

Stochastic Processes - 1
Dr. S. Dharmaraja
Department of Mathematics
Indian Institute of Technology - Delhi

Lecture - 60
Compound Poisson Process and Examples

Next I am going to give some more process related to the Poisson process.

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Non-homogenous Poisson Process

Let $n(t)$ denote the number of customers arriving in $(0, t)$.
The arrival process $\{N(t), t \geq 0\}$
has a Poisson distribution
$$P(N(t) = i) = \frac{(\lambda(t))^i e^{-\lambda(t)}}{i!}, \quad i=0,1,\dots$$

where
$$\lambda(t) = \int_0^t \lambda(x) dx$$

The first one is the non-homogeneous Poisson process, let $N(t)$ you know the number of customers arriving in the interval 0 to t , the arrival process has a Poisson distribution but here the change instead of the mean arrival rate is a constant mean arrival rate is a constant λ but here it is a function of t $\lambda(t)$ is the cumulative rate t time t that is a change from the Poisson process then this stochastic process is called non homogeneous Poisson process.

Instead of mean arrival rate is a constant, here the $\lambda(t)$ is a function of t therefore this stochastic process is called non homogeneous Poisson process.

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Compound Poisson Process

Consider a Poisson process $\{N(t), t \geq 0\}$

Let X_i denote the number of customers arriving in i 'th arrival.

Let $X(t)$ denote the total number of customers arriving during the interval $(0, t)$.

$$X(t) = X_1 + X_2 + \dots + X_{N(t)}$$

Then $\{X(t), t \geq 0\}$ is a Compound PP.
 If $P\{X_i = i\} = 1, \forall i$, then $\{X(t), t \geq 0\}$ is a PP.



Second one Compound Poisson process using Poisson process one can develop a little complicated stochastic process related to the arrival that is called compound point of Poisson process, consider a Poisson process n of t then you define a random variable X_i denote the number of customers arriving at the i th time point of arrival, X_1 denotes how many arrivals takes place at the time of first arrival first arrival time point.

Making a new random variable X of t that involves t that denotes the total number of customers arriving during the interval 0 to t that means it is going to be how many arrival takes place in the first time point X_1 , how many arrival takes place at the second time of arrival that is X_2 and so on $+ X$ of t X of X of X of X N t , here N t is a random variable and how many arrival takes place that is X_i 's altogether that is going to be the total number of arrivals.

And X_2 will be what is the second time of arrival how many arrival takes place therefore I am the X_i 's are independent and identically distributed random variables with some distribution function G independent of the Poisson process N of t , so this is nothing but a random sum because these are all the random variable and how many random variables you are going to add that depends on the value of N of t over the t .

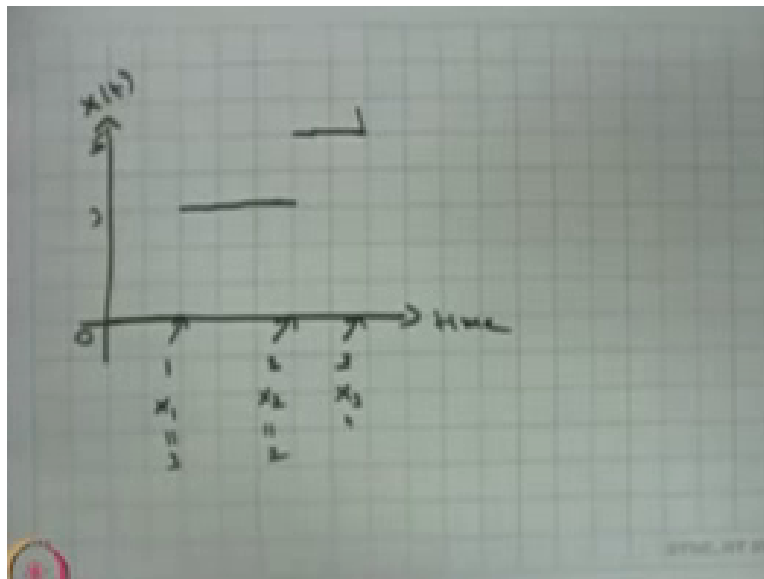
Therefore, this is a random sum of X_i 's with N of t , obviously these two are independent X_i 's are independent of N of t , and since it is a number of our customers arrival during the in the i th time

point, therefore X_i 's are discrete random variable - X_i 's are discrete random variable and N of t is also Poisson process, therefore X of t is going to be a discrete state continuous time stochastic process.

And we are using a Poisson process to get these stochastic process therefore it is called a compound Poisson process. one can deduce Poisson process from a compound Poisson process by substituting each X_i takes a value only one unit that means the number of customers arriving at the i th time point is going to be only one that means if I make a P probability of X_i takes the value only one that probability is one for all i .

Then I will have only one value possible till N of t , then this is going to be a Poisson process, suppose the probability of X_i is supposed the X_i 's are going to be a discrete random variable with the possible values are 0 1 2 and so on, then the X of t is going to be a Poisson process

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I can make a simple example path for the compound Poisson process this is over the time and this is over the N X of t , suppose these are all the time points in which arrival time point so this is the first arrival time point and this is the second arrival time point and this is the third arrival time point It can be anywhere in the continuous time therefore this is called discrete state continuous time stochastic process.

So here I am relating with the random variable X_1 , this X_2 random variable this is X_3 so till the first arrival till the first time of arrival the number of customers in the system is 0, at the first time of arrival the X_1 suppose you think you make the assumption X_1 takes a value 3 therefore this will be incremented by 3 till the second arrival, at the time of second arrival suppose you assume that this takes value 2 with some probability.

Probability of X_2 takes the value 2 is greater than 0 so you have assumed the value 2, it can take in any other value also so it is incremented by 2 till it takes a third arrival the value is so this is 0 this is 3 then $3 + 2$, 5 at this time whatever be the number of arrival accordingly this can take some value. So the difference between the compound Poisson process.

And the Poisson process in the Poisson process the increment will be only - only one-unit increment over the time whenever the time it which the arrival occur - arrival time approves, whereas here wherever the time of arrival time approves the number of customers entered that need not be 1, it can be more than or equal to 1 so that is the way the jump goes, therefore this is called a compound Poisson process.

So we have seen two variations of Poisson process one is a non-homogeneous Poisson process the other one is a compound Poisson process, so before I go to the, I complete let me give the solution for the first example - second example which I started that is this question.

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Example 2

Consider the car insurance claims reported to insurer. Assume that, the average rate of occurrence of claims 10 per day. Also, assume that this rate is constant throughout the year and at different times of day. Further assume that in a sufficiently short time interval there can be at most one claim. What is the probability that there are less than 2 claims reported on a given day? What is the probability that time until the next reported claim is less than 2 hours?

The car insurance problem we have discussed two problems, the first problem is related to the bus stand bus arrival issues and this is the car insurance problem, so in this problem we have not assume the Poisson process, but the problem is related to the Poisson process one can assume it is a form of Poisson process, because you see the assumptions the average rate of occurrence of claims is a 10 per day.

Also the rate is a constant and in a very small interval of time at most one claim can happen the questions are what is the probability that there are less than 2 claims reported on a given day, since the increments are stationary so any day you can think of with the only the interval what is the probability that the time until the next reported claim is less than 2 hours, so this is related to use the exponential distribution because the inter arrival time are exponential distribution.

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Assume that
 $\{N(t), t \geq 0\}$ is a PP

$$P[N(t) < 2] = P[N(t) = 0] + P[N(t) = 1]$$

$$= e^{-10} + 10e^{-10} = 11e^{-10}$$

$$P[N(t) = k] = \frac{e^{-10t} (10t)^k}{k!}$$

$$T = 0 \Rightarrow P\left(\frac{10}{20}\right) \quad P(T < 2) = 1 - e^{-\frac{20}{10}}$$

So for the first question you can assume that - you can assume that the N of t is nothing but a number of a - number of insurance car insurance claims reported to the insurer that has Poisson process you can assume that N of t is a Poisson process based on the assumptions given in the problem, once you assume that this is a Poisson process the question is what is the probability that there are less than 2 claims reported on a given day.

So the given day 2 days you can shift in to 0 to 2 days itself, because of the increments are stationary so the question is nothing but what is the probability that N of 1 is less than 2 in a given day a day, so what is the probability that N of 1 is less than 2, that is nothing but what is the probability that N of 1 is equal to 0 or N of 1 is equal to 1.

Therefore, the probability is added so you substitute since it is N of t is a Poisson process the probability mass function of N of t is equal to k that is e power minus lambda, here the lambda is 10 per day, 10 times t and 10 times t power k by k factorial, so this is the probability mass function for the random variable N of t for fixed t . Therefore, N of 0, N of 1 is equal to 0, that is nothing but e power minus 10 here the t is 1, 1 day + N of 1 is equal to 1.

You substitute here therefore you will get 10 times e power minus 10, so the answer is 11 times e power minus 10, numerically you can get what is a value, so the probability that there are two claims reported on a given day is a 11 times e power minus 10.

The second question what is the probability that time until the next reported claim is less than 2 hours, so this is equivalent of the next reporting claim is less than 2 hours that means the residual time of the next claim that is going to happen the one claim is going to happen less than 2 hour that means you can use the inter arrival time that is exponential distribution with the parameter lambda here the lambda is a 10 or 10 by 2 hours.

Therefore, you should convert the values so it is a 10 divided by 24 claim can happen at any day throughout 24 hours, therefore 10 per day therefore it is 10 divided by 24 per hour, so that T is a exponentially distributed with a parameter 10 by 24 now the question is what is the probability that the time until the next report claim is less than 2 hours that means what is the probability that T is less than 2.

That is nothing but - that is nothing but since it is exponential distribution and you know the CDF of the random variable T so the probability of T is less than 2 is nothing but a 1 minus e power minus 2 so 2 times so 20 by 24, so the answer is 1 minus e power minus 20 by 24 that is the probability that the next report - report claim is going to be occur before 2 hours, so with these I have completed two examples also so.

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Summary

- **Poisson process is explained.**
- **Some important properties are discussed.**
- **Simple examples are illustrated.**

So in this lecture we have discussed a Poisson process and we have illustrated two examples for the Poisson process also, some important properties also discussed in this. The next class I am going to discuss the applications of CTMC in queuing models.

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Reference Books

- **J Medhi, "Stochastic Processes", 3rd edition, New Age International Publishers, 2009.**
- **Kishor S Trivedi, "Probability and Statistics with Reliability, Queuing and Computer Science Applications", 2nd edition, Wiley, 2001.**
- **S Karlin and H M Taylor, A First Course in Stochastic Processes, 2nd edition, Academic Press, 1975.**



These are all the reference books. Thanks.