Stochastic Processes - 1
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Lecture – 58
Definition of Poisson Process

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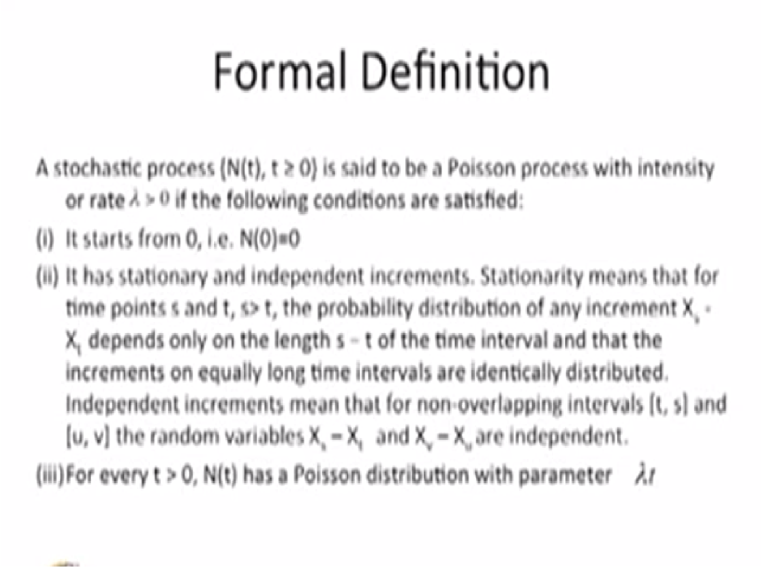

Formally we define Poisson process as follows. A stochastic process n of t, t greater than or equal to zero is said to be a Poisson process with the intensity or rate lambda greater than zero in the following conditions are satisfied. First condition, it starts from zero, that is n of zero is equal to zero. Second condition, the increments or stationary and independent, stationarity means that for time points s and t, s greater than t.

The probability distribution of any increment n of s minus n of t depends only on the length s minus t of the time interval and that the increments on equally long time intervals are identically distributed. Independent increments means that for any non-overlapping intervals, t, s and u, v, the random variables n of s minus n of t and n of v minus n of u are independent. For t greater than zero, n of t has a Poisson distributed random variable with a parameter lambda t.

And the difference of the random variables defined over non-overlapping intervals are independent. Lambda t is a cumulative rate t time t. The exercise x i's are independent and identically distributed random variables with some distribution function g independent of the

Poisson process n of t, t greater than or equal to zero. The t is Markov in nature because the two queues act independently and are themselves mm1 queuing system, which satisfied the Markov property.

Assuming that each queue behaves as the mm1 queue, the details of the proof can be found in the reference books because queue i j's are obtained by differentiating the p i j's. For every t greater than zero n of t has a Poisson distribution with the parameter lambda t. Like that you can go for many more increments also.

For illustration, I have made it with the two increments, that means the occurrence of a arrival during this non-overlapping intervals are independent and stationary mean it is a time in variant only the length matters and not the actual time. Third one, for every t, no of t has a Poisson distribution with the parameter lambda t. So the Poisson logic is coming into the fourth condition only.

The first condition is started zero, increments are stationary and increments are independent. The third condition for fixed t, n of t is a Poisson distribution random variable with the parameter lambda t. Therefore, this stochastic process is called a Poisson process. Now we can relate the way we have done the derivation. We have taken care these three assumptions starting at time zero, zero.

Increments are stationary that we have taken and increments are independent that is non-overlapping intervals are independent. Then we have derived, we are getting the distribution of the random variable n of t is a Poisson distributed random variable. Therefore, this is a Poisson process.

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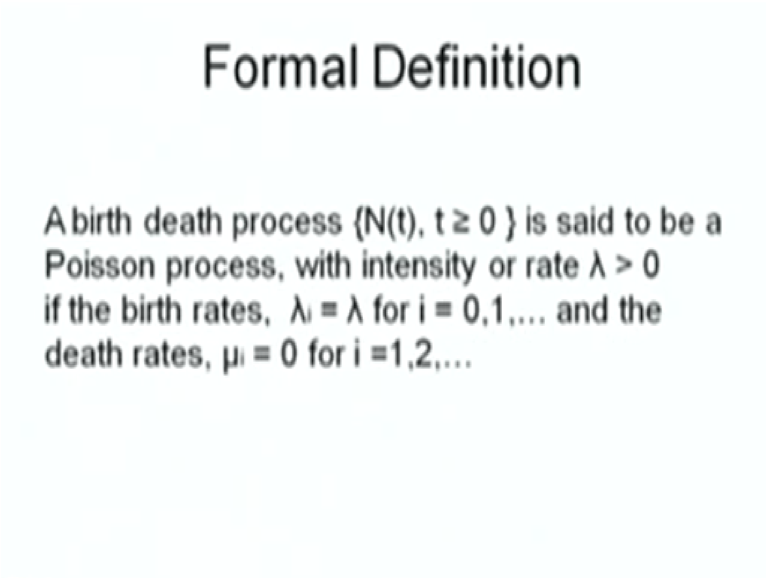

The another way of defining the Poisson process, we can start with the birth death process. You know that birth death process is a special case of a Continuous-time Markov Chain. It is a special case of a Markov process also. So you can think of a stochastic process, then the special cases of Markov process. Then the special cases are Continuous-time Markov Chain. Then you have a special case, that is a birth death process.

So you can define the Poisson process from the birth death process also. A birth death process n of t is said to be a Poisson process with intensity or rate lambda if a birth rates are constant for all i and the death rates are zero. You start from the birth death process with all the birth rates are same, that means it is a special case of pure birth process in which birth rates are constant for all the states. And the death rates are zero.

Then also you will get the Poisson process.

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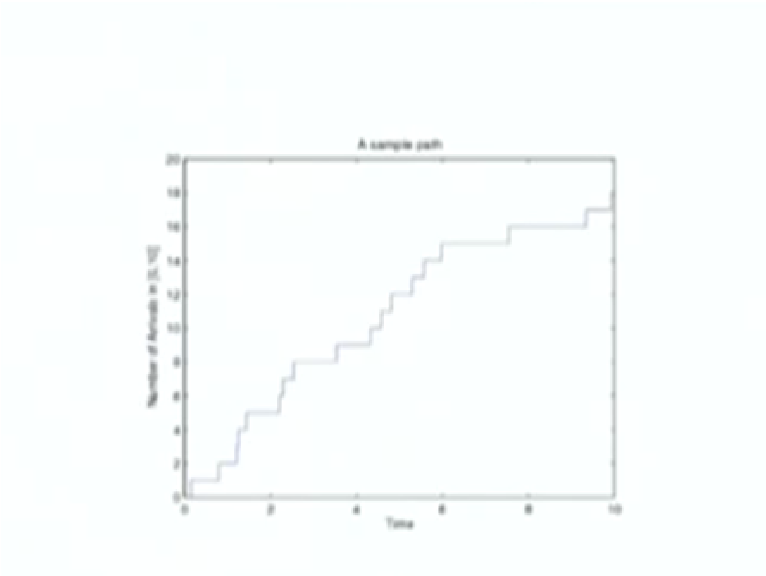

Here I am giving a sample path for the Poisson process, so this is a created using the MATLAB write the simple code of a Poisson process. Then, you develop the sample path, that means at time zero, the system at zero, at some time one arrival takes place. Therefore, the system land up 1, therefore, the y axis is nothing but the n of t. So at this time one arrival takes place therefore the number of customers in the system, number of arrivals till this time, that is 1.

So it is a right continuous function. The value at that point and the right limit is same as both are same which is different from the left limit of the arrival (()) (06:30), arrival time of a (()) (06:32). So the system was in the state 1, till the next arrival takes place. So suppose the arrival takes place here, then the n of t value is true at this time point in which the arrival (()) (06:46) and the right limit and so on.

So this is the way, therefore, the system at any time it will be the same value or it will be incremented by only one unit. The Poisson process sample path will be with the one unit step by increment at any time, there is no way the two steps the system can move forward at even in a very small interval of time, the system will move into the only one step, that you can visualise here.

Therefore, you can go back to the assumptions which we have started the derivation n of zero is equal to zero in a very small interval of time utmost one event can takes place and the difference of the random variables defined over non-overlapping intervals are independent. And increments are also stationary. So those things you cannot able to visualise in the sample path. So this is just one sample path over the time and n of t.

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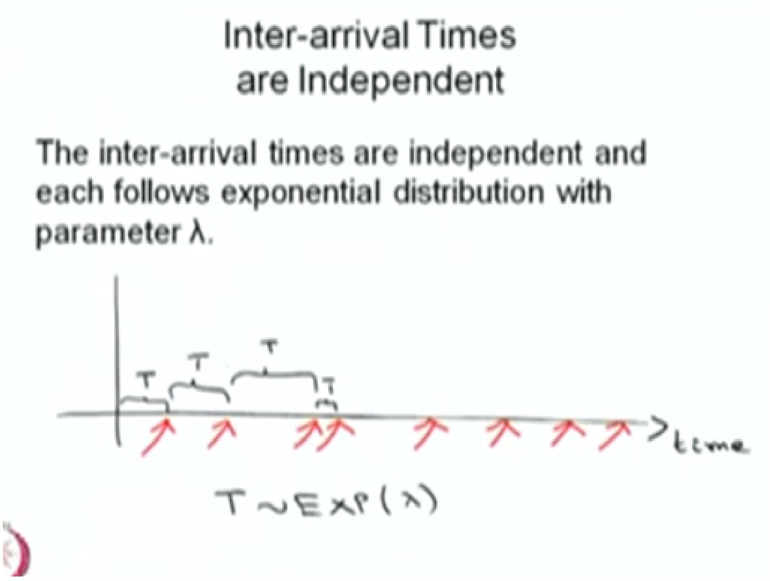

The second one inter-arrival times are independent, as well as we can conclude the inter-arrival times are exponentially distributed also. The inter-arrival times are independent and each follow exponential distribution with the parameter lambda. What is the meaning of inter-arrival times, a time zero, the system is in the state zero. First arrival occurs at this point, second arrival occurs this time point and third, fourth and so on.

The inter-arrival times means what is the time taken for the first arrival, then what is the interval of time taken for the first arrival to the second arrival and second to the third and so on. So that is the inter-arrival time. So whenever you have a Poisson process that means the arrival of event occur over the time that follows a Poisson process.

then this inter-arrival time, suppose I make it as a random variable capital t and those random variables going to follow exponential distribution with the same parameter lambda and all the inter-arrival times also independent. That means these are all identically distributed random variable, I can go for different random variable label also x1, x2, x3, x4 and so on. So all those random variables are IAD random variables and each follows exponential distribution with the parameter lambda.

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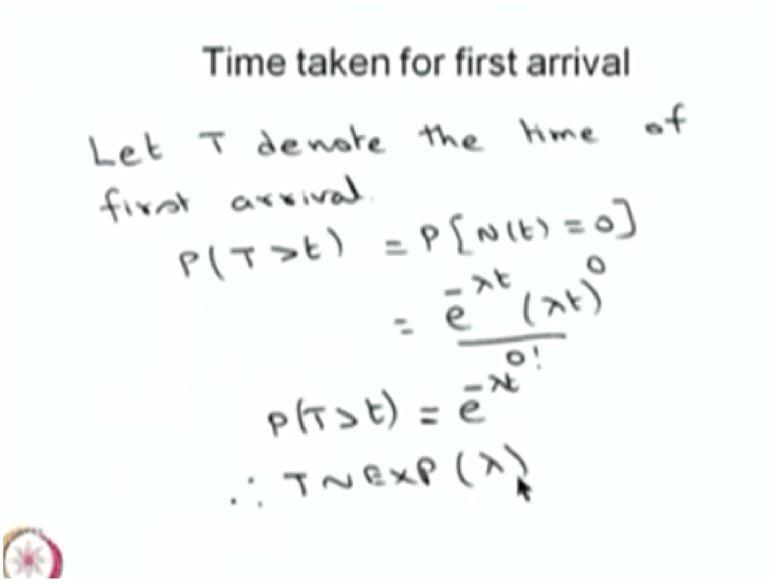

So this can be proved easily. Let me start giving the proof for the first arrival time. That means the first one from zero to the first arrival, like that you can go for the other arrivals also using the other properties or you can use the multidimensional random variable distribution concept and use the function of a random variable and you can get the distribution also.

But here I am finding the distribution for the first arrival. So let t denote the time of first arrival. My interest is to find out what is a distribution of T. I know that this is going to be a continuous random variable because it is a time, so anytime the first arrival can occur. So, since it is a continuous random variable, I can find out the CDF of the random variable or complement CDF.

So here I am finding first the complement CDF using that I am going to find out the distribution. Let me start with the probability that the first arrival is going to takes place after time t. What is the meaning of that, the first arrival is going to occur after time t. That means till time t there is no arrival. So both the events are equivalent events. The probability of t greater than t that is same as the probability of n of t is equal to zero.

That means no event takes place till time t because the n of t denotes the number of arrival of customers during the interval zero to small t, both are closed, zero to 1, zero to t. Therefore, n of t equal to zero that means till time t nobody turned up, that is equivalent of the first arrival is going to takes place after t. I do not know the distribution of capital t but I know what is the probability of n of t equal to zero.

Therefore, I am writing this relation. So once I substitute the probability mass at zero for the random variable n of t, just now we have proved that n of t for fixed t is a Poisson distribution random variable with a parameter lambda times t. Therefore, I know what is the probability mass at zero. So substitute the probability mass function with the zero. I will get e power minus lambda t that is a complement CDF of the random variable capital t.

Once I know the complement CDF, I can find out of the CDF, from the CDF I can compare the CDF of some standard continuous random variable. I can conclude this is nothing but exponential distribution with the parameter lambda because this is a complement CDF at time t. Therefore, it is a lambda times t. So I conclude the distribution of a capital t is exponential distribution with the parameter lambda.

That means the first time of arrival, this random variable that is a continuous random variable and the continuous random variable follows exponential distribution with the parameter lambda. Since, I know the increments are independent, increments are stationary and so on, I can use the similar logic for inter-arrival time of this time also, then that is also going to follow exponential distribution. Since the increments are independent.

So this is the first time and this is second time. Therefore, the inter-arrival times also going to be independent. That means, whenever you have a Poisson process, that means the arrival occurs over the time in a very small interval, maximum one arrival takes place and the probability of one arrival in that small interval is lambda times delta t, from that you will get the lambda. So you can conclude that is a Poisson distribution, Poisson process.

So ones the arrival follows a Poisson process, the inter-arrival times are exponential and independent. So from the Poisson process, one can get the inter-arrival times are exponential distribution and independent. The converse also true, that means if some arrival follows with the inter-arrival times exponential and exponential distribution and all the inter-arrival times are independent.

Then you can conclude the arrival process is going to form a Poisson process. That means arrival process and Poisson process implies the inter arrival times are exponentially distributed and are independent. Similarly, inter-arrival times are independent as well as exponentially distributed with the parameter lambda, then the arrival process is a Poisson process with a parameter lambda.