Stochastic Processes - 1
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Lecture - 64
Burke's Theorem and Simulation of M/M/1 queueing Model

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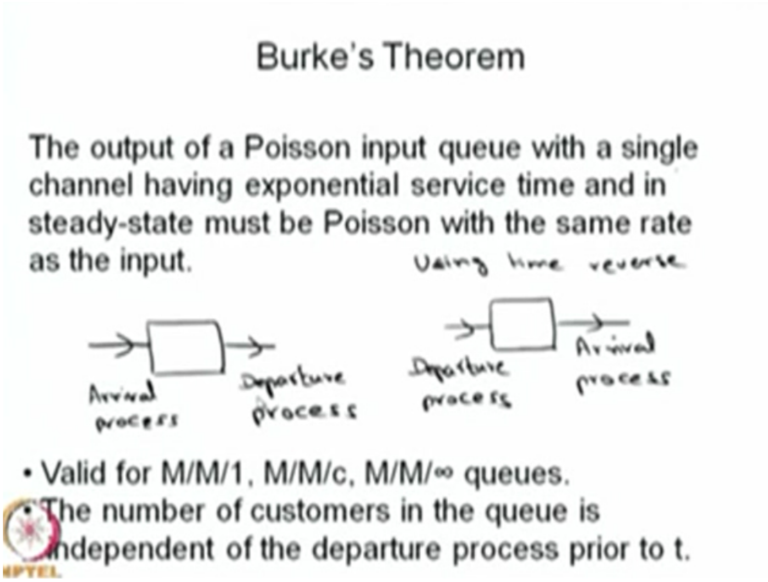

Here I am giving the concept of output process. The arrival follows the Poisson process for the MM1 queue and the service is exponentially distributed which is independent of the arrival process and the customers leave the system. Now the question is what is the distribution of the departure process. That means, what is the inter departure time.

After first customer leaves how much time it takes for the second customer leave the system and then the third customer how much time it takes for the inter departure time and therefore what is the distribution of the departure process. That is given by the Burke's Theorem. The output of a Poisson input queue with a single channel having exponential service time and in steady state must be a Poisson with the same rate as the input.

So whenever you have a system in which the arrival follows the Poisson and whenever the system has a single channel and the service time is exponentially distributed, in a longer run the departure process is also going to be a Poisson process and the rate will be the same rate as the arrival process. So this can be proved, but here I am giving the interpretation using the time reverse process, because in a steady state this model is going to satisfy the time reverse model.

Therefore, this stationary distribution exists and if we make this MM1 queuing model, the underlying birth death process satisfies the time irreversibility equation. Therefore, using the time reverse you can conclude the departure process, you can reverse it and that is going to be independent of the arrival process and this is also going to be again Poisson process.

So using the time reverse process one can prove the departure process is independent of the arrival process and departure process is also Poisson process with their same rate as the arrival rate. And even though I said it is the single channel having exponential service time and this is valid for MM1 queue, the multi server Markovian queue as well as the infinite server Markovian queue also.

So all those models can be combined with the single channel having exponential service time. Whether it is a single server or multi server or infinite server, this results hold good. And the next result is the number of customers in the queue is independent of the departure process prior to it that it also satisfies.

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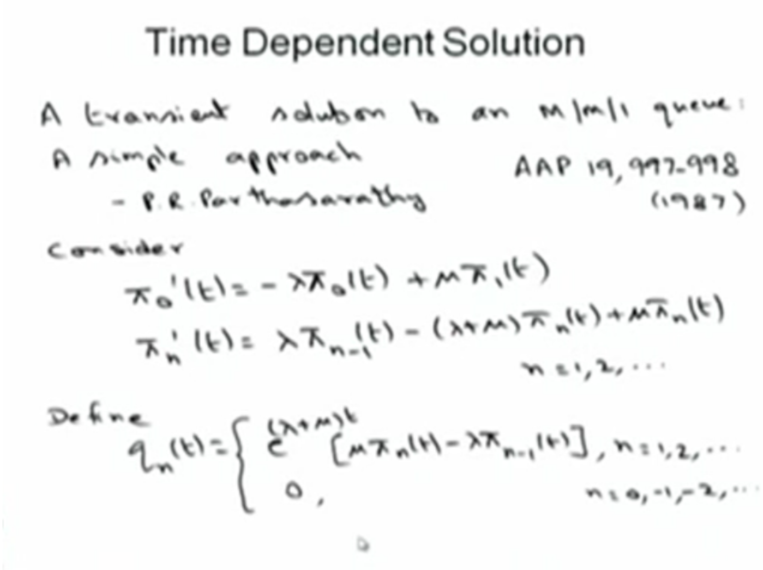

Now we are giving the time dependent solution of a MM1 queue. There are many more methods to find out the time dependent solution for a MM1 queue. It started with the spectral method and the combinatorial method and also the difference equation method. Like that there are many more methods in the literature to find out the time dependent solution and here I am presenting the time depending the time dependent solution by P.R. Parthasarthy and this work has appeared in the advanced applied probability volume number 19, 1987.

So in this paper he has considered the system of differential equation that is nothing but the forward Kolmogorov equation and making a simple function q n of t, that is the difference of pi n with the multiplication of e power lambda plus mu of t. So once you use this definition, once you convert this system of difference equation with q n of t by making a proper generating function.

That is of the form n is equal to minus infinity to infinity q n of t times s power n, therefore this is sort of generating function in terms of q n of t where q n of t is for n is equal to one to infinity. This is of difference of mu times pi n minus lambda times pi n minus one, multiplication e power this function and for n is equal to zero minus 1,2 and so on zero.

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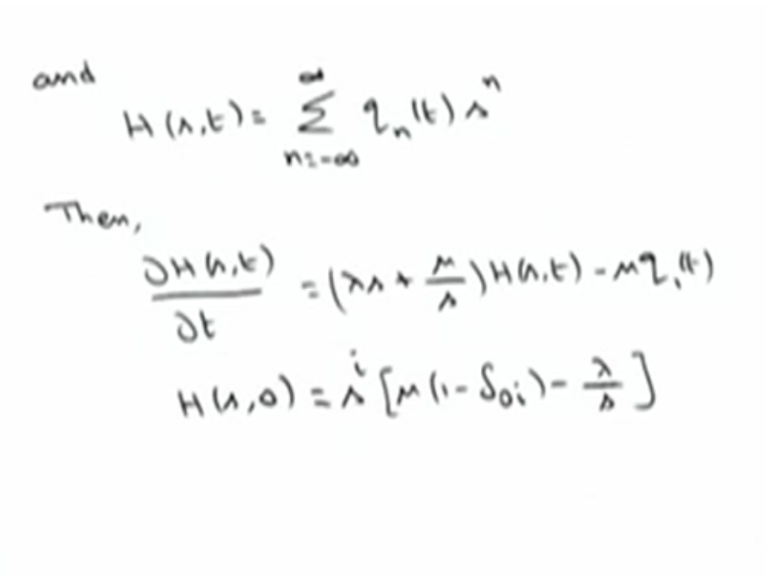

Therefore, you have a generating function. So you can covert the whole difference differential equation in terms of pi n multiplied by the one partial differential equation with the initial condition also changes because if you assume that the I customers in the system at times zero and this is going to be initial condition for function H of s comma t at t equal to zero.

So now the question is you have to solve this equation with this initial condition, this PD using this initial condition.

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The time dependent solution is

$$\pi_n(t) = \frac{e^{-(\lambda+\mu)t}}{\mu}\sum_{k=1}^{n} q_k(t)\left(\frac{\lambda}{\mu}\right)^{n-k} + \left(\frac{\lambda}{\mu}\right)^{n} \pi_0(t)$$

$n = 1, 2, \ldots$

and

$$\pi_0(t) = \int_0^t q_1(y)\, e^{-(\lambda+\mu)y}\, dy + \delta_{0i}$$

where

$$q_n(t) = \mu \beta^{n-i}(1-\delta_{0i})\left[I_{n-i}(\alpha t) - I_{n+i}(\alpha t)\right] + \lambda \beta^{n-i-1}\left[I_{n+i+1}(\alpha t) - I_{n-i-1}(\alpha t)\right]$$

$\alpha = 2\sqrt{\lambda\mu}$; $\beta = \sqrt{\frac{\lambda}{\mu}}$; $I_n(t)$: modified Bessel function

So use some identity of modified Bessel function one can get the solution pi n of t in terms of pi not, where pi not you can get it in terms of Q1 where all the q n satisfies this equation that is in terms of the modified Bessel function. So one can see the complete solution in this paper.

But here I am giving the very simple approach of getting the time dependent solution for the MM1 queue by changing this system of differential equation multiplied by one PD the initial condition and solve that PD and obtaining the pi n and pi not in terms of modified Bessel Function. Before I go to the summary let me give the simulation of MM1 queue.

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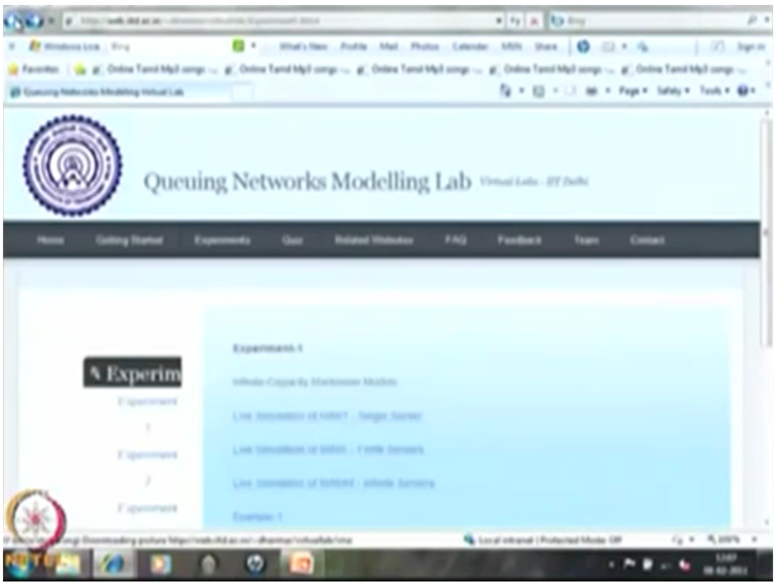

So this is the queuing network modeling lab. So in this queuing network modeling lab, one can simulate the queuing network models.

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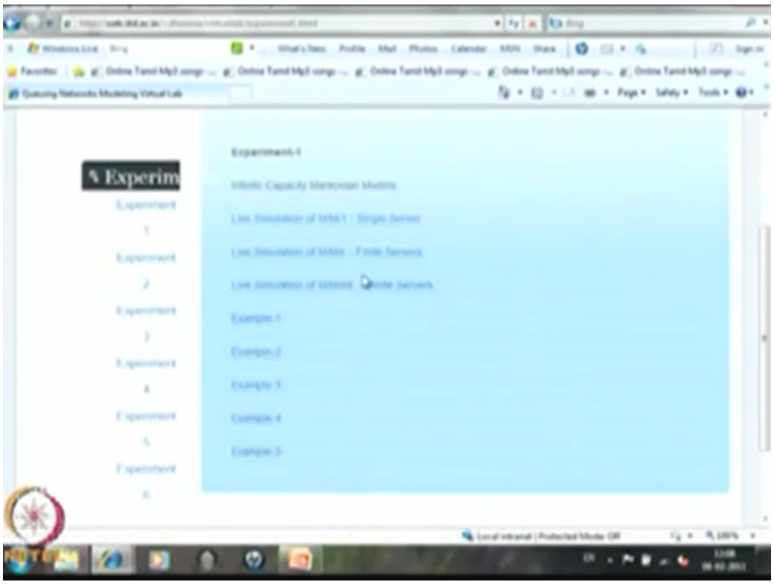

So in this I am going to explain how to simulate the MM1 queue and first experiment is nothing but live stimulation of MM1 queue single server as well as you can simulate a multi server queue model and you can go for the infinite server model also. So here I am simulating the MM1 queuing model.

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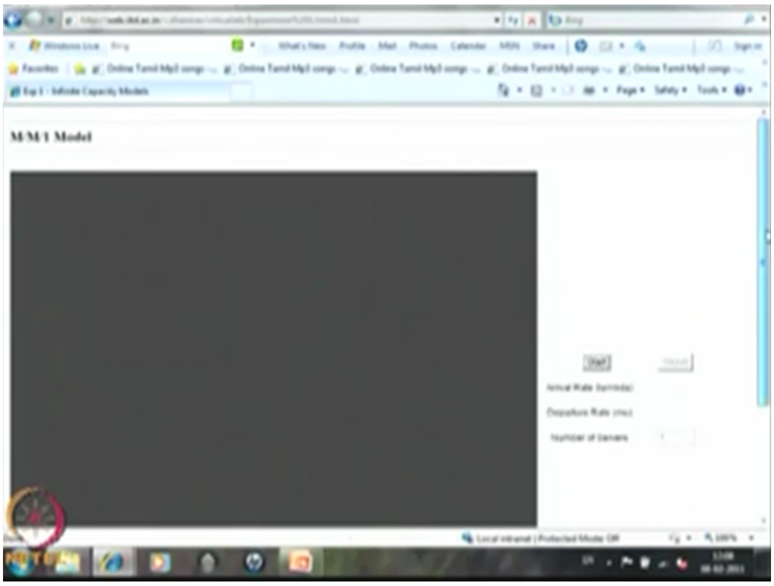

So to simulate the MM1 queuing model, you need the information about the inter arrival time that is exponential distribution you need a parameter lambda, the value of lambda as well as you need a value of mu, that mu is nothing but the service rate. So suppose you supply the arrival rate, suppose the arrival rate is two and the departure rate is 5, the number of servers is, it is MM1 queue therefore it is already one is placed, it is the number of servers.

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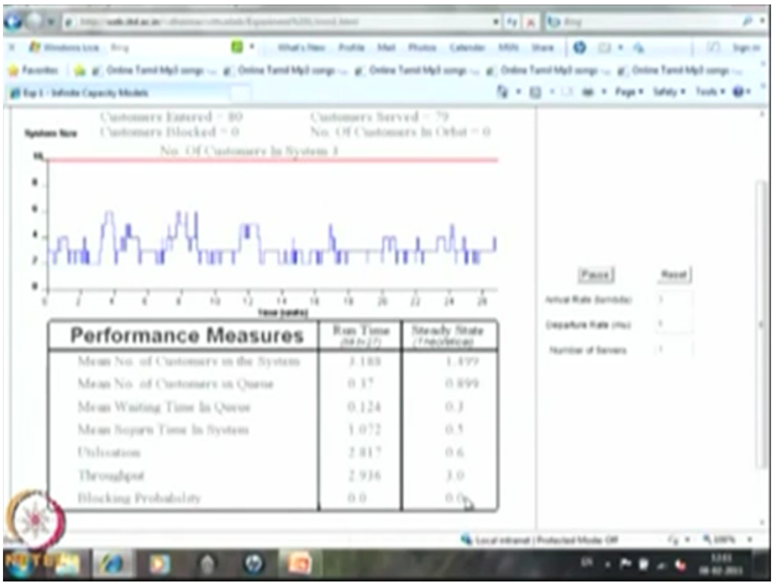

So you can start. So this is a way this system increases. So this is the actual simulation goes with that, it is the time x axis and y is the number of customers in this system and here the information is how many customers entered till this time, that is 15 customers entered and nobody is blocked because it is a MM1 queuing system.

Therefore all the customers who are entering it will be queued and how many customers are served during this time and the number of customers in the orbit this is nothing to do with the MM1 queue, this is for the retrading queues and now how many customers are in the system at this time and here this table gives the performance measures, the one we have calculated the average number of customers in the system, E of r.

And the average number of customers in the queue E of this is mean number of customers in the system that is E of n, the mean number of customers in the queue E of q, mean waiting time that is E w, mean sojourn time in the system, sojourn time, spending time, response time all are the same, the mean sojourn time in the system is nothing but F r. So this is nothing but the E of r, this is nothing but E of w, this is nothing but E of q.

And this is nothing but the E of n and the utilization is nothing but what is the probability that - so here I am giving the run time, what is the average values till this time and what is the result is going to be in longer run in a steady state and blocking probability is here zero because the system is infinite capacity model, therefore there is no one blocked, therefore the blocking probability is zero.

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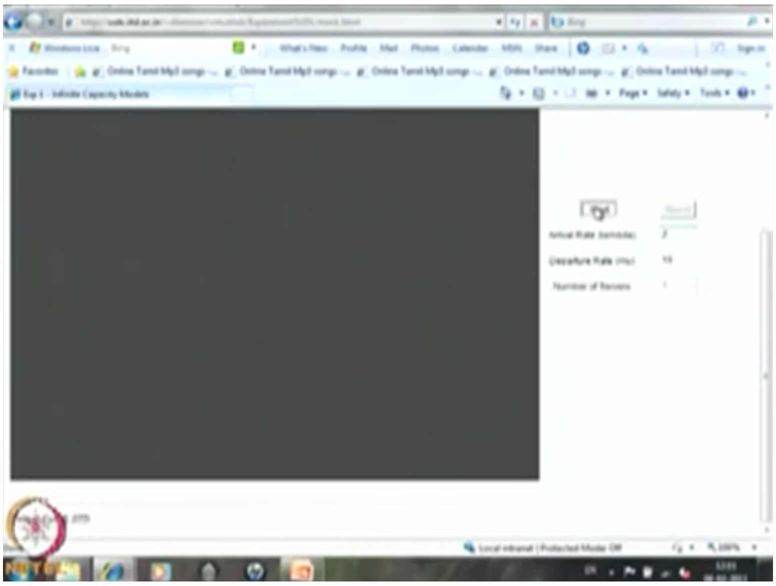

So this is the way we can reset and you can give some other values and you can start again. And you get another simulation also and initially it gives the fixed steady state results in the steady state theoretical result and the run time is nothing but what is the result over the time. With this let me complete the simulation.

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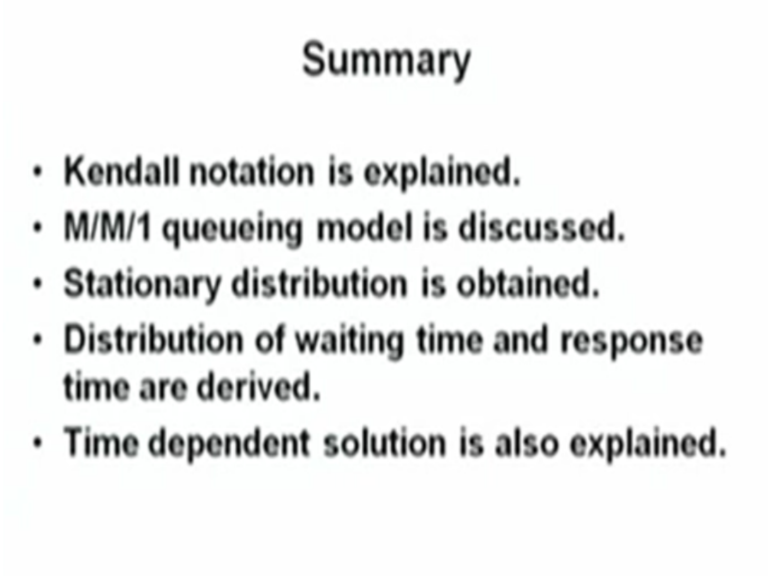

So in the summary, we have started with the Kendall notation and MM1 queue is discussed. Stationary distribution, waiting time distribution, waiting time distribution is discussed for the MM1 queue and also the time dependent solution and I have given the simulation of MM1 queue also.

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Reference Books

- Gross D and C M Harris, "Fundamentals of Queueing Theory", 3rd edition, Wiley, 1998.
- J Medhi, "Stochastic Models in Queueing Theory", 2nd edition, Academic Press, 2002.
- Kishor S Trivedi, "Probability and Statistics with Reliability, Queuing and Computer Science Applications", 2nd edition, Wiley, 2001.

These are all the reference books.