Stochastic Processes - 1
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Lecture - 63
Little's Law, Distribution of Waiting Time and Response Time

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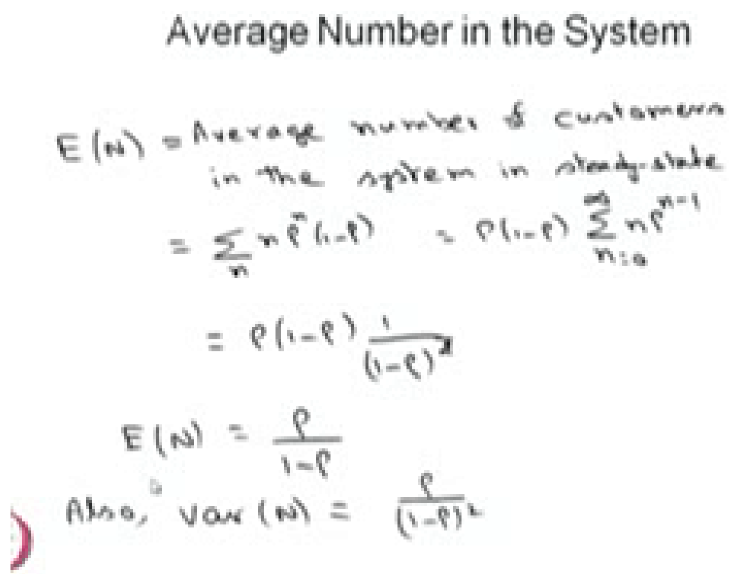

Other than stationary distribution one can find out the average measures also in the system, so suppose you make E of N that is nothing but the average number of customers in the system in steady state, since you know the probability distribution substitute pi n's here therefore n times pi n summation over n, that he is going to be the average number of customers in the system, if you do little simplification you will get rho divided by 1 minus rho, where rho is less than 1.

So this is an average number of customers in the system and also one can get a variance of the number of customers in the system also for that you have to find out the E of N square then using that formula you can get the variance of N also, so here we are getting a mean and variance of number of customers in the system in steady state.

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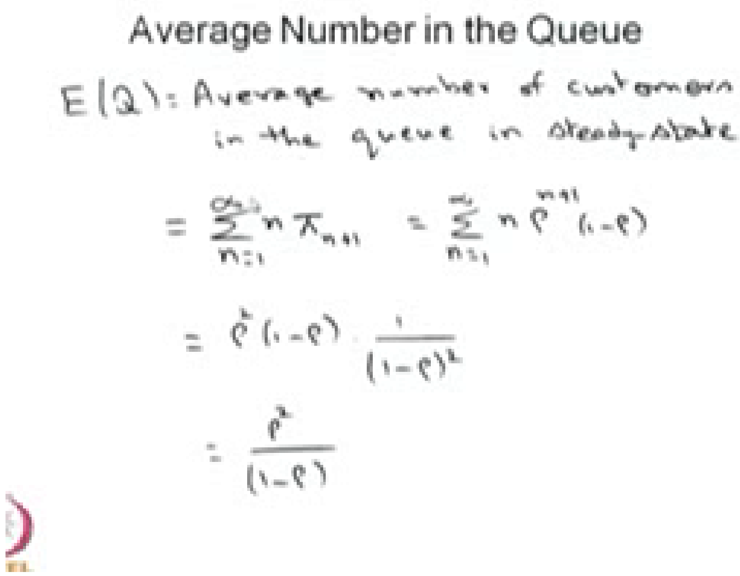

Also one can find average number in the queue, so the letter Q is a random variable, and here we are finding the expectation of Q, that is average number of customers in the queue that means before getting the service, how many customers in the system we have only one server in the system and whenever the service is going on and all other arriving customers will be queued, that means when n + 1 customers in the system n people are in the queue.

Therefore, summation n times pi n + 1, do the simplification you will get average number of customers in the queue also substitute the pi n + 1, from the one I have discussed in the stationary distribution.

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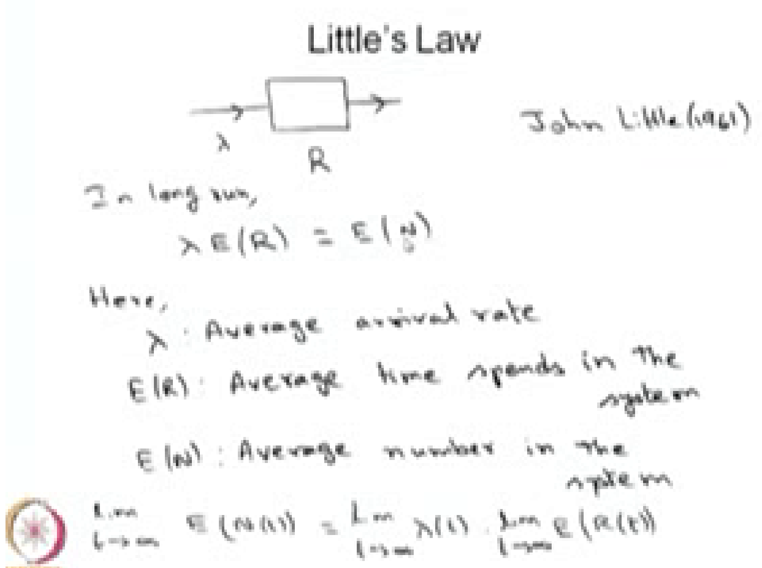

Here I am going to relate the average measures using the Little's law this is proven by John little 1961, this is valid for any system in which arrival comes into the pattern with the arrival rate lambda, and R is a time spend in the system and leave the system after the service or whatever things are over, then in a longer than one can say the arrival rate multiplied by the average time spent in the system that is same as average number in the system.

So this relation is valid for whatever be the underlying distribution - whatever be the underlying distribution of the service, underlying distribution of the arrival what it says if you have a system in which the arrival rate is mean arrival rate is lambda and the mean time spent in the system is expectation of R than that product will give average number in the system.

Since indirectly it says whenever the system has a long run in a stable system the expectation of a average number of customers during the interval 0 to t, as t tends to infinity that is going to be have a limit expectation of a N and the arrival rate lambda of t, that also has the mean arrival rate as t tends to infinity that is also going to be a sum having a limit constant lambda.

And similarly the average spent by the customers in the system at any time t and if you make t tends to infinity that expectation quantity also has a limit, therefore you will have a lambda times expectation of R is same as the expectation of N, now using this Little's law I am going to find out the - the measures for the MM1 queue model.

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R (W) : time spend (waiting time) by any customer

Using Little's formula,

$$E(R) = \frac{E(N)}{\lambda} = \frac{\rho}{\lambda(1-\rho)}$$

$$E(R) = \frac{1}{\mu - \lambda}$$

Now,

$$E(W) = E(R) - \frac{1}{\mu} = \frac{\rho}{\mu - \lambda}$$

So suppose R denotes the time spent in the system by the customer and W denotes the waiting time by any customer in the system, then I can use the Littles formula, the Little's law in the previous one, so if I know the mean arrival rate and if I know mean number of customers in the system in a longer run using these I can find out the average time spent in the system.

If I know to use the to using Little's law If I know the average number of customers in the system longer run and if I know the arrival rate then I can find out the average time spent in the system in a longer run, similarly I can once I know the average time spent in the system if I subtract the average time of my own service then that is going to be the average time waiting in the queue

So this is the average time waiting in the queue that is same as average time spent in the system minus my own average service time. in the MM1 queue model the service time is exponentially distributed with a parameter mu, therefore the average is 1 by mu, so the difference will give the average time waiting in the queue by any customer not only the average measures for the MM1 queue one can find out the actual distribution for R as well as W also.

Because this is a very simplest Markovian queuing model, whereas for all other models it is little complicated but still one can get it, so this is the easy model in which one can find out the distribution of the time spent in the system as well as the time R as well as the waiting time by a customer in the queue.

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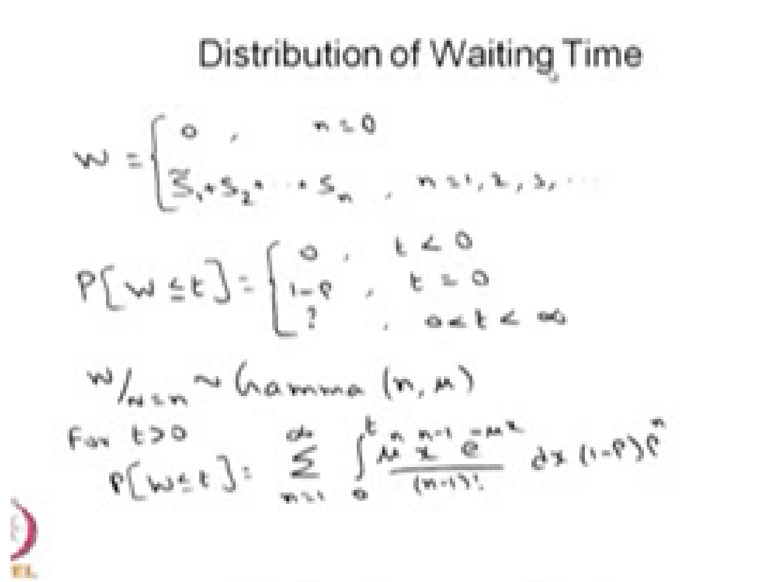

First let us go for finding out the distribution of waiting time, waiting time means if no one in the system when you arrive then you are waiting time is 0, you are immediately going to get the service so the service time is your time spent in the system usually the time spent in the system is the time of your service + time of waiting time, so here I am finding the only the distribution of waiting time first so whenever the system is 0, you are waiting time is 0.

whenever no customer in the system in the waiting time is 0, whenever more than or equal to one customer in the system then the waiting time is same as the remaining service time for the customer who is under service + the customers in the queue before you join in the queue so those peoples service time addition + the residual are the remaining service time of the customer who the first customer who is under service.

So this total time is the waiting time whenever the system is non-empty, whenever the system is empty then the waiting time is 0, therefore the W is a random variable either it takes the value 0 or it takes a value greater than 0 based on the time - time of service of previous n people in the ahead of you, therefore W is a mixed random variable which has the probability mass function at 0 as well as a density function between the intervals 0 to infinity.

So let me try for finding out the CDF of this random variable, so the CDF is going to be 0, till 0 at 0 it has the CDF 1 minus rho, because when the waiting time is 0 that is equivalent of no one in the system so in the long run no one in the system that is a pi naught.

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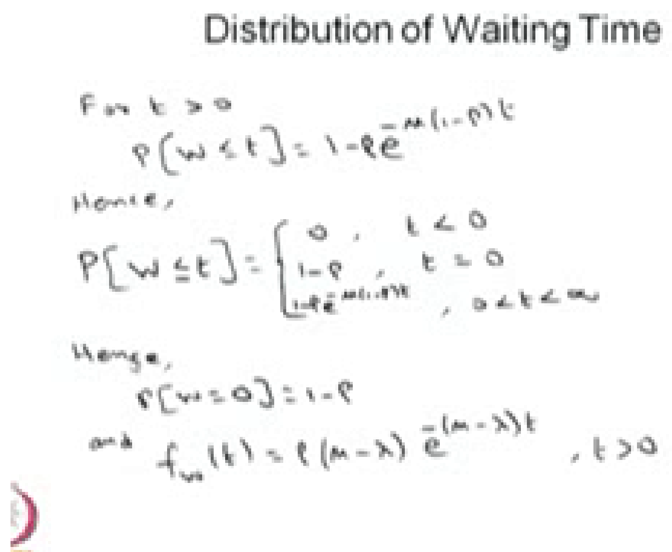

And the pi naught probabilities - pi naught probability is 1 minus rho, system is empty in a longer run that is 1 minus rho, therefore the CDF at 0 that is same as 1 minus rho of that is pi naught between the interval 0 to infinity when we have to find out the distribution of W.

Whenever n customers before you - before you join in the system that conditional distribution the distribution of a W given the number of customers in the system is n that distribution is nothing but the service time of the n customers, the first customers remaining service time, the service time of the first customers is exponential distribution the residual or remaining service time of the first customer that is also exponential distribution because of memoryless property.

So this is exponential distribution, this is second customer service times that is exponential distribution and similarly, for the n th customer also service time is exponentially distributed and the way we made assumption all the service times are independent and each one is exponentially distributed with a parameter mu, therefore this is a sum of n independent exponentially distributed random variables.

Therefore, the sum of n exponentials that is going to be a gamma distribution with the parameters n and mu, there are many ways of finding out the distribution but here I am just explaining through the distribution concept this is sum of n independent exponential distribution therefore you can conclude it is gamma distribution with a parameter - parameters n and mu, once you know the conditional distribution our interest is to find out the unconditional one.

That means for t is greater than 0 and CDF at the point t, that is nothing but what is the conditional density probability density and what is the probability of n customers in the system that multiplication with the possible n will give the CDF between the interval 0 to t, so I have a density function of a gamma distribution probability density function with a parameters n and mu.

And this is a probability density function multiplied and integration between 0 to t that will give the CDF and unconditional multiplied by probability of n customers in the system that with the summation that will give the unconditional, therefore the CDF is going to be summation n is equal to 1 to infinity integration 0 to t of the probability density function of a gamma distribution multiplied by n customer in the system.

If you do the simplification you will get the 1 minus e power minus mu times 1 minus rho times t, therefore we can substitute here - here I made a mistake so here it is multiplied by rho, so 1 minus rho times - 1 minus rho times - so 1 minus rho times e power minus rho e power minus mu times 1 minus rho, that is going to be the, so once you are getting the CDF you can conclude this is a mixed random variable with the probability mass at 0 is 1 minus rho.

And the density function between the interval 0 to infinity that is a rho times mu minus lambda times e power minus mu minus lambda times t, that is the probability density function for a distribution of waiting time.

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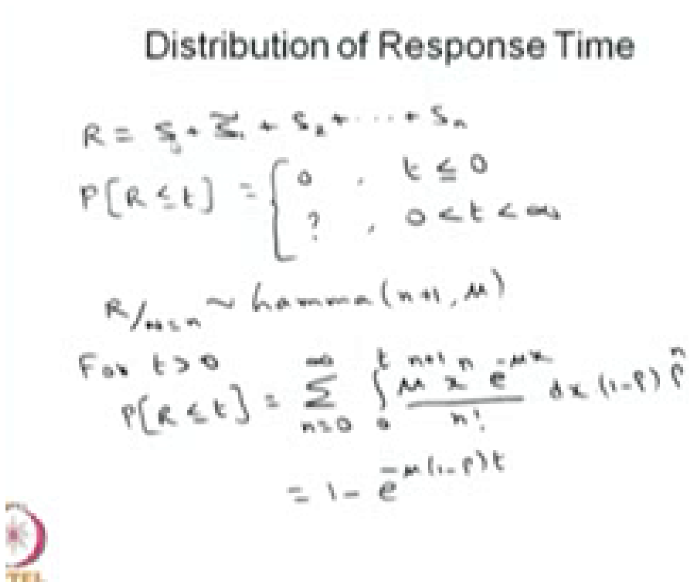

Similarly, one can get the distribution of response time also or the total time spent in the system, the total time spent in the system that is nothing but that's a random variable and there is a dual service time of the first customer who is in the system + all the remaining n customers in the system in the queue + your own service time, therefore here this is not a mixed random variable this is a continuous random variable.

Because your service time is a continuous random variable which is exponentially distributed, therefore the R is going to be sum of your own service + the remaining service of the first person in the system if and so on, till the n th customer who is in the queue, therefore this is the CDF of the random variable R, here also one can argue when n customer in the system before him who enter into the system that is a sum of exponential independent random variable and so on.

Therefore, this is going to be a gamma distribution with the parameters n + 1 mu and for t greater than 0, find out the CDF using the first conditional then unconditional multiplied by 1 minus rho times rho power n summation over n is equal to 0 to infinity, because there is a possibility no one in the system or 1 customer, 2 customer and so on, therefore the running index is 0 to infinity, do the simplification you will get 1 minus e power minus rho times this.

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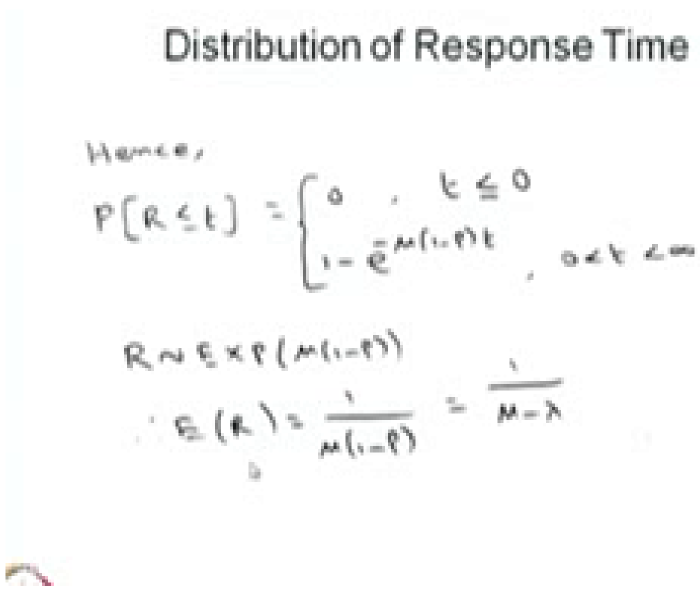

Therefore, you can substitute here and if you see the CDF is same as the CDF of exponential distribution with a parameter that is mu times 1 minus rho, therefore you can conclude the total time spent in the system is exponentially distributed with a parameters mu times 1 minus rho, if you find out the average time that is going to be 1 divided by the parameter that is this.

The same thing you got it in the average response time from the Little's formula using once you know the value of lambda and expected number in the system using Little's law you got expectation of time spent in the system that is same result, so here we are getting first finding the distribution of time spent in the system or response time then we are finding the average time.