Stochastic Processes - 1
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Lecture – 56
Steady State Distributions, Pure Birth Process and Pure Death Process

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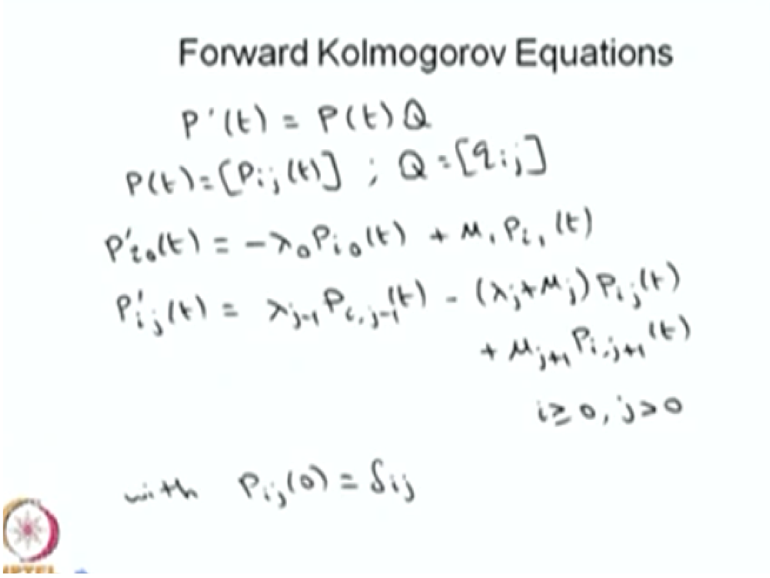

We are discussing the forward Kolmogorov equation of a special case of Continuous-time Markov Chain that is the birth death process. For a birth death process, the Q matrix is a tridiagonal matrix. Therefore, you will have the equations from the forward Kolmogorov equation, you will have a only two terms in the right hand side for the first equation and you will have only three terms, the diagonal element and two off-diagonal elements.

Therefore, the first equation one can discuss first, the p dash of i comma zero that is nothing, but the system is not mode from the state zero, moving from the state zero that rate is lambda naught, therefore not moving minus lambda naught times the probability and (()) (00:47), the system can come from the state one with the rate mu 1. Therefore, mu 1 times p i comma 1 of t.

For all other equations either the system comes from the previous state with the rate lambda j minus 1 or it comes from the forward one state with the rate mu j plus 1 or not moving anywhere. So these are all the all possibilities therefore with these three possibilities you have a three terms in the right hand side and that is the net rate for any strategy. So if you solve this equation with this initial condition, Kronecker delta i comma j, you will have the solution of a p i comma j.

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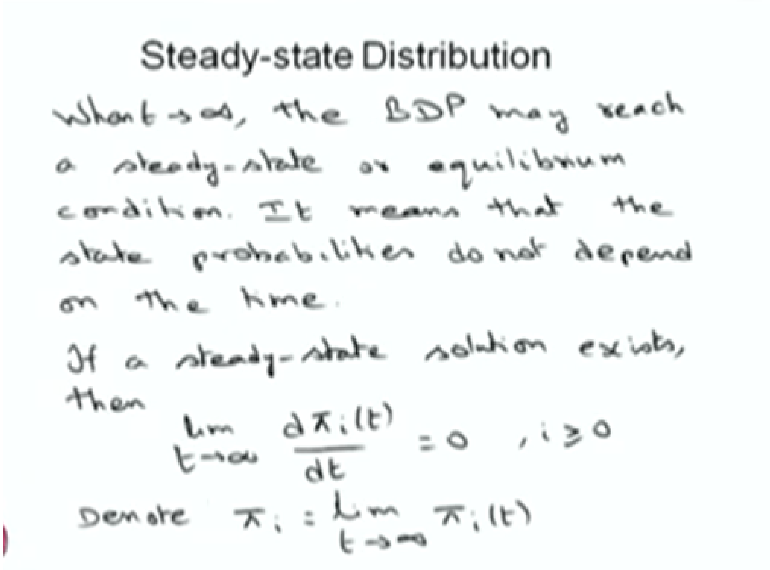

Here I am discussing the steady state distribution, the way I have discussed the limiting distribution that is a limit t tends to infinity, probability of i comma j of t exist, then it is called the limiting distribution and the stationary distribution is nothing but for the DTMC is pi is equal to p summation of pi is equal to 1. For the CTMC pi q is equal to zero and the summation of pi i is equal to 1. That is going to be the steady state distribution, stationary distribution.

Now I am discussing the steady state distribution, that is nothing but when t tends to infinity the birth death process may reach steady state or equilibrium condition. That means the state probability is does not depend on time. That is a meaning of a steady state distribution. As t tends to infinity, whenever we say the birth death process reaches a steady state or at equilibrium, that state probability does not depend on time.

That means a if a steady state solution exist, since the state probability does not depend on time t, the derivative of the time dependant state probability at time t, that derivative at t tends to infinity becomes zero. If the steady state solution exists. Since the state probability is does not depend on time t as t tends to infinity, I can write as a pi i is a limit t tends to infinity of pi i of t.

So this is different from the way we discussed earlier that conditional probability p i j of t.

But using p i j of t, one can find out what is pi j of t, pi i of t.

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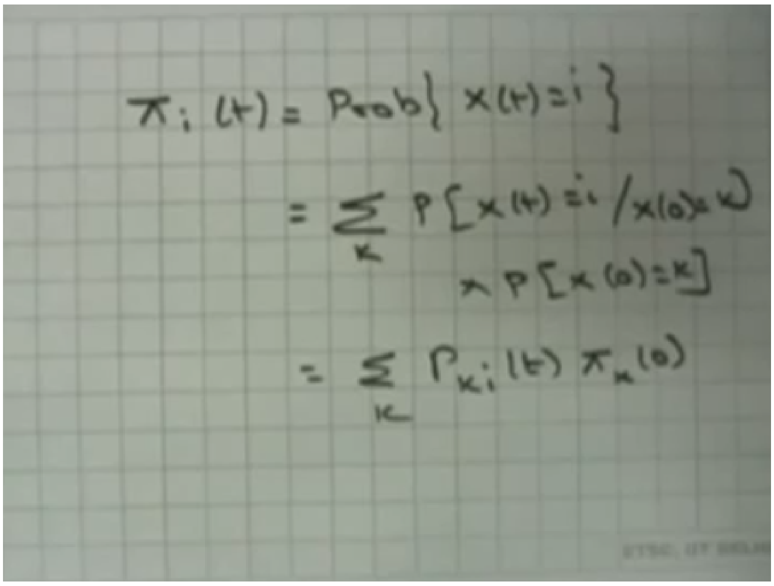

That is nothing but the pi of t that I have given in the first lecture for the CTMC. The pi i of t that is nothing but what is the probability that the system will be in the state i at times. That is same as what is the probability that the system will be in the state i given that it was in the state some k at times zero multiplied by what is the probability that it was in the state k at times k.

That is nothing but summation of k and this is nothing but the transition probability and this is nothing but the initial probability vector element. So using p k i of t or p i j of t that is a conditional probability, one can get the unconditional probability. This is nothing but the distribution of x of t.

So this is the probability mass function, probability mass at state i. So now what I am defining, whenever the steady state distribution exists, that means it is independent of time t. Therefore, as t tends to infinity the pi i of t can be written as the pi i and whenever the steady state solution exists, I can use limit t tends to infinity, the derivative of a pi i of t that is going to be zero.

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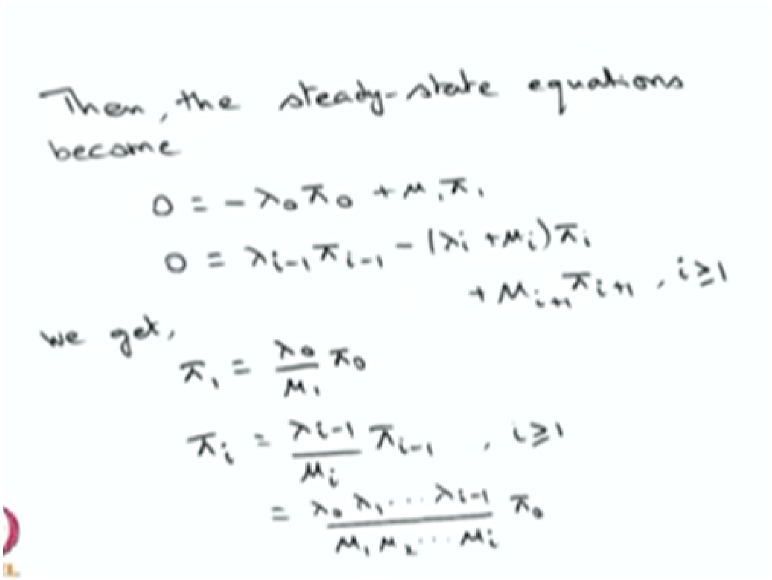

Therefore, I am going to use these two to get the steady state probabilities for the birth death process. Since, as a t tends to infinity, the derivative of pi i j of t is equal to zero, therefore, all the left hand side in the forward Kolmogorov equation that is going to be zero, the right hand side you will have a as a t tends to infinity, the pi i of t, that can be written as the pi zero and pi one.

So the way we write the conditional probability for p i j with the Kolmogorov forward equation, you can write the similar equation for the unconditional probability pi i also. So now I am putting the left hand side zeros because of this condition, limit t tends to infinity, the derivatives equal to zero and the right hand side I am using as a t tends to infinity, this probability is nothing but the pi i.

Therefore, it is going to be minus lambda naught times pi naught plus mu 1 times pi and all other equation as a three terms. In this homogeneous equation and you need a one normalising condition. So from this homogeneous equation, I can get regressively pi i in terms of pi naught. So from the first equation, I can get a pi 1 in terms of pi naught and the second equation.

I can get a pi 2 in terms of first pi 1 then I can get a pi 1 in terms of pi naught. Therefore, regressively I can get pi i in terms of pi naught for all i greater than or equal to 1.

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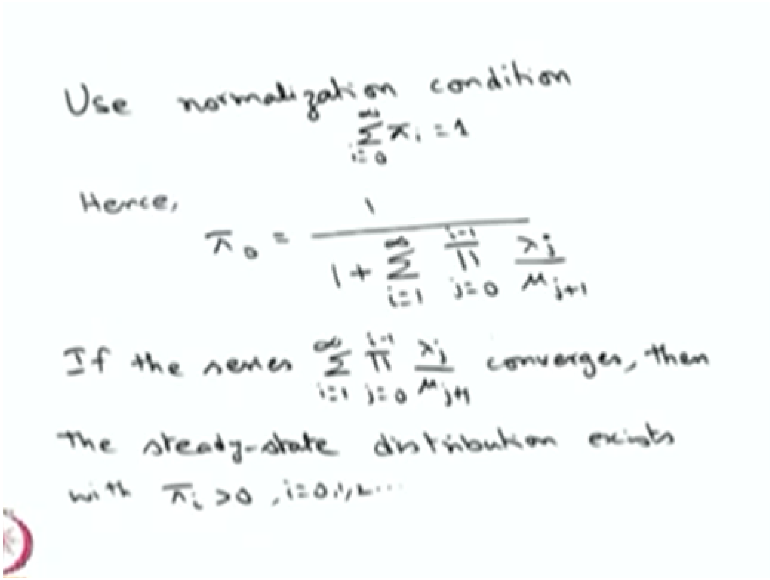

Now I can use a normalising condition, summation of pi i is equal to 1, therefore, I will get a pi naught is equal to 1 divided by summation of this many terms in the product form. Since we need a steady state probabilities and all the pi i are in terms of pi naught. As long as the denominator is converges, you will have a pi zero is greater than zero. So ones the pi zero is greater than zero, then you will get all the pi i with the summation of pi i is equal to 1.

So whenever these series converges, then I will have a steady state distribution with the positive probability and a summation of probability is going to be 1. So this is the condition for a steady state distribution for a birth death process because we started with a birth death process forward Kolmogorov equation using these two conditions we have simplified into this form and use a normalising condition and get the pi naught.

As long as the summation is or the series is converges, then you will have the steady state. If the series diverges, that means by substituting the values for the lambda i's and mu i’s and if the series denominator series diverges, then the pi naught is going to be zero in turn all the pi i’s are is equal to zero therefore the steady state distribution will not exist if the denominator series diverges.

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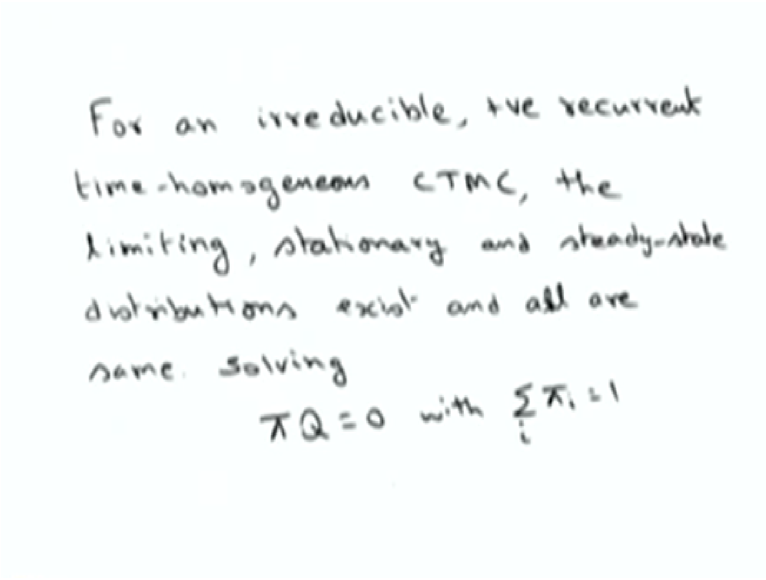

I am going to give a one simple result, for a irreducible positive recurrent time homogeneous CTMC, we know that a limiting distribution exist a stationary distribution exist. Now I am including the steady state distribution also exist, I have given for a steady state distribution for the birth rate process, not for the CTMC but here I am giving the result for the CTMC. All the three distribution exist and all are going to be same.

Whenever the CTMC is at time homogeneous irreducible positive recurrent, all these three distributions are same and one can evaluate, one can solve this two equation by q is equal to zero and with the summation of pi is equal to 1, you can get the limiting distribution, stationary distribution or steady state or equilibrium distribution.

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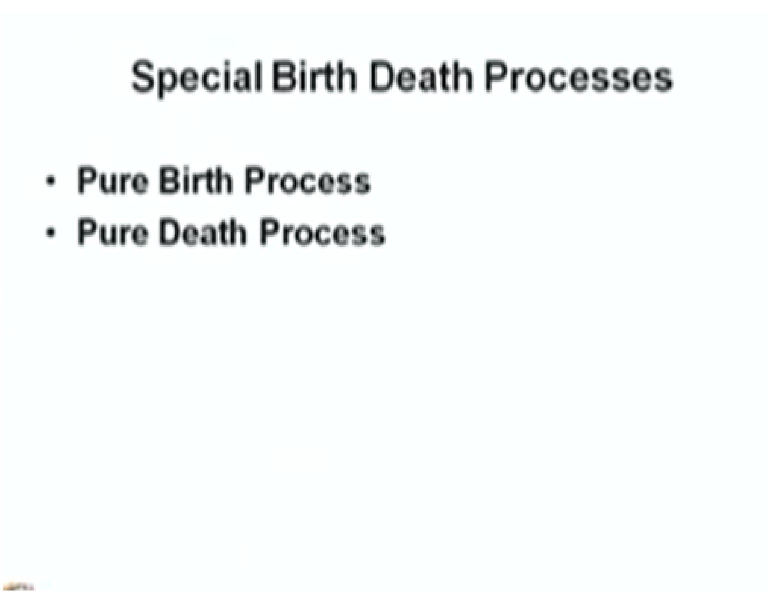

As a special case of birth death process, I am going to discuss these two process in this lecture.

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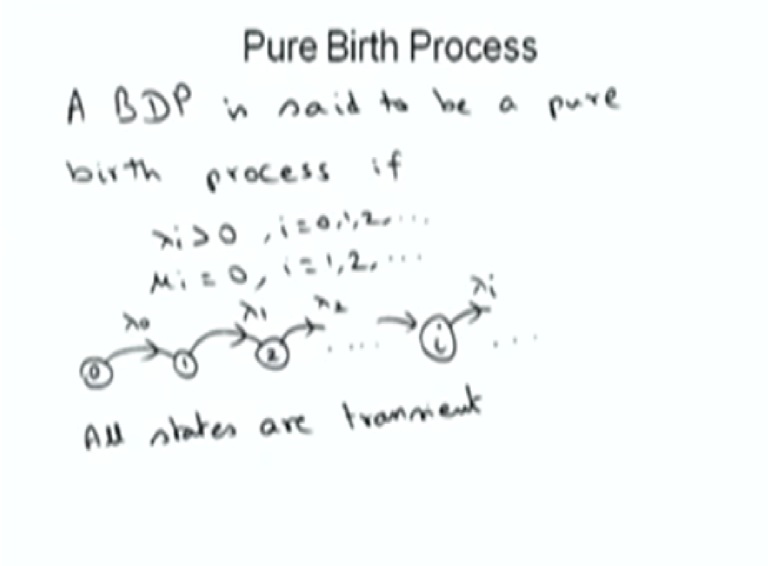

Whenever, we say the birth death process is a pure birth process, that means all the death rates are going to be zero, we started with a birth death process with the only lambda i's are greater than zero and the mu i's are going to be zero, then it is going to be call it as a pure birth process. There is a one special case of pure birth process with the lambda i's are going to be constant, that is lambda, that is a Poisson process.

I am going to discuss in the next lecture and in these pure birth process, these lambda i's are the function of i. Here all the states are transient states.

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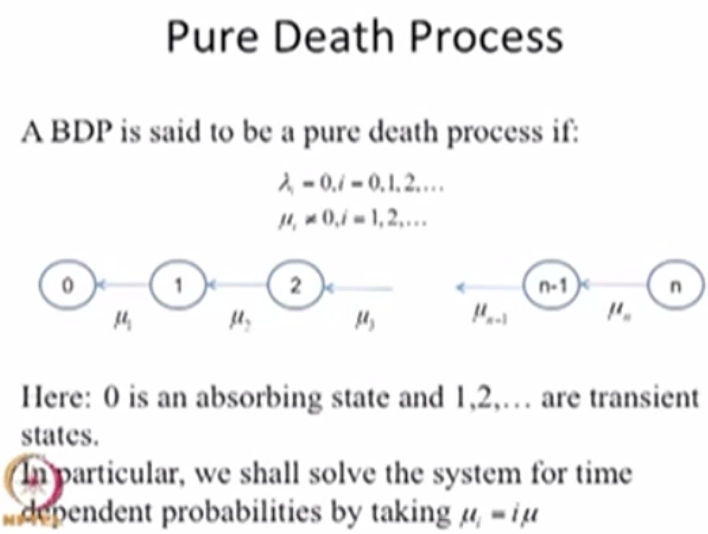

Here I am discussing the pure death process. A birth death process is said to be a pure death process if the birth rates are zero and the death rates are non zero. In particular, we shall obtain the time dependent probabilities of a pure death process in which the death rates mu i's are equal to i times mu.

As I given the example, as a fourth example in the birth death process, this state zero is a observing barrier. Therefore, the state zero is a observing state and all other states are going to be transient state. And here the limiting distribution exists and one can also find time dependent probabilities for this model.

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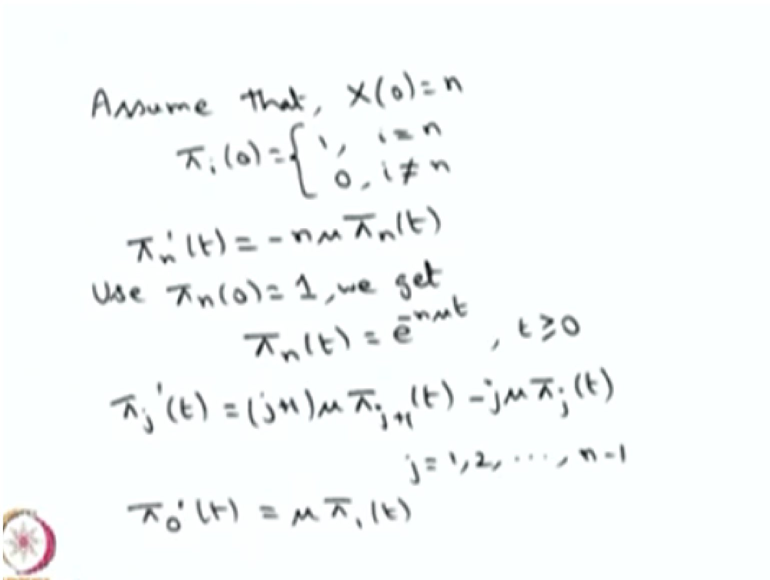

Suppose you start with the assumption, the system time zero, in the system is in the state n, at times zero the systems in the state n at times zero. With that assumption, I can frame the equation that is the pi n dash of t is equal to minus n times mu of pi n of t.

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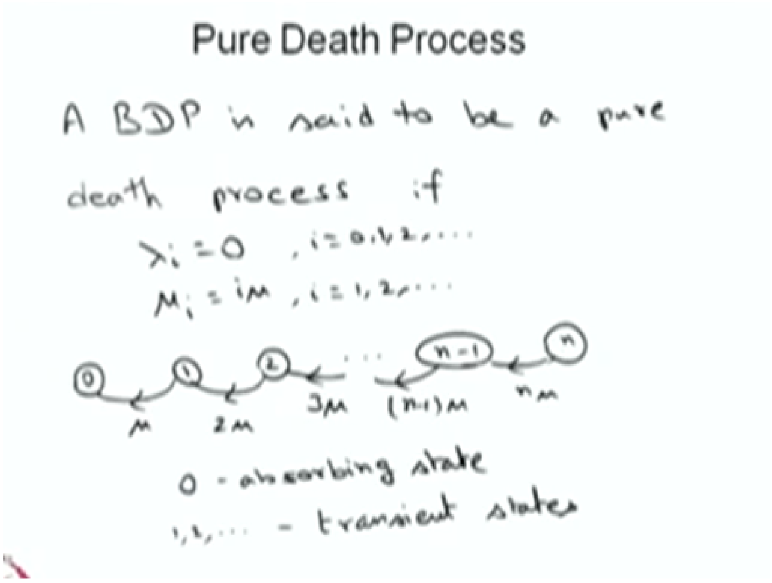

That means the rate in which the system is in the state n that is nothing but not moving to the state n minus 1 with the rate n minus n times mu. Therefore, the equation for the state n that is a pi n dash of t that is equal to not moving from the state n therefore minus that outgoing rate that is n times mu being the state is n therefore pi n of t. I can use the initial condition pi n of zero is equal to 1, so I will get pi n of t.

For the second equation, I have to go for what is the equation for the state n minus 1. So the pi n minus 1 dash t, that is nothing but either the system come from the state n or not moving from the state n minus 1. Therefore, system coming from the state n that is a n mu times the system being the state n minus n minus 1 times mu pi n minus 1 of t. So we will have a two terms in the right hand side coming from the one forward state or not moving from the same state.

So you will have a two terms for j is equal to 1 to n minus 1. For the last state, that is the state zero, the system comes from the state 1. Since the state is observing states, there is no second term. So it is going to be mu times pi n of t. So you know pi n of t, use the pi n of t in the equation for n minus 1 and get the pi n minus 1, like that you find out till pi 1. Use the pi 1 to get the pi zero of t. Use the recursive way.

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Use $\pi_n(t) = e^{-n\mu t}$

$$\frac{d}{dt}\left(e^{(n-1)\mu t}\pi_{n-1}(t)\right) = n\mu\pi_n(t)e^{(n-1)\mu t}$$

$$\pi_{n-1}(t) = n e^{-(n-1)\mu t}\mu\int_0^t e^{-n\mu x}e^{(n-1)\mu x}dx$$

$$\pi_{n-1}(t) = n e^{-(n-1)\mu t}(1-e^{-\mu t})$$

Recursively,

$$\pi_j(t) = \binom{n}{j}(e^{-\mu t})^j(1-e^{-\mu t})^{n-j}$$

So using the recursive way, you will get the pi j of t is equal to n c j combination n c j and e power minus mu times t power j, this is survival probability of system being in the state and 1 minus e power minus mu of t n minus j. Suppose the system being in the state j, that means from the state n this many combinations would have come and the survival probability is e power minus mu times t and that power.

So this is nothing but the probability p power j and one minus p power n minus j. Therefore, this pi j follows the binominal distribution with the survival probability e power minus mu t being in the state j.

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Summary

- Limiting, stationary and steady-state distributions are discussed.
- Birth death process is introduced.
- Some important results in BDP are explained.
- Pure birth and pure death processes are discussed.
- Some examples are illustrated.

So for the pure death process, I have explained the time dependent probabilities of being in the state j, that is a unconditioned probability. So with this the summary of this lecture is, I have discussed the limiting stationary and a steady state distribution, I have introduced a birth death process. Some important results also discussed.

And at the end, I have discussed the pure birth and pure death process also. In the next lecture, I am going to explain the important pure birth process that is the Poisson process.

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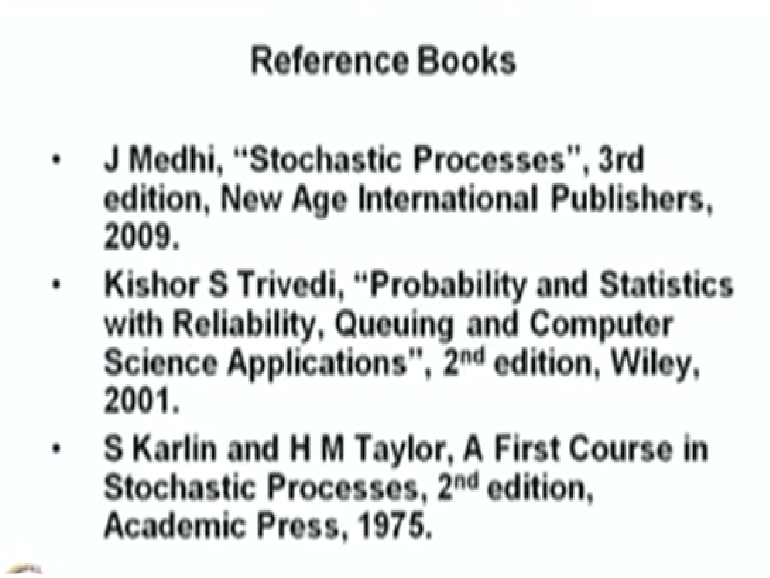

And these are all the reference books. Thanks.