Stochastic Processes - 1
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Lecture – 57
Introduction to Poisson Process

This is a model 5 Continuous-time Markov Chain, lecture 3, Poisson process. In the first two lectures, we have discussed the Continuous-time Markov Chain, definition, Kolmogorov differential equation, Chapman-Kolmogorov equations and infinitesimal generator matrix, then we have discussed some properties also.

In the lecture 2, we have discussed the birth death process and their properties and also we have discussed the special cases of birth death process, pure birth process and death process. In this lecture, we are going to discuss Poisson process and its application. So let me start with the Poisson process definition, then I give some properties in the Poisson process and I also present some examples.

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Poisson process is a very important stochastic process. Whenever something happens in some random way occurrence of some event and if it satisfies a few properties, then we can model using a Poisson process and Poisson process has some important properties whereas the other stochastic processes will not be satisfied with those properties.

Therefore, the Poisson process is a very important stochastic process for the many modellings in applications like telecommunication or wireless networks or any computer systems or anything, any dynamical system in which the arrival comes in some pattern and satisfies few properties.

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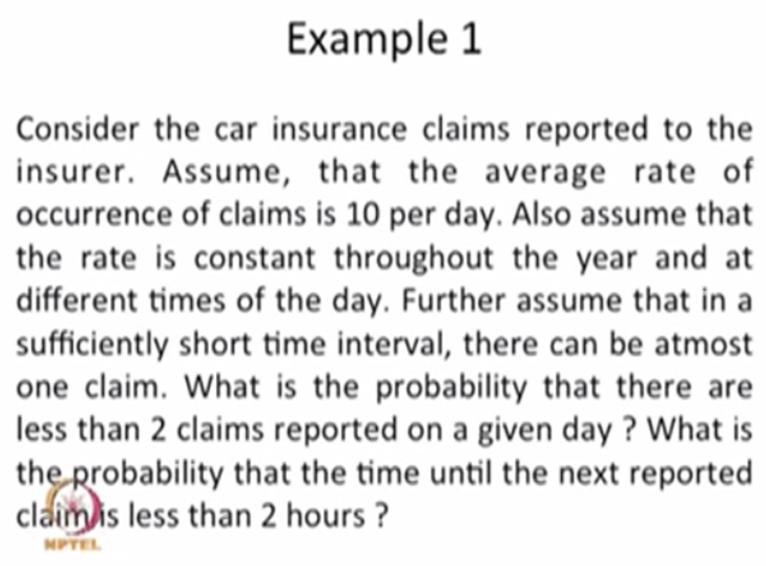

So before moving into the actual definition of Poisson process, I am going to give one simple example and through this example, I am going to relate the Poisson process definition then later I am going to solve the same example also. This is the example number 2, example 1 I have something else.

Consider a car insurance claims reported to insurer. It need not be car insurance, you can think of any motor car, motor insurance or any particular type of vehicle or whatever it is. Assume that the average rate of occurrence of claims 10 per day. It is a average rate per day. Therefore, it is a rate, per day the average rate is 10. Also assume that this rate is a constant throughout the year and at a different times of a day.

So even though these quantities are average quantities, there is a possibility some day, there is no claim reported at all or the there are someday more than some 30, 40 claims reported. And all the possibilities are there but we make the assumption, the average rate is a constant throughout the year at the different times of a day also. Further assume that, in a sufficiently short time interval there can be utmost one claim.

Suppose you think of a very small interval of 1 minute or 5 minutes or whatever, a very small quantity comparing to the because here I have given the average rate is 10 per day. Therefore, whatever the time you think of a very negligible, in that the probability of a or it is sufficiently small interval of time, there is a possibility of only maximum one claim can be reported.

The question is what is the probability that there are less than two claims reported on a given day, what is a probability that less than two claims reported means what is a probability that in a given day either no claim or one claim. Also we are asking the second question, what is the probability that the time until the next reported claim is less than 2 hours. Suppose, some time one claim is reported, what is the probability that the next time is going to be reported before 2 hours.

We started with this problem, the car insurance claims reported, therefore the claims are nothing but some event and these events are occurring over the time. Suppose you make the assumption of a sufficiently smaller interval of time utmost one claim can happen and average rate of occurrence of claim is a constant throughout the time.

So with this assumption, one we can think of a sort of arrival process, pure birth process, satisfying some condition and that may lead into Poisson process. So this same example, we are going to consider it again also. Now I am going for definition of a Poisson process. How one can derive the Poisson process.

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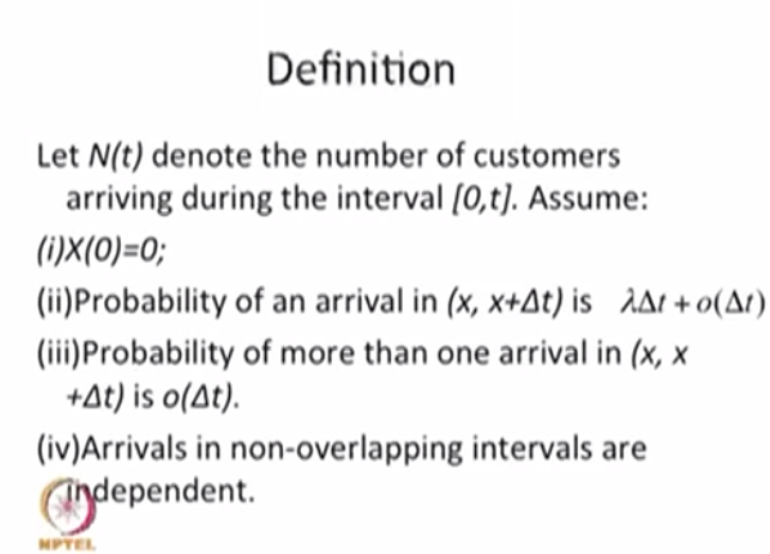

Poisson process is a stochastic process with some conditions. So how one can derive the Poisson process. For that let me start with the random variable n of t, that denotes the number of customers arriving during the interval 0 to time t. That means how many arrivals takes place in the interval 0 to t. That means for fixed t, n of t is a random variable. Over the time these n of t collection, that is a stochastic process.

I am making some four assumptions. With these assumptions, I am going to conclude the n of t is going to be a stochastic process. The first assumption, not x of zero, n of zero is equal to zero. A times zero, the number of customers is zero, n of zero is equal to zero. It is a wrong n of zero. Second one, the probability of arrival in a interval x to x plus delta t that is a lambda times delta t where lambda is strictly greater than zero.

That means probability that only one arrival is going to take place in an interval of delta t that probability is a lambda times delta t. For us very, very small interval delta t. It is independent of x that means it is a increments are stationary. That property I am going to introduce in this assumption. The probability of more than one arrival in the interval x to x plus delta t is negligible.

That means utmost maximum one arrival can occur in a very small interval of time, that is the assumption, that I am specifying in third one. The fourth assumption arrivals in non-overlapping intervals are independent. That means, if the arrival occurs in some interval and another some non-overlapping interval, then those arrivals are going to be form a independent. That means there is no dependency over the non-overlapping intervals arrivals going to occur or not.

So with these four assumptions n of zero is equal to zero and probability of one arrival is lambda times delta t in a small interval, more than one arrival occurrence in a interval delta t, where delta t is very small, that is the probability is negligible and non-overlapping intervals arrival are independent.

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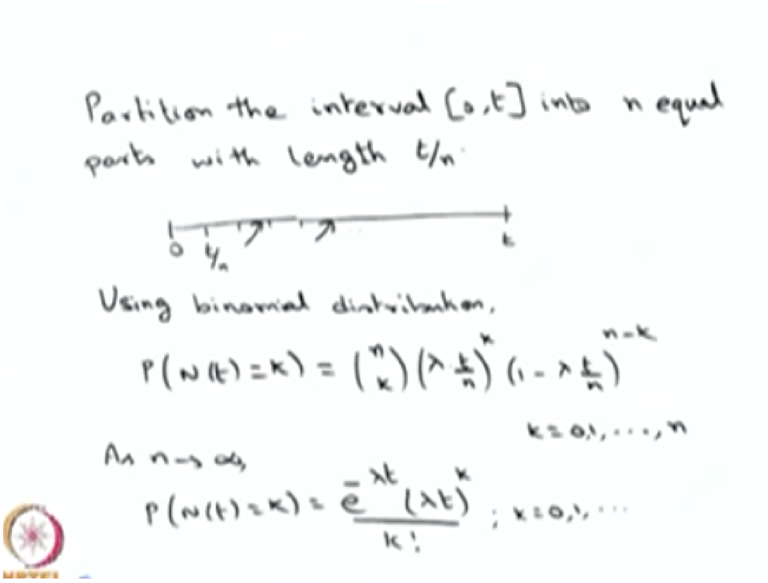

So with this derivation, I am going to find out the distribution of n of t. To find the distribution of n of t, first I am doing partitioning the interval zero to t into n equal parts with the length t divided by n. The way I use the, the way I partition the interval zero to t into n pieces, such that t by n is going to be very small interval, so that means I have to partition that interval zero to t in such a way that the t by n is going to be as small as.

Therefore, I can use those assumption of a probability of occurring one arrival in that interval of length t by n, that probability is lambda times t by n and the probability of not occurring a event in that interval t by n is 1 minus lambda time t by n. So I can use those concepts for that I have to partition the interval zero to t into n parts. It sufficient to larger n, therefore t by n is going to be smaller.

Now, since I partition these intervals into n pieces, n parts, I can think of at each part, I can think of a binomial or Bernoulli distribution at each pieces, therefore, all the non-overlapping intervals occurrence are independent therefore I can think of it is accumulation of a n independent Bernoulli trials. Since it is a n independent Bernoulli trials for each intervals t by n of length t by n.

Therefore, the total number of event occur in the interval zero to t by partitioning into n equal parts. This is a sort of what is a probability that k event occurs in the interval zero to in the time duration zero to t as a n partition. So out of n equal parts, what is the probability that k events occur in the interval zero to t. That is nothing but since it is each interval is going to form a Bernoulli distribution with a probability p is lambda times t by n.

Therefore, the total number is going to be binomial distribution with the parameters n and p where p is a lambda times t by n. Therefore, this is the probability mass function of a k event occurs out of n equal parts. Therefore, n c k lambda times t by n power k, one minus lambda times t by n power n minus k. Now the running index for k goes from zero to n. That means there is a possibility no event takes place in the interval zero to t or maximum of n interval, n event takes place in all n intervals.

So this is for sufficiently large n such that the t by n is smaller. We take n tends to infinity to understand the limiting behaviour of the scenario as the partition becomes final. Now I can go for n tends to infinity, what will happen, as n tends to infinity, if you do the simplification here as n tends to infinity, that simplification I am not doing in this presentation as a limit n tends to infinity, the whole thing will land up e power minus lambda t, lambda t power k by k factorial.

Now the k running index is a 0, 1, 2 and so on. This you can use the concept the binomial distribution as n tends to infinity and p tends to zero, your n into p becomes the lambda. So that will give the Poisson distribution. The limiting case of a binomial distribution is a Poisson distribution.

So using that logic, these binomial distribution mass as n intends to infinity. This becomes a Poisson distribution mass function. So this is nothing but the right hand side is the probability mass function for a Poisson distribution with a parameter lambda times t. And this is a random variable n of t for a fixed t. Therefore, for fixed t n of t is a Poisson distributed random variable with a parameter lambda times t where lambda is greater than zero.

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i.e., $P[N(t)=k] = \frac{e^{-\lambda t}(\lambda t)^{k}}{k!}\ ;\ k=0,1,\dots$

$$E(N(t)) = \lambda t$$

$$Var(N(t)) = \lambda t$$

Therefore, we can conclude the stochastic process related to the n of t or fixed t n of t is a Poisson distribution. Therefore, the stochastic process n of t over the t greater than or equal to zero that is nothing but a Poisson process. So from the Poisson distribution, we are getting Poisson process because each random variable is a Poisson distributed with a parameter lambda times t.

Therefore, that collection of random variable is a Poisson process with the parameter lambda t. Since it is a Poisson distributed random variable for fixed t, you can get the mean and variants and all other moments also by using the probability mass function of n of t.