are lambda because it is a infinite source population, so all the lambda n are going to be lambda, whereas the death rates are state dependent.

That is going to be n times mu, lies between 1 to c, from c to k onwards it is going to be mu. I have not drawn the state transition diagram for MMcK, but you can visualize the way we have MM1n and MMc model, so it is a combination of that, that is going to be the state transition diagram. Since it is a finite capacity model, it is easy to get the steady state and the equilibrium solution.

So first you solve pi q sequel to zero that means you write pi n in terms of pi not and use a normalizing constant summation of pi i is equal to one, using that you will get pi not. So I have not written here. So use the normalizing constant summation of pi i is equal to one, get the pi not, then substitute pi not here therefore you will get pi n in terms of pi n completely.

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1. $E(N) = \sum_{n=1}^{K} n \pi_n$

2. $E(Q) = \sum_{n=c}^{K} (n-c) \pi_n$

3. $E(R) = \frac{E(N)}{\lambda_{eff}}$; $\lambda_{eff} = \lambda(1-\pi_K)$

4. $E(W) = \frac{E(Q)}{\lambda_{eff}}$

After that you can get all other average measures, the way I have explained MM1n and the MMc infinity, the combination of that you can get the average number of customers in the system, average number of customers in the queue. That is n minus c times pi n, combination, the summation goes from c to K and the average time spend in this system.

Since it is a finite capacity you have to find out the lambda effective, effective arrival rate, that his one minus, its capacity is capital K, therefore one minus pi K and that is the probably that the system is not full. So the effective arrival rate is lambda times one minus pi K, substitute here and get the average time spend in the system and similarly you can find out the average time spend in the Q hours using the little's formula.

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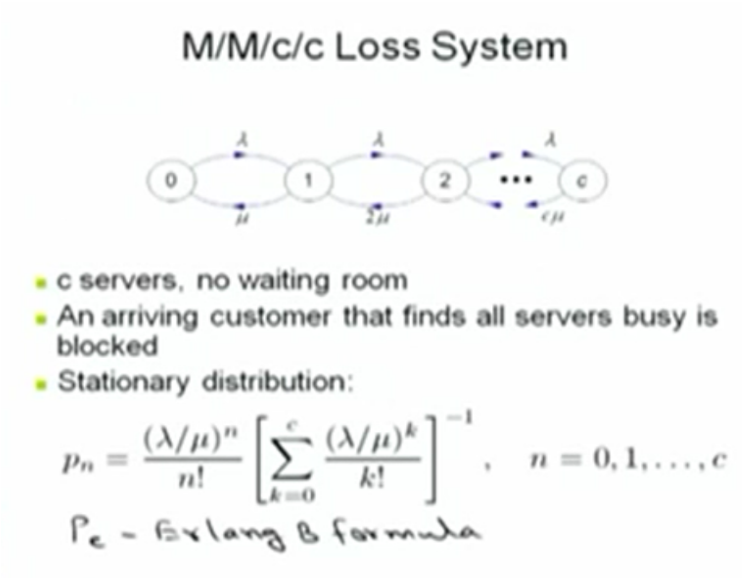

Now I am moving into the fourth simple Markovian queuing model. First I started with the MMC infinity, MM1n, then I did MMcK and now I am going for capital K is equal to c that is loss system. It is not a queuing system because we have c servers and the capacity of the system is also c.

Example is you can think of parking lot which has some c parking lots and the cars coming into the system that is if you make the assumption is inter arrival time is exponentially distributed and the car spending time in each parking lot that is exponentially distributed then the parking lot problem can be visualized as the MMc loses too. So here we have a c identical servers, no waiting room.

So since it is a capacity and c waiting room you think of self service with the capacity c, that also you can visualize. So here the inter arrival times are exponentially distributed and service by each server, that is exponentially distributed with the parameter mu therefore the system goes from two to one, one to zero, so on, it is going to be how many customers in the system and completing their service.

Therefore, the time is exponentially distributed with the sum of those parameters accordingly. Therefore, it is going to be one mu, two mu till c mu. Since it is a finite capacity and so on, it is a reducible model, positive recurrent. So this listed probability exists, limiting probability also exists and that is same as the equilibrium probabilities also. Therefore, by using P q is equal to zero and summation of p i is equal to one you can get the steady state or the equilibrium probabilities, that is p n.

The p suffix c, that is nothing but the probability that the system is full and that is same known as the Erlang B formula. So this is also useful to design the system for a given or what is the optimal c such that you can minimize the probability that the system is full, for that you need this formula therefore to do the optimization problem over the c and here we denote p suffix c, that is Erlang B formula. Whereas Erlang C formula comes from the MMcK model for the last system we will get the Erlang B formula.

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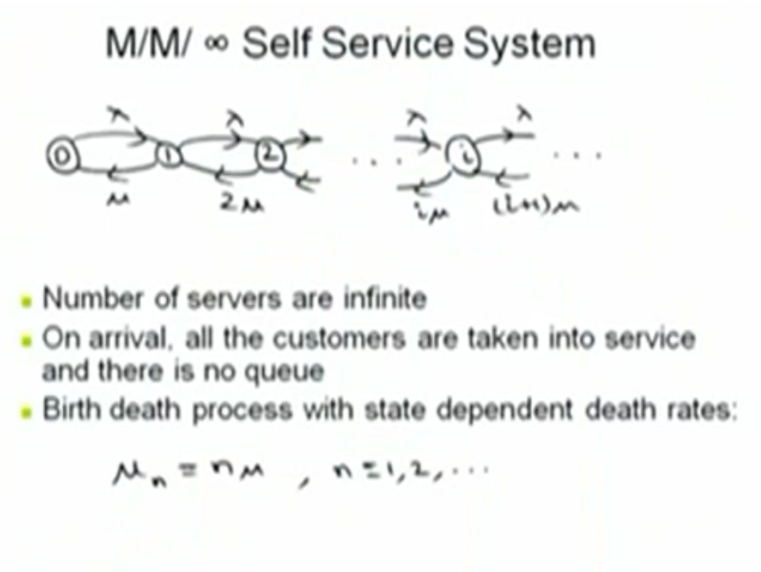

The fifth model, that is MM infinity, it is not a queuing model, because servers are infinite and limited servers in the system and therefore the customers whoever enters he will get it immediately serviced. The service will be started immediately whereas the service time is exponentially distributed with the parameter mu by each server, all the servers are identical.

The number of servers are infinite here. Therefore, you will have the underlying stochastic process for the system size that is a birth death process with the birth rates are lambda because the population is from the infinite source, the death rates are one mu, two mu and so on because the number of servers are infinite. So the model which I have discussed in today's lecture, all the five models are the underlying stochastic processes, birth death process.

This is simplest Markovian queuing models.

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Steady-state Distribution

$$\pi_n = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n \pi_0 , \quad n = 1, 2, \cdots$$

Using $\sum_{i=0}^{\infty} \pi_i = 1$, $\pi_0 = e^{-\frac{\lambda}{\mu}}$

Hence,

$$\pi_n = \frac{e^{-\frac{\lambda}{\mu}} \cdot \left(\frac{\lambda}{\mu}\right)^n}{n!} , \quad n = 0, 1, 2, \cdots$$

$N \sim \text{Poisson}\left(\frac{\lambda}{\mu}\right)$

You can get the steady state distribution, use the same theory of birth death process and if you observe these steady state probabilities is of the same Poisson, it is of the form that is probability mass function of a Poisson distribution. Therefore, you can conclude in a steady state, number of customers in the system that is Poisson distributed with the parameter Lambda by mu.

Because the probability mass function for the pi n is same as the probability mass function of exponentially distributed random variable with the parameter lambda by mu.